Uma formulação integral para as teorias de Yang-Mills

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Plan of the talk

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Integral formulation of Maxwell's theory

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Generalized non-abelian Stokes theorem

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Conserved charges: gauge invariance, etc

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Gauge and Integrable theories

Maxwell's equations

Maxwell's equations

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \qquad \vec{\nabla} \times \vec{B} - \mu_0\,\varepsilon_0\,\frac{\partial\,\vec{E}}{\partial\,t} = \mu_0\,\vec{J}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{E} + \frac{\partial\,\vec{B}}{\partial\,t} = 0$$

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$$\mu_0\, \varepsilon_0 = 1/c^2$$

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$$E_i = F_{0i}$$

$$B_i = -\frac{1}{2\, c}\, \varepsilon_{ijk} F_{jk}$$

$$j^\mu \equiv \frac{1}{\varepsilon_0} \left(\rho\, ,\, -\frac{1}{c}\, J^i \right)$$

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$$\mu_0\, \varepsilon_0 = 1/c^2$$

$$\widetilde{F}_{\mu\nu} = \frac{1}{2}\, \varepsilon_{\mu\nu\rho\sigma}\, F^{\rho\sigma}$$

Integral form of Maxwell's equations

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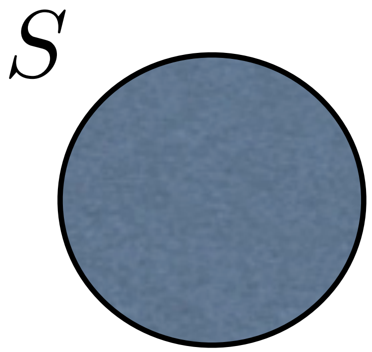
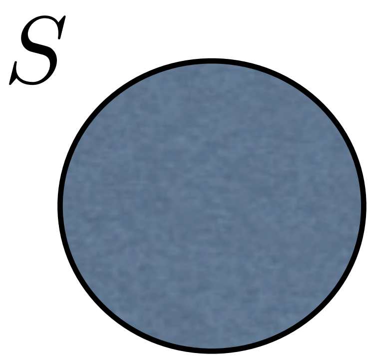


Closed two dimensional surface in space-time

Integral form of Maxwell's equations



Closed two dimensional surface in space-time

$$\Phi(S) = \oint_S F_{\mu\nu} \frac{\partial\, x^\mu}{\partial\, \sigma} \frac{\partial\, x^\nu}{\partial\, \tau}\, d\sigma\, d\tau$$

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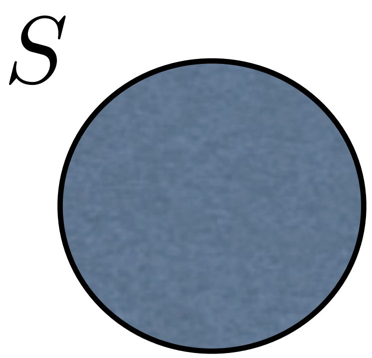

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Maxwell's equations ⟷

$$\Phi(S) = 0$$

$$\widetilde{\Phi}(S) = -\frac{Q}{\varepsilon_0}$$

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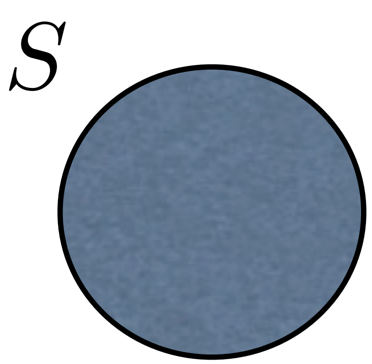

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Maxwell's equations $\longleftrightarrow$

$$\Phi(S) = 0$$

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charge inside S

$$Q = -\varepsilon_0 \int_V \widetilde{J}_{\mu\nu\rho} \frac{\partial\, x^\mu}{\partial\, \sigma} \frac{\partial\, x^\nu}{\partial\, \tau} \frac{\partial\, x^\rho}{\partial\, \zeta}\, d\sigma\, d\tau d\zeta$$

$$\widetilde{J}_{\mu\nu\rho} = \varepsilon_{\mu\nu\rho\sigma} j^\sigma$$

$S \equiv$ closed spatial surface

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$$\Phi(S) = \oint_S F_{ij} \frac{\partial\, x^i}{\partial\, \sigma} \frac{\partial\, x^j}{\partial\, \tau}\, d\sigma\, d\tau = -c \oint_S \vec{B} \cdot d\vec{\Sigma} = 0$$

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$$F_{ij} = -c\, \varepsilon_{ijk}\, B_k \qquad d\vec{\Sigma} = \varepsilon_{ijk} \frac{\partial\, x^i}{\partial\, \sigma} \frac{\partial\, x^j}{\partial\, \tau}\, d\sigma\, d\tau \qquad \vec{\nabla} \cdot \vec{B} = 0$$

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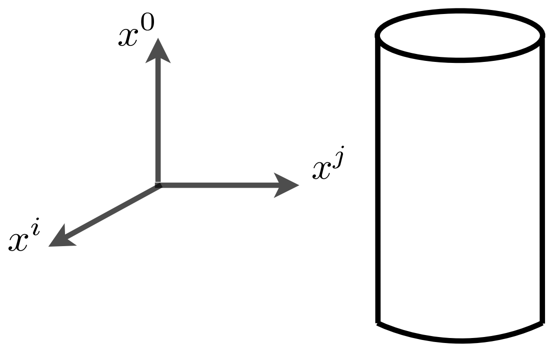
$$\widetilde{F}_{ij} = -\varepsilon_{ijk} \, E_k$$

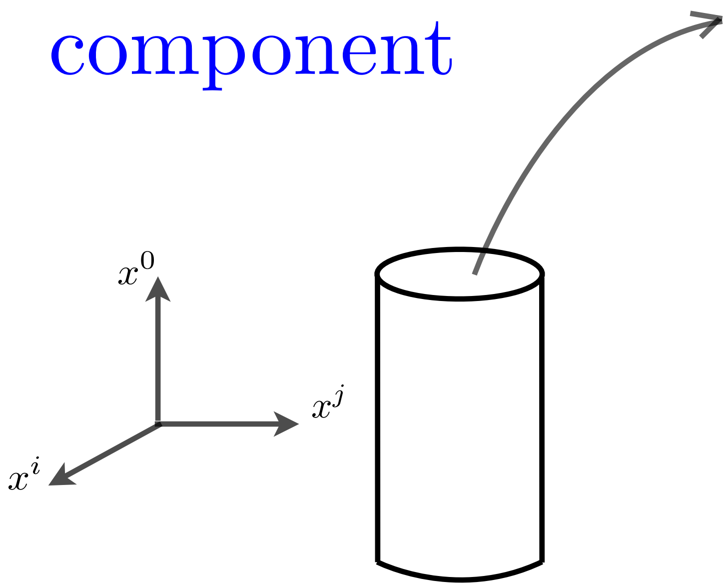
$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

S with a time component



S with a time component

$$\Phi(\text{disc}) = \int_{\text{disc}} F_{ij} \frac{\partial x^i}{\partial \sigma} \frac{\partial x^j}{\partial \tau} \, d\sigma \, d\tau = -c \int_{\text{disc}} \vec{B} \cdot d\vec{\Sigma}$$



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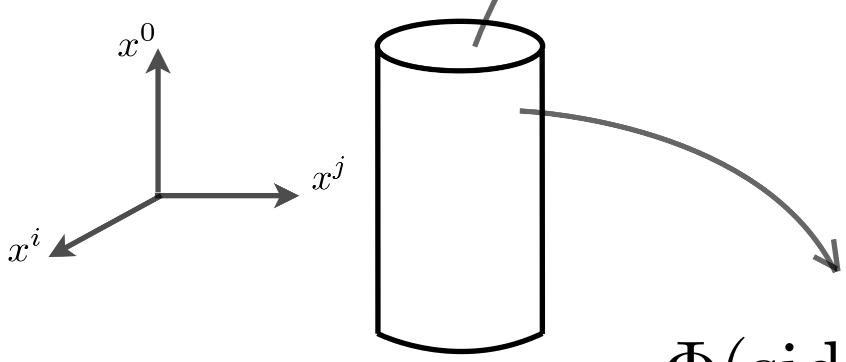
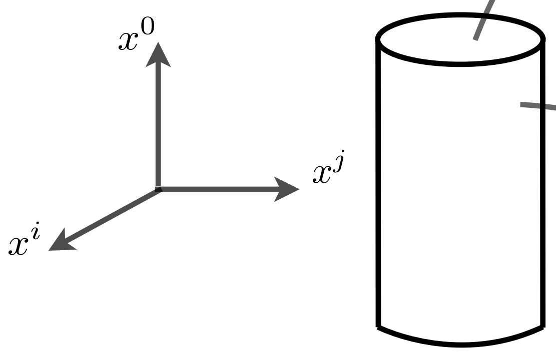


$$\Phi(\text{side}) = \int_{\text{side}} F_{k0} \frac{\partial\, x^k}{\partial\, \sigma} \frac{\partial\, x^0}{\partial\, \tau}\, d\sigma\, d\tau = -\int dx^0 \oint \vec{E} \cdot d\vec{l}$$

S with a time component



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Then

$$\Phi(S) = 0 = -c \int_{\text{top disc}} \vec{B} \cdot d\vec{\Sigma} + c \int_{\text{botton disc}} \vec{B} \cdot d\vec{\Sigma} - \int dx^0 \oint \vec{E} \cdot d\vec{l}$$

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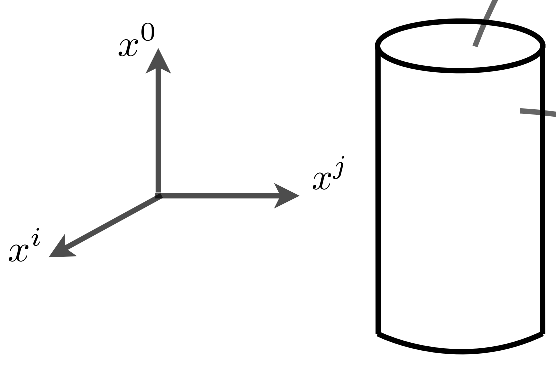

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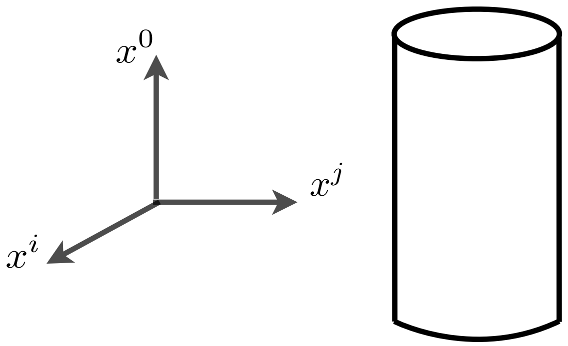
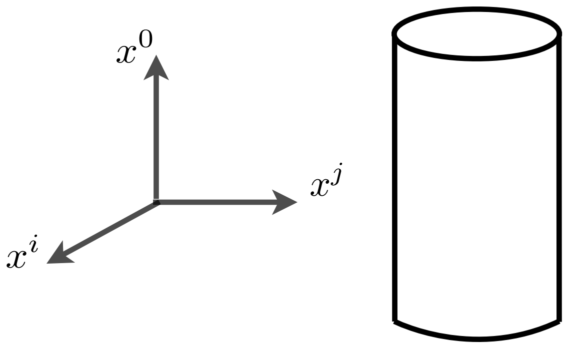
In the limit $\delta x^0 \to 0$

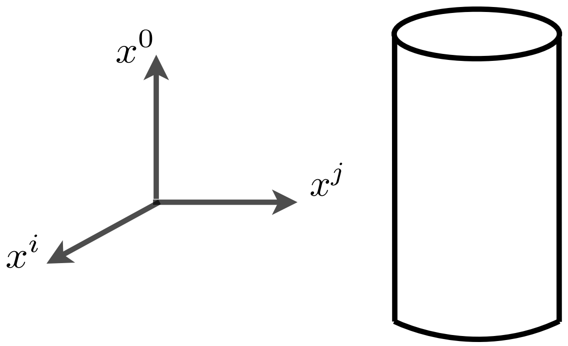
$$\frac{d}{d\,t} \int \vec{B} \cdot d\vec{\Sigma} = -\oint \vec{E} \cdot d\vec{l} \qquad \rightarrow \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$


x^0
x^j
x^i



Analogously

$$
\begin{aligned}
\widetilde{\Phi}(S) &= -\int_{\text{top disc}} \vec{E}\cdot d\vec{\Sigma} + \int_{\text{botton disc}} \vec{E}\cdot d\vec{\Sigma} + c\int dx^0 \oint \vec{B}\cdot d\vec{l} \\
&= \frac{1}{c\,\varepsilon_0}\int dx^0 \int \vec{J}\cdot d\vec{\Sigma}
\end{aligned}
$$



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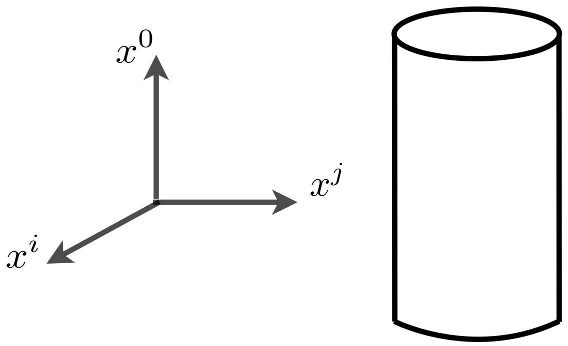
$$= \frac{1}{c\,\varepsilon_0}\int dx^0 \int \vec{J}\cdot d\vec{\Sigma}$$

In the limit $\delta x^0 \to 0$

$$\oint \vec{B}\cdot d\vec{l} - \varepsilon_0\,\mu_0 \frac{d}{dt}\int \vec{E}\cdot d\vec{\Sigma} = \mu_0 \int \vec{J}\cdot d\vec{\Sigma}$$



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$$\vec{\nabla}\times\vec{B} - \mu_0\,\varepsilon_0\,\frac{\partial\,\vec{E}}{\partial\,t} = \mu_0\,\vec{J}$$

Summarizing

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Maxwell's eqs. are equivalent to

$$\oint_S \left[\alpha\, F_{\mu\nu} + \beta\, \widetilde{F}_{\mu\nu}\right] \frac{\partial\, x^\mu}{\partial\, \sigma}\, \frac{\partial\, x^\nu}{\partial\, \tau}\, d\sigma\, d\tau = -\beta\, \varepsilon_0\, \int_V \widetilde{J}_{\mu\nu\rho}\, \frac{\partial\, x^\mu}{\partial\, \sigma}\, \frac{\partial\, x^\nu}{\partial\, \tau}\, \frac{\partial\, x^\rho}{\partial\, \zeta}\, d\sigma\, d\tau d\zeta$$

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V is the volume inside S

$$\left(\widetilde{J}_{\mu\nu\rho} = \varepsilon_{\mu\nu\rho\sigma}\, j^{\sigma}\right)$$

Summarizing

Maxwell's eqs. are equivalent to

$$\oint_S \left[\alpha\, F_{\mu\nu} + \beta\, \widetilde{F}_{\mu\nu}\right] \frac{\partial\, x^\mu}{\partial\, \sigma}\, \frac{\partial\, x^\nu}{\partial\, \tau}\, d\sigma\, d\tau = -\beta\, \varepsilon_0 \int_V \widetilde{J}_{\mu\nu\rho}\, \frac{\partial\, x^\mu}{\partial\, \sigma}\, \frac{\partial\, x^\nu}{\partial\, \tau}\, \frac{\partial\, x^\rho}{\partial\, \zeta}\, d\sigma\, d\tau d\zeta$$

V is the volume inside S

$$\left(\widetilde{J}_{\mu\nu\rho} = \varepsilon_{\mu\nu\rho\sigma}\, j^\sigma\right)$$

Maxwell's eqs. are recovered in the limit where S is infinitesimal

Yang-Mills equations

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with

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Yang-Mills equations

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$$J_\mu = \bar{\psi}\,\gamma_\mu\,R(T_a)\,\psi\,T_a$$

Yang-Mills equations

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Gauge covariant under the transformations

$$\begin{aligned} A_{\mu} &\rightarrow g\, A_{\mu}\, g^{-1} + \frac{i}{e}\, \partial_{\mu} g\, g^{-1} \\ F_{\mu\nu} &\rightarrow g\, F_{\mu\nu}\, g^{-1} \\ J^{\mu} &\rightarrow g\, J^{\mu}\, g^{-1} \end{aligned}$$

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$$j_\nu \equiv \partial^\mu F_{\mu\nu} = J_\nu - i\,e\,[\,A_\mu\,,\,F_{\mu\nu}\,] \qquad \rightarrow \qquad \partial^\mu j_\mu = 0$$

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Charges

$$Q = \int d^3x\, \partial^i F_{i0} = \int d^3x\, \vec{\nabla} \cdot \vec{E} = \int d\vec{\Sigma} \cdot \vec{E}$$

$$\widetilde{Q} = \int d^3x\, \partial^i \widetilde{F}_{i0} = -\int d^3x\, \vec{\nabla} \cdot \vec{B} = -\int d\vec{\Sigma} \cdot \vec{B}$$

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Under a gauge transformation

$$Q \rightarrow \int d\vec{\Sigma} \cdot g\, \vec{E}\, g^{-1} = g\, Q\, g^{-1}$$

if g is constant at infinity

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$$\widetilde{j}_\nu \equiv \partial^\mu \widetilde{F}_{\mu\nu} = -i\, e \left[\, A_\mu \,,\, \widetilde{F}_{\mu\nu} \,\right] \qquad \rightarrow \qquad \partial^\mu \widetilde{j}_\mu = 0$$

Charges

$$Q = \int d^3x\, \partial^i F_{i0} = \int d^3x\, \vec{\nabla} \cdot \vec{E} = \int d\vec{\Sigma} \cdot \vec{E}$$

$$\widetilde{Q} = \int d^3x\, \partial^i \widetilde{F}_{i0} = -\int d^3x\, \vec{\nabla} \cdot \vec{B} = -\int d\vec{\Sigma} \cdot \vec{B}$$

Under a gauge transformation

$$Q \rightarrow \int d\vec{\Sigma} \cdot g\, \vec{E}\, g^{-1} = g\, Q\, g^{-1} \longrightarrow$$

eigenvalues of Q are gauge invariant

(the equality holds) if g is constant at infinity

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Steven Weinberg
vol. II, pag. 12-13

15.3 Field Equations and Conservation Laws

Using Eq. (15.2.9) for the matrix $g_{\alpha\beta}$ in Eq. (15.2.3), the full Lagrangian density is

$$\mathscr{L} = -\tfrac{1}{4}\, F_{\alpha\mu\nu} F_\alpha{}^{\mu\nu} + \mathscr{L}_M(\psi, D_\mu\psi)\,, \tag{15.3.1}$$

where in the absence of gauge fields $\mathscr{L}_M(\psi, \partial_\mu\psi)$ would be the 'matter' Lagrangian density. We could, in principle, include a dependence of $\mathscr{L}_M$ on $F_{\alpha\mu\nu}$ as well as higher covariant derivatives $D_\nu D_\mu \psi$, $D_\lambda F_{\alpha\mu\nu}$, etc., but we exclude these non-renormalizable terms here for the same reason as in electrodynamics: as discussed in Section 12.3, such terms would be highly suppressed at ordinary energies by negative powers of some very large mass. For this reason the standard model of the weak, electromagnetic and strong interactions has a Lagrangian of the general form (15.3.1).

The equations of motion of the gauge field are here

$$\partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu A_{\alpha\nu})} = -\partial_\mu F_\alpha{}^{\mu\nu} = \frac{\partial \mathscr{L}}{\partial A_{\alpha\nu}}$$
$$= -F_\gamma{}^{\nu\mu} C_{\gamma\alpha\beta} A_{\beta\mu} - i\,\frac{\partial \mathscr{L}_M}{\partial D_\nu \psi}\, t_\alpha \psi$$

and so

$$\partial_\mu F_\alpha{}^{\mu\nu} = -\mathscr{J}_\alpha{}^\nu\,, \tag{15.3.2}$$

where $\mathscr{J}_\alpha{}^\nu$ is the current:

$$\mathscr{J}_\alpha{}^\nu \equiv -F_\gamma{}^{\nu\mu} C_{\gamma\alpha\beta} A_{\beta\mu} - i\,\frac{\partial \mathscr{L}_M}{\partial D_\nu \psi} t_\alpha \psi\,. \tag{15.3.3}$$

The current $\mathscr{J}_\alpha{}^\nu$ is conserved in the ordinary sense

$$\partial_\nu \mathscr{J}_\alpha{}^\nu = 0\,, \tag{15.3.4}$$

as can be seen either from the Euler–Lagrange equations for ψ and the invariance equivalent (15.2.2) or, more easily, directly from the field equations (15.3.2).

The derivatives in Eqs. (15.3.2) and (15.3.4) are ordinary derivatives, not the gauge-covariant derivatives D_ν, so the gauge invariance of these equations is somewhat obscure. It can be made manifest by rewriting Eq. (15.3.2) in terms of the gauge-covariant derivative of the field strength

$$D_\lambda F_\alpha{}^{\mu\nu} \equiv \partial_\lambda F_\alpha{}^{\mu\nu} - i\,(t^A_\beta)_{\alpha\gamma} A_{\beta\lambda} F_\gamma{}^{\mu\nu}$$
$$= \partial_\lambda F_\alpha{}^{\mu\nu} - C_{\alpha\gamma\beta} A_{\beta\lambda} F_\gamma{}^{\mu\nu}\,. \tag{15.3.5}$$

Then Eq. (15.3.2) reads

$$D_\mu F_\alpha{}^{\mu\nu} = -J_\alpha{}^\nu\,, \tag{15.3.6}$$

where $J_\alpha{}^\nu$ is the current of the matter fields alone

$$J_\alpha{}^\nu \equiv -i \frac{\partial \mathscr{L}_M}{\partial D_\nu \psi}\, t_\alpha \psi\,. \tag{15.3.7}$$

This is gauge-covariant, if $\mathscr{L}_M$ is gauge-invariant. Also, by operating on Eq. (15.3.6) with D_ν, using the commutation relation

$$[D_\nu, D_\mu] F_\alpha^{\rho\sigma} = -i\,(t^A{}_\gamma)_{\alpha\beta} F_{\gamma\nu\mu} F_\beta{}^{\rho\sigma} = -C_{\gamma\alpha\beta} F_{\gamma\nu\mu} F_\beta{}^{\rho\sigma}\,,$$

we see that $J_\alpha{}^\nu$ satisfies a gauge-covariant conservation law

$$D_\nu J_\alpha{}^\nu = 0\,, \tag{15.3.8}$$

rather than the ordinary conservation law (15.3.4) obeyed by the full current $\mathscr{J}_\alpha{}^\nu$. Also, it is straightforward (using Eq. (15.1.5)) to derive the identities:

$$D_\mu F_{\alpha\nu\lambda} + D_\nu F_{\alpha\lambda\nu} + D_\lambda F_{\alpha\mu\nu} = 0\,, \tag{15.3.9}$$

which hold whether or not the gauge fields satisfy the field equations.

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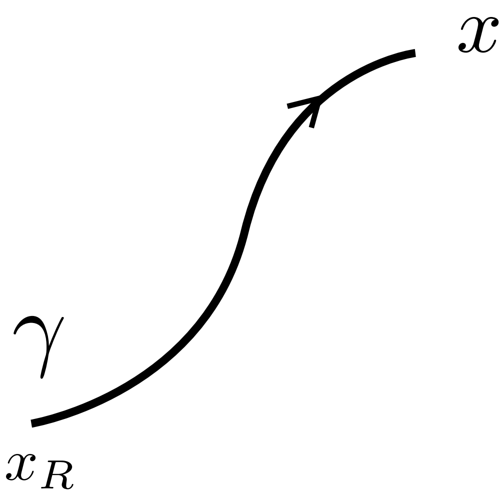
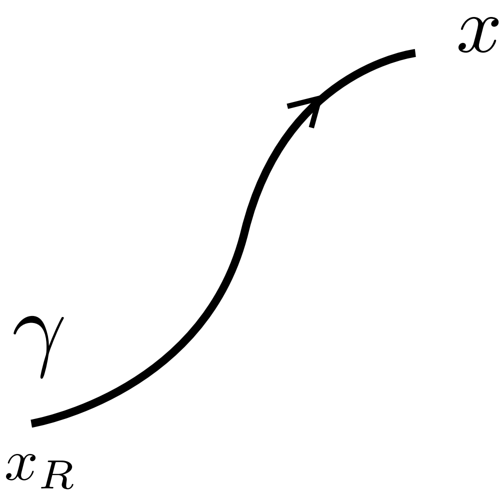
The Wilson line

The Wilson line

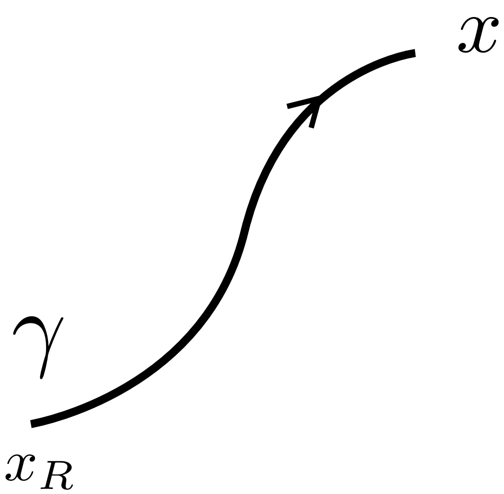


The Wilson line



$$\frac{d\,W}{d\,\sigma} + i\,e\,A_\mu\,\frac{d\,x^\mu}{d\,\sigma}\,W = 0$$

The Wilson line



$$\frac{d\,W}{d\,\sigma} + i\,e\,A_\mu\,\frac{d\,x^\mu}{d\,\sigma}\,W = 0$$

$$W(\gamma) = P_1\,e^{-i\,e\,\int_\gamma A_\mu\,\frac{dx^\mu}{d\sigma}\,d\sigma}\,W_R$$

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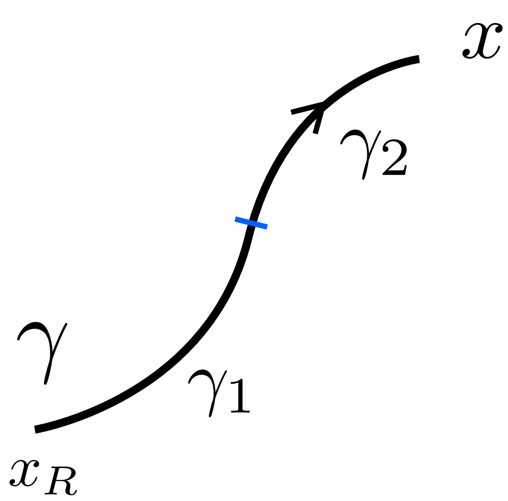

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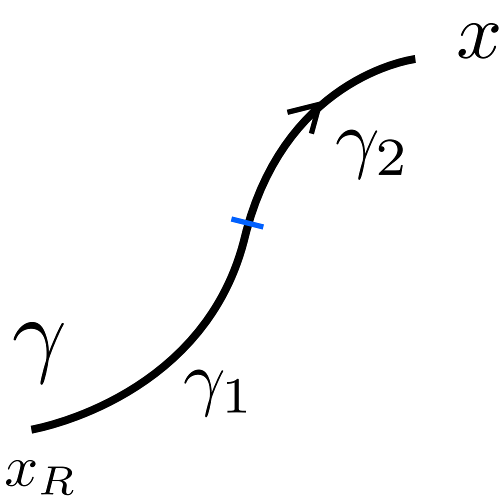
1) $W(\gamma) = W(\gamma_2)\,W(\gamma_1)$

The Wilson line



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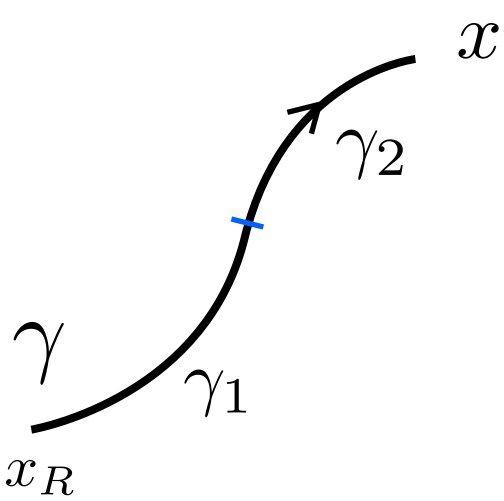
1) $W(\gamma) = W(\gamma_2)\,W(\gamma_1)$

2) $A_\mu \to g\,A_\mu\,g^{-1} + \frac{i}{e}\,\partial_\mu g\,g^{-1}$ then $W \to g(x)\,W\,g^{-1}(x_R)$

The Wilson line



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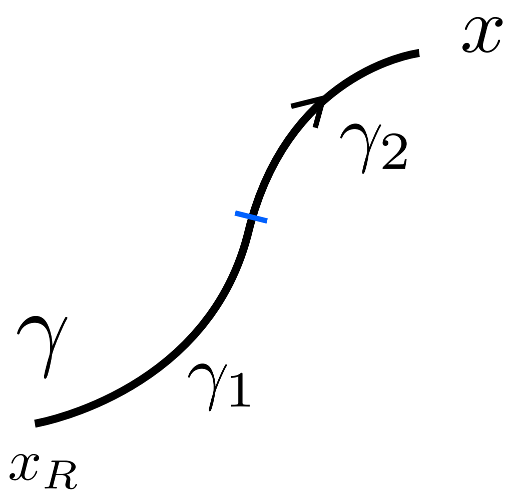
3) Suppose $B_{\mu\nu} \to g\,B_{\mu\nu}\,g^{-1}$

The Wilson line



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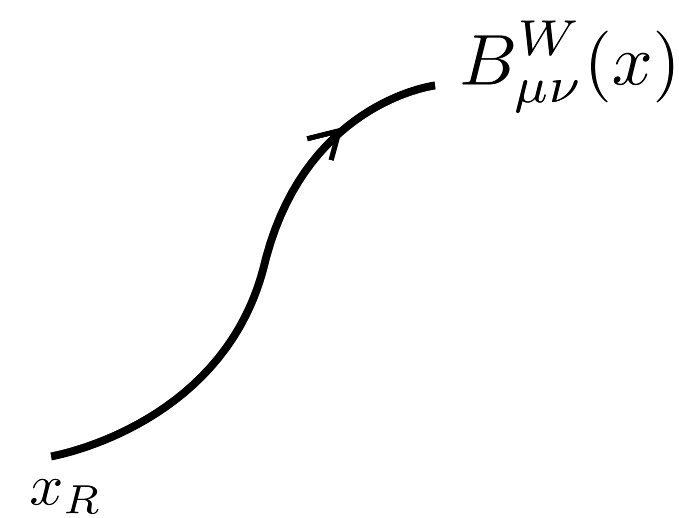
3) Suppose $B_{\mu\nu} \to g\,B_{\mu\nu}\,g^{-1}$

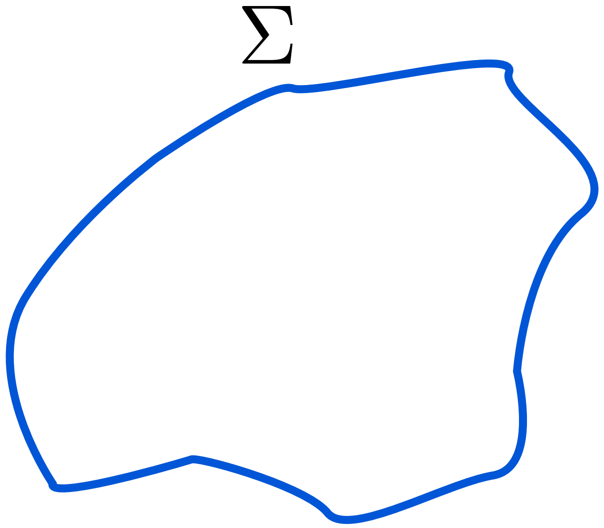
Then $B^W_{\mu\nu} \equiv W^{-1}\,B_{\mu\nu}\,W$ satisfies

$$B^W_{\mu\nu} \to g(x_R)\,B^W_{\mu\nu}\,g^{-1}(x_R)$$



Integrating on surfaces

Integrating on surfaces



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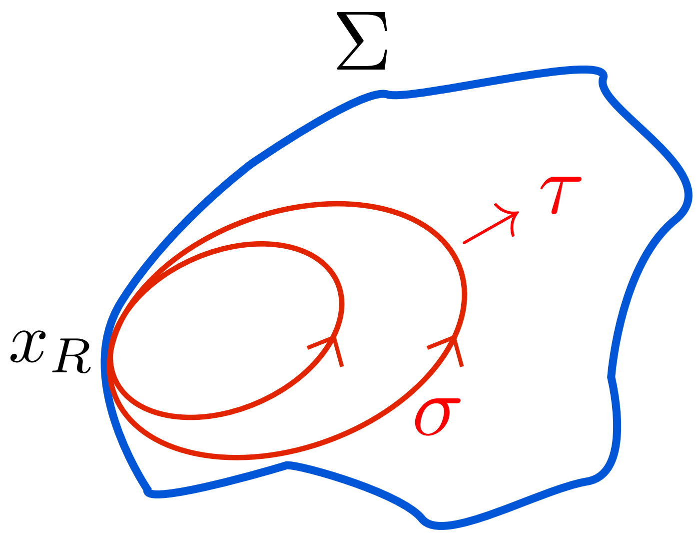
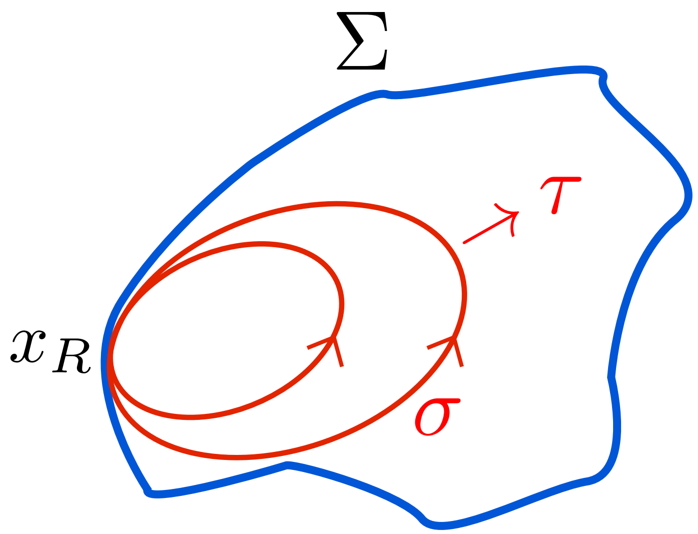


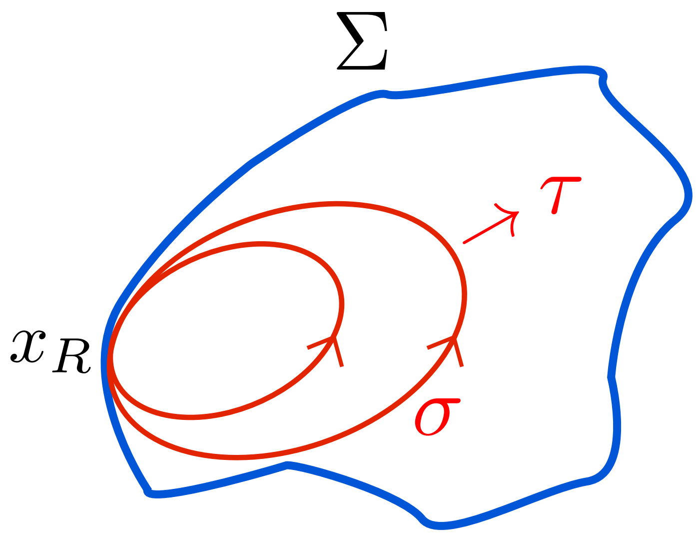
Integrating on surfaces



$$\frac{d\,V}{d\,\tau} - V\,T\,(A, B, \tau) = 0$$

$$T\,(B, A, \tau) \equiv \int_0^{2\,\pi} d\sigma\; W^{-1}\, B_{\mu\nu}\, W\, \frac{d\,x^{\mu}}{d\,\sigma}\, \frac{d\,x^{\nu}}{d\,\tau}$$

Integrating on surfaces



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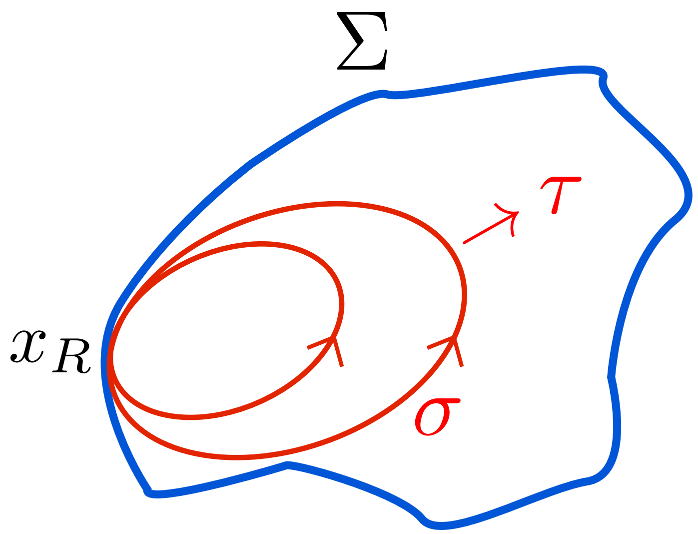
It is a surface ordered integral

$$V(\Sigma) = V_R\;P_2\,e^{\int_\Sigma d\sigma d\tau\; W^{-1}\,B_{\mu\nu}\,W\,\frac{d\,x^\mu}{d\,\sigma}\,\frac{d\,x^\nu}{d\,\tau}}$$

Integrating on surfaces



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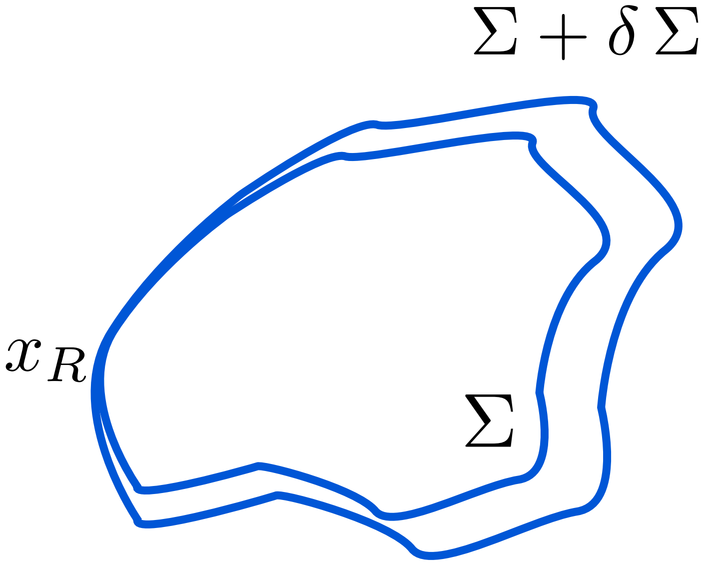
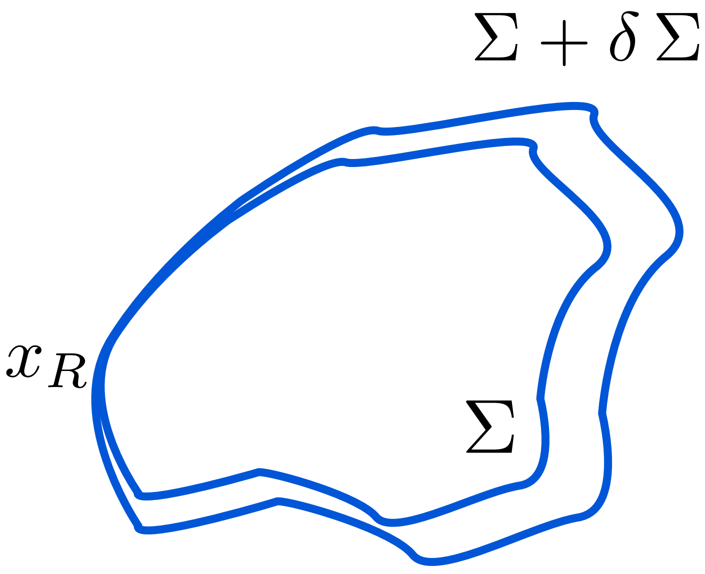
$$A_\mu \to g\, A_\mu\, g^{-1} + \frac{i}{e}\,\partial_\mu g\, g^{-1}$$

$$B_{\mu\nu} \to g\, B_{\mu\nu}\, g^{-1}$$

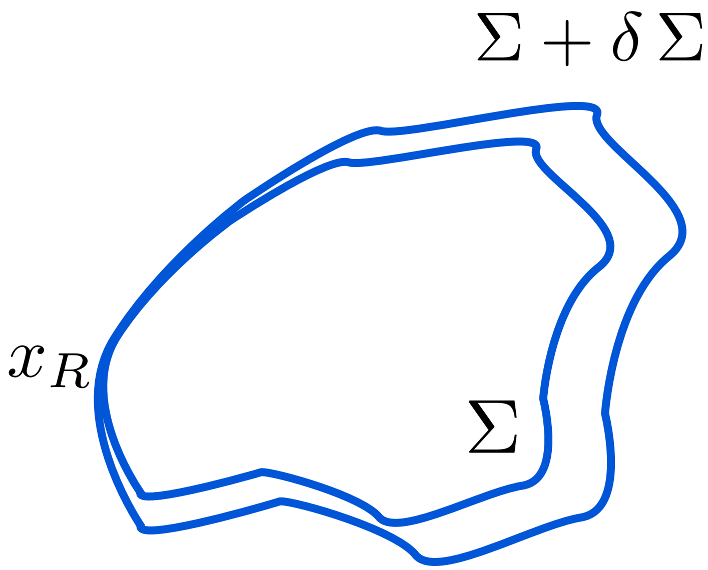
$$\to \begin{cases} T\,(B, A, \tau) \to g(x_R)\, T\,(B, A, \tau)\; g^{-1}(x_R) \\ V \to g(x_R)\, V\, g^{-1}(x_R) \end{cases}$$



$\Sigma + \delta\Sigma$
x_R
Σ



What happens to V when one changes the surface



What happens to V when one changes the surface

$$
\begin{aligned}
\delta V\, V^{-1} \equiv & \int_0^{2\,\pi} d\tau \int_0^{2\,\pi} d\sigma\, V\left(\tau\right)\, \Big\{ \\
& W^{-1}\, \left[D_\lambda B_{\mu\nu} + D_\mu B_{\nu\lambda} + D_\nu B_{\lambda\mu}\right]\, W \frac{d\,x^\mu}{d\,\sigma}\, \frac{d\,x^\nu}{d\,\tau}\, \delta x^\lambda \\
& - \int_0^{\sigma} d\sigma' \left[\, B^W_{\kappa\rho}\left(\sigma'\right) - ieF^W_{\kappa\rho}\left(\sigma'\right),\, B^W_{\mu\nu}\left(\sigma\right)\right] \frac{dx^\kappa}{d\sigma'}\frac{dx^\mu}{d\sigma} \\
& \times \left(\frac{d\,x^\rho\left(\sigma'\right)}{d\,\tau}\delta x^\nu\left(\sigma\right) - \delta x^\rho\left(\sigma'\right)\,\frac{d\,x^\nu\left(\sigma\right)}{d\,\tau}\right)\Big\}\, V^{-1}\left(\tau\right)
\end{aligned}
$$

Integrating on volumes

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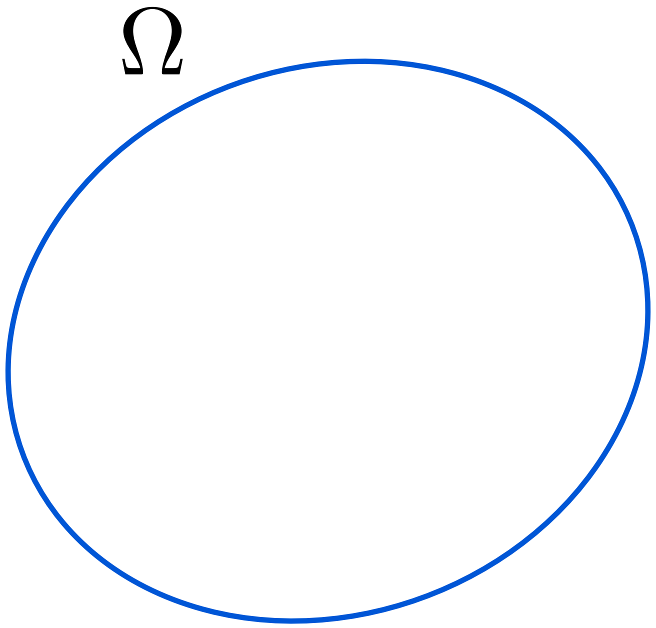

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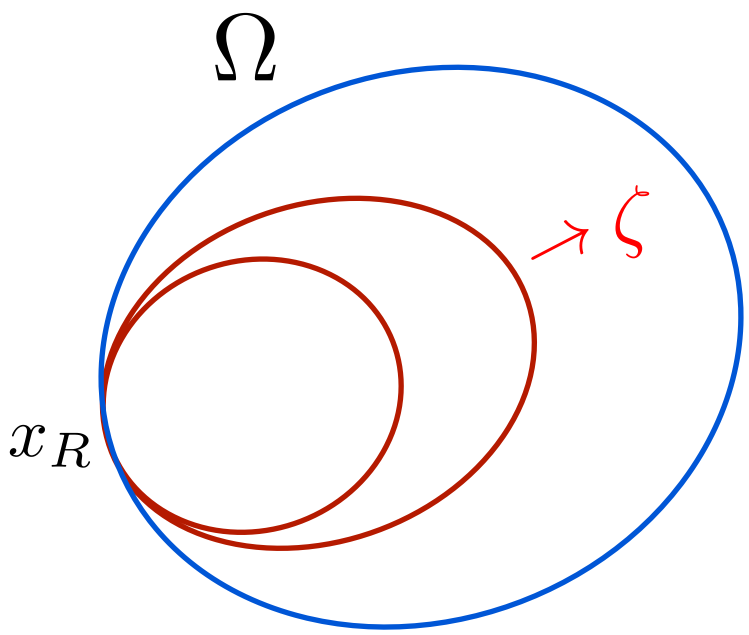
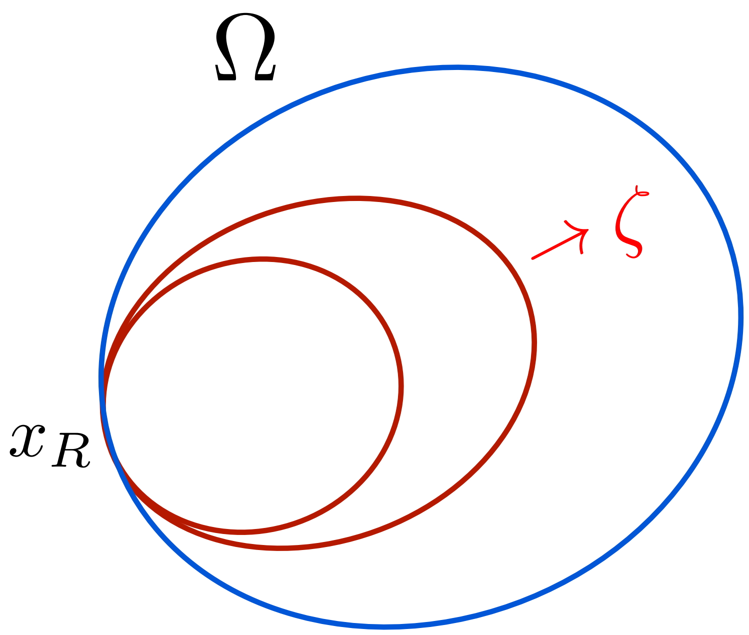
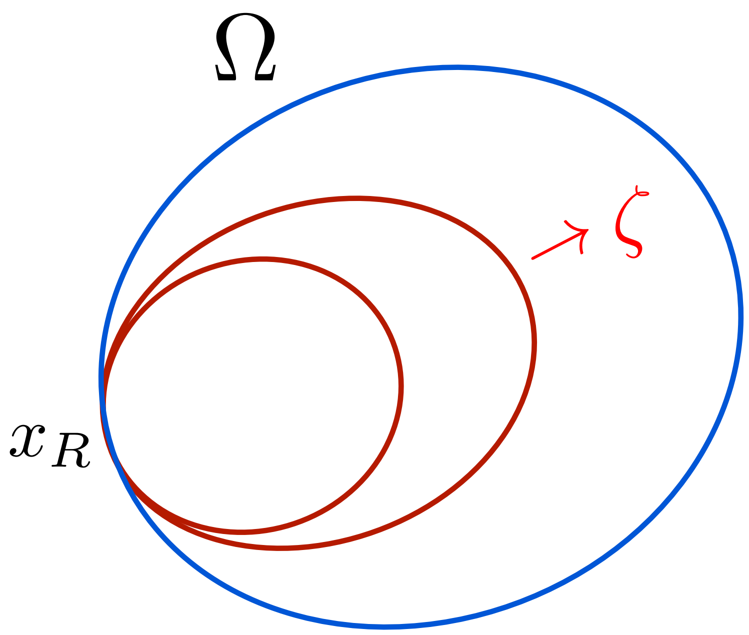




Integrating on volumes

$$\frac{dV}{d\zeta} - \mathcal{K}V = 0$$

Integrating on volumes



$$\frac{d\,V}{d\,\zeta} - \mathcal{K}\,V = 0$$

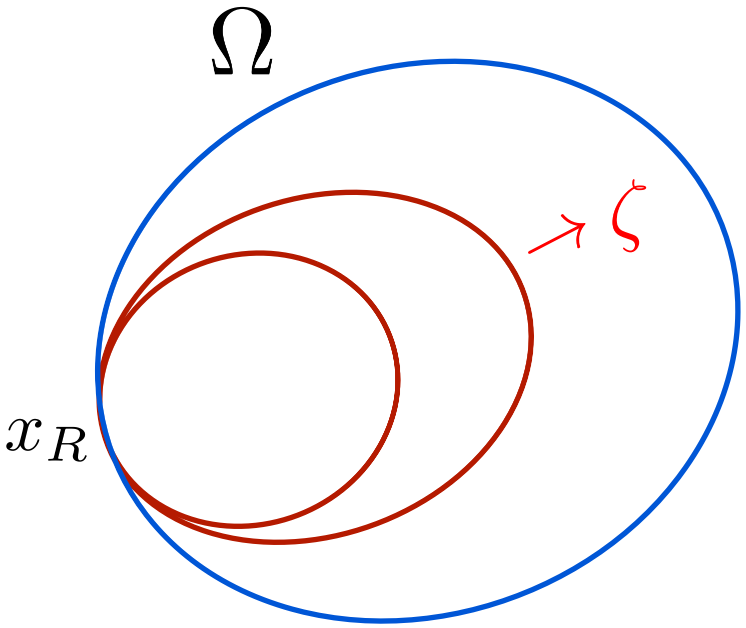
$$\mathcal{K} \equiv \int_0^{2\,\pi} d\tau \int_0^{2\,\pi} d\sigma\, V\,(\tau)\;\Big\{$$

$$W^{-1}\,\left[D_\lambda B_{\mu\nu} + D_\mu B_{\nu\lambda} + D_\nu B_{\lambda\mu}\right]\,W\,\frac{d\,x^\mu}{d\,\sigma}\,\frac{d\,x^\nu}{d\,\tau}\,\frac{d\,x^\lambda}{d\,\zeta}$$

$$-\int_0^{\sigma} d\sigma'\,\left[\,B^W_{\kappa\rho}\,(\sigma') - ie F^W_{\kappa\rho}\,(\sigma')\,,\,B^W_{\mu\nu}\,(\sigma)\,\right]\,\frac{dx^\kappa}{d\sigma'}\,\frac{dx^\mu}{d\sigma}$$

$$\times\left(\frac{d\,x^\rho\,(\sigma')}{d\,\tau}\,\frac{d\,x^\nu\,(\sigma)}{d\,\zeta} - \frac{d\,x^\rho\,(\sigma')}{d\,\zeta}\,\frac{d\,x^\nu\,(\sigma)}{d\,\tau}\right)\Big\}\,V^{-1}\,(\tau)$$

Integrating on volumes



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Ordered volume integral $\rightarrow$ $\qquad V(\partial\Omega) = P_3\, e^{\int_\Omega \mathcal{K}}\, V_R$

The generalized non-abelian Stokes theorem

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$$V_R \, P_2 e^{\int_{\partial\Omega} d\tau d\sigma W^{-1} B_{\mu\nu} W \frac{dx^\mu}{d\sigma} \frac{d\,x^\nu}{d\,\tau}} = P_3 e^{\int_\Omega d\zeta \, \mathcal{K}} \, V_R$$

The generalized non-abelian Stokes theorem

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$$\frac{d\,V}{d\,\tau} - V\,T\,(A, B, \tau) = 0$$

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O. Alvarez, L. A. Ferreira and J. Sanchez Guillen,
Nucl. Phys. B **529**, 689 (1998) [arXiv:hep-th/9710147].
Int. J. Mod. Phys. A **24**, 1825 (2009) [arXiv:0901.1654 [hep-th]]

The integral equations for Yang-Mills

The integral equations for Yang-Mills

$$P_2 e^{ie\int_{\partial\Omega} d\tau d\sigma \left[\alpha F^W_{\mu\nu} + \beta \widetilde{F}^W_{\mu\nu}\right] \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}} = P_3 e^{\int_\Omega d\zeta d\tau V \mathcal{J} V^{-1}}$$

The integral equations for Yang-Mills

$$P_2 e^{ie\int_{\partial\Omega} d\tau d\sigma \left[\alpha F_{\mu\nu}^W + \beta \widetilde{F}_{\mu\nu}^W\right] \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}} = P_3 e^{\int_\Omega d\zeta d\tau V \mathcal{J} V^{-1}}$$

$$\mathcal{J} \equiv \int_0^{2\pi} d\sigma \left\{ ie\beta \widetilde{J}_{\mu\nu\lambda}^W \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\zeta} + e^2 \int_0^\sigma d\sigma' \right.$$
$$\times \left[\left((\alpha - 1) F_{\kappa\rho}^W + \beta \widetilde{F}_{\kappa\rho}^W \right)(\sigma'), \left(\alpha F_{\mu\nu}^W + \beta \widetilde{F}_{\mu\nu}^W \right)(\sigma) \right]$$
$$\left. \times \frac{d\,x^\kappa}{d\,\sigma'}\frac{d\,x^\mu}{d\,\sigma} \left(\frac{d\,x^\rho(\sigma')}{d\,\tau}\frac{d\,x^\nu(\sigma)}{d\,\zeta} - \frac{d\,x^\rho(\sigma')}{d\,\zeta}\frac{d\,x^\nu(\sigma)}{d\,\tau} \right) \right\}$$

The integral equations for Yang-Mills

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$$\left. \times \frac{d\,x^\kappa}{d\,\sigma'}\frac{d\,x^\mu}{d\,\sigma} \left(\frac{d\,x^\rho(\sigma')}{d\,\tau}\frac{d\,x^\nu(\sigma)}{d\,\zeta} - \frac{d\,x^\rho(\sigma')}{d\,\zeta}\frac{d\,x^\nu(\sigma)}{d\,\tau} \right) \right\}$$

$$B_{\mu\nu} \to \alpha F_{\mu\nu} + \beta \widetilde{F}_{\mu\nu}$$

The integral equations for Yang-Mills

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$$\times \left[\left((\alpha - 1) F_{\kappa\rho}^{W} + \beta \widetilde{F}_{\kappa\rho}^{W} \right)(\sigma'), \left(\alpha F_{\mu\nu}^{W} + \beta \widetilde{F}_{\mu\nu}^{W} \right)(\sigma) \right]$$
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direct consequence of Stokes theorem
and Yang-Mills eqs.

It implies the local Yang-Mills equations

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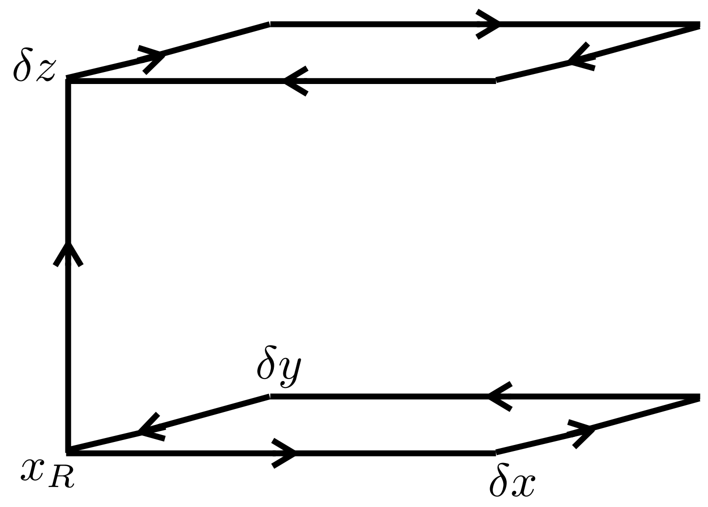
Infinitesimal cube



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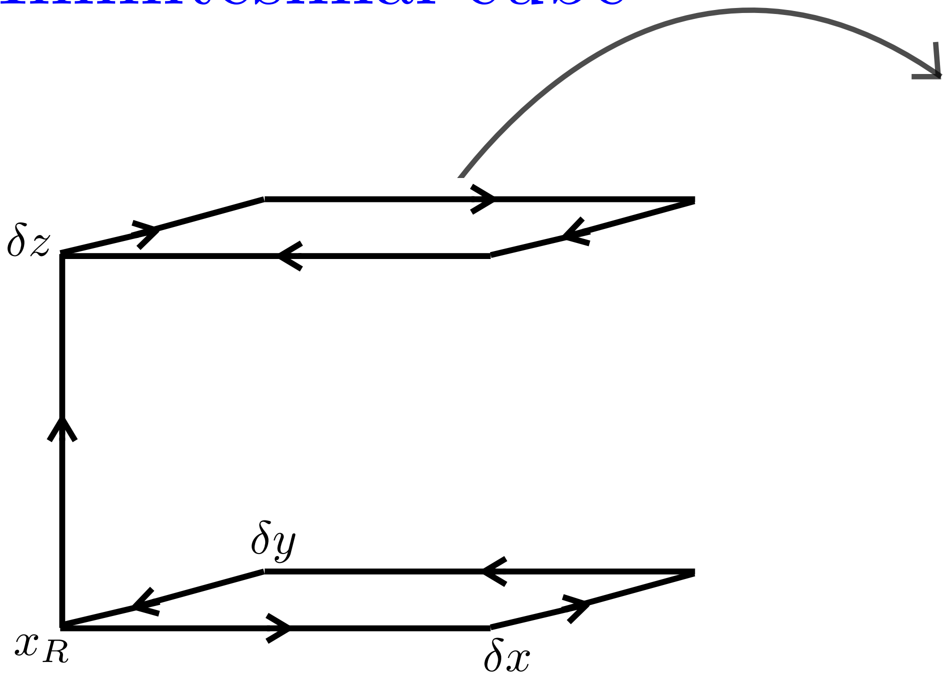

$$ie\left(W^{-1}\left[\alpha F_{\mu\nu}+\beta\widetilde{F}_{\mu\nu}\right]W\right)_{(x_R+\delta z^{\lambda})}\delta x^{\mu}\delta y^{\nu}$$

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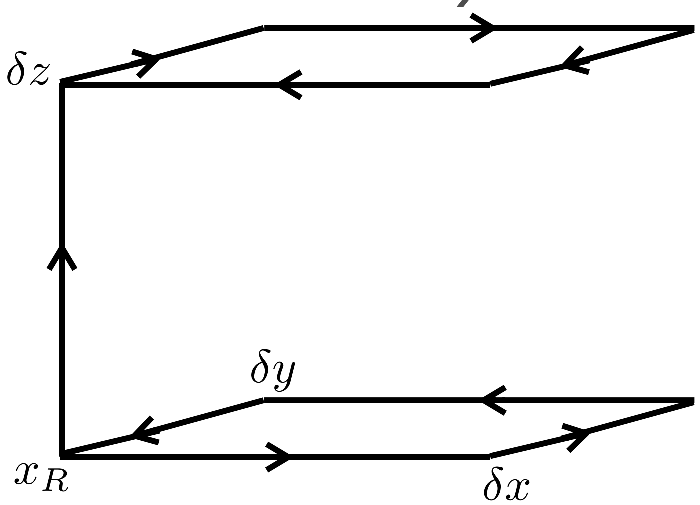
Infinitesimal cube



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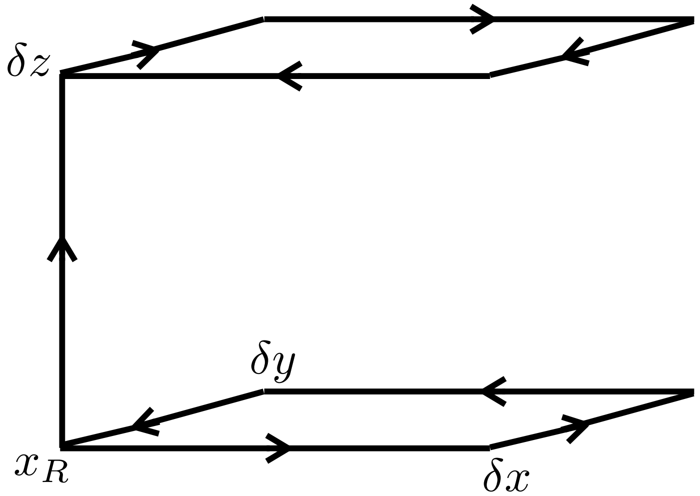

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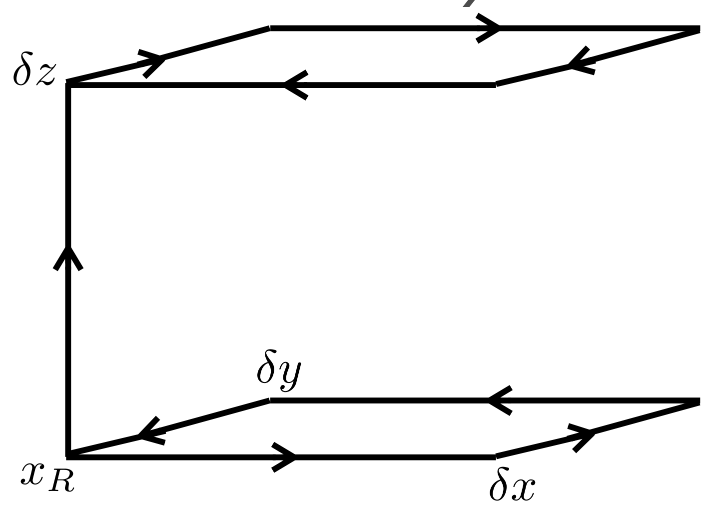

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Taylor expanding and adding other faces

$$P_2 e^{ie\int_{\partial\Omega}d\tau d\sigma\left[\alpha F^W_{\mu\nu}+\beta\widetilde{F}^W_{\mu\nu}\right]\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}}\sim 1+ie\left[D_\lambda\left[\alpha F_{\mu\nu}+\beta\widetilde{F}_{\mu\nu}\right]_{x_R}+\text{cyclic perm.}\right]\delta x^\mu\delta y^\nu\delta z^\lambda$$

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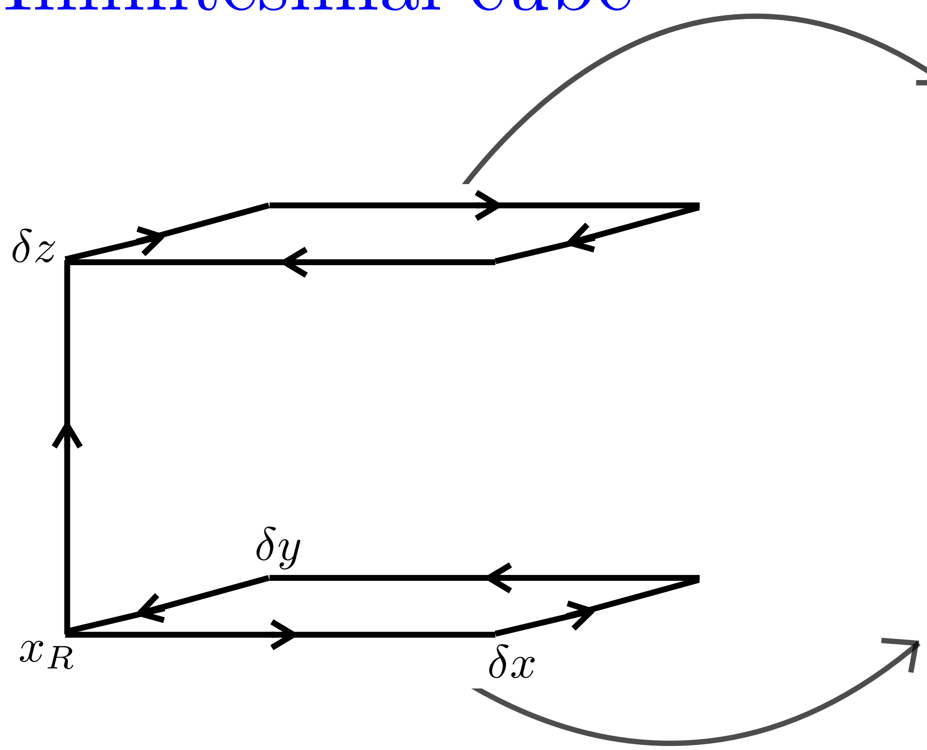
Infinitesimal cube



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The volume integral gives

$$P_3 e^{\int_\Omega d\zeta d\tau V\mathcal{J}V^{-1}}\sim 1+ie\beta\widetilde{J}_{\mu\nu\lambda}\delta x^\mu\delta y^\nu\delta z^\lambda$$

Consequences

$$P_2 e^{ie\int_{\partial\Omega} d\tau d\sigma \left[\alpha F^W_{\mu\nu} + \beta \widetilde{F}^W_{\mu\nu}\right] \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}} = P_3 e^{\int_\Omega d\zeta d\tau V \mathcal{J} V^{-1}}$$

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Consider Maxwell theory on a space-time $\mathcal{S} \times \mathbb{R}$, with $\mathcal{S}$ being a spatial closed submanifold. Then

Consequences

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$$e^{i\,e\,\beta\,\int_{\mathcal{S}} d\zeta d\tau d\sigma \widetilde{J}_{\mu\nu\lambda}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\zeta}} = 1$$

Consequences

$$P_2 e^{ie\int_{\partial\Omega} d\tau d\sigma\left[\alpha F^W_{\mu\nu}+\beta \widetilde{F}^W_{\mu\nu}\right]\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}} = P_3 e^{\int_\Omega d\zeta d\tau V \mathcal{J} V^{-1}}$$

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$$q = \frac{2\pi n}{e\beta}$$

Loop space

Loop space

$$L\Omega = \left\{ \gamma : S^2 \to \Omega \,|\, \text{north pole} \to x_R \in \partial\Omega \right\}$$

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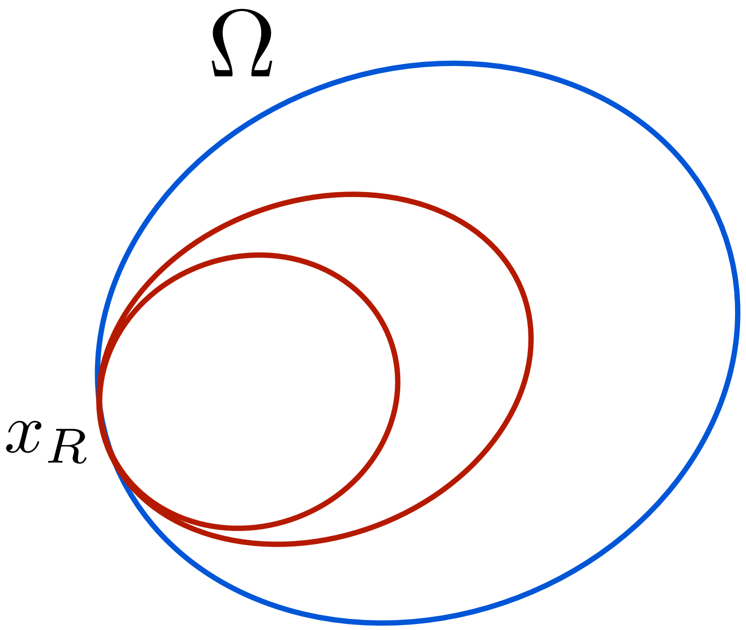

Volume in space-time

Loop space

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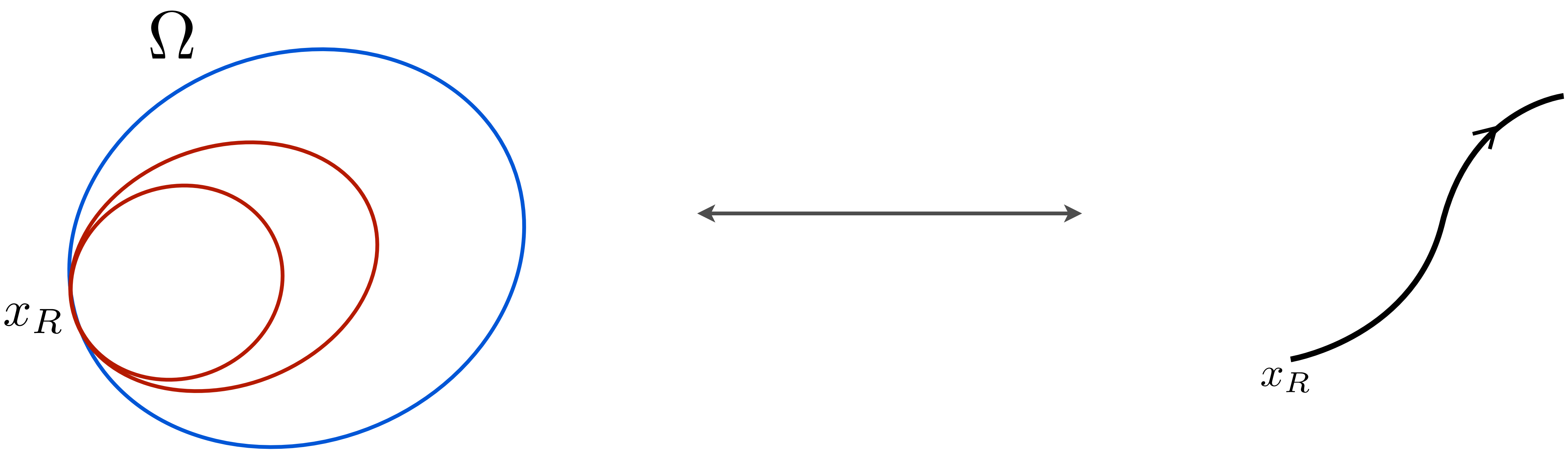


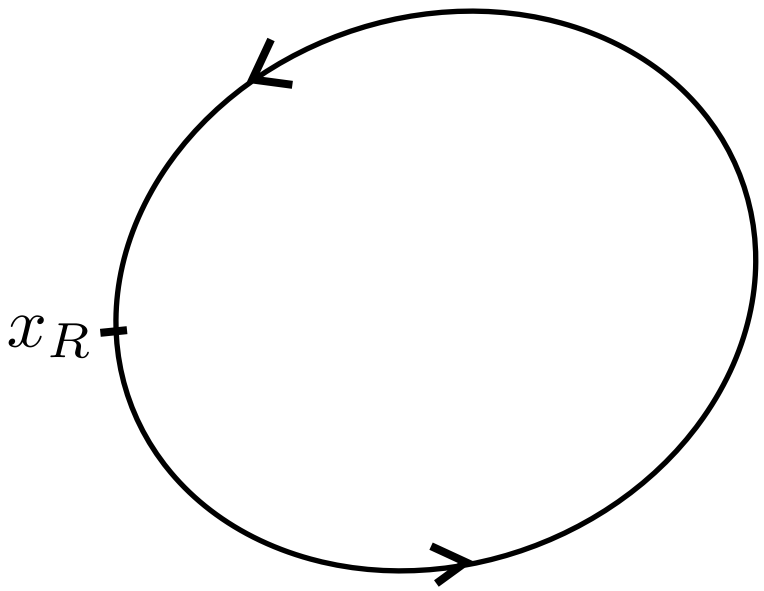
Volume in space-time

Path in loop space

Path independency

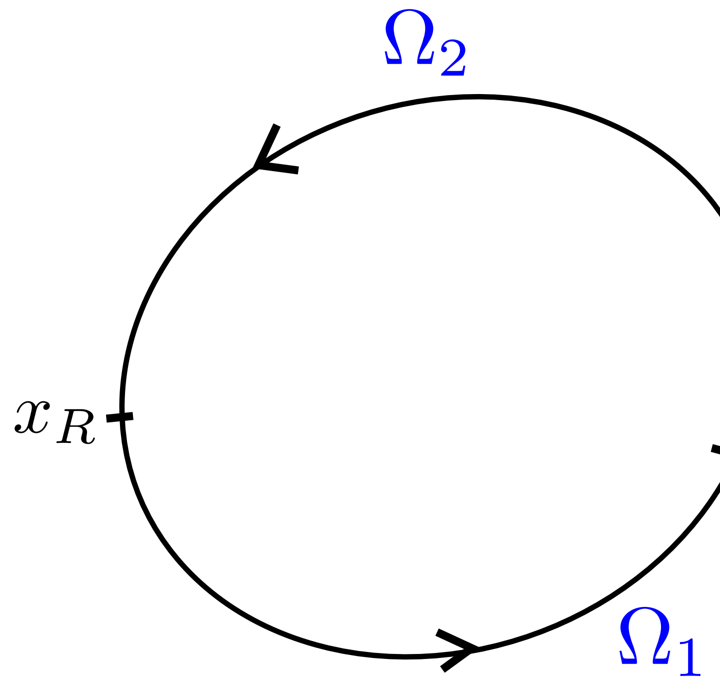
Path independency

Closed volume (path)



Path independency

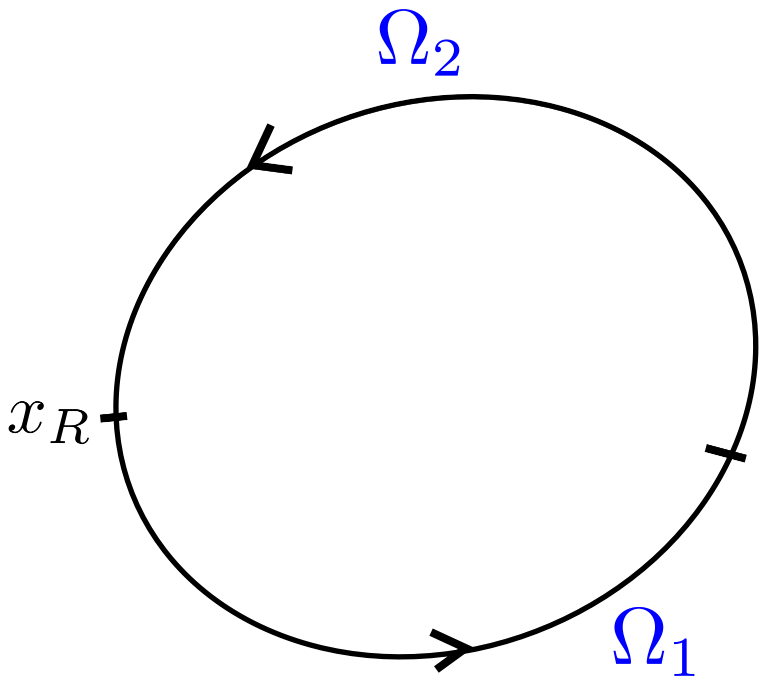
Closed volume (path)



$$\Omega_c = \Omega_1 + \Omega_2$$

Path independency

Closed volume (path)

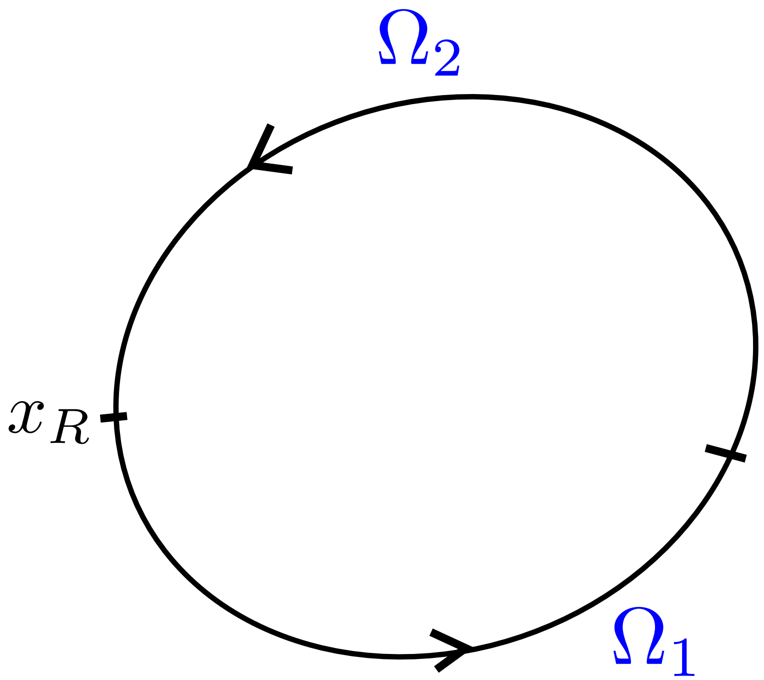


$$\Omega_c = \Omega_1 + \Omega_2$$

$$\mathbb{1} = P_3 e^{\oint_{\Omega_c} d\zeta d\tau V \mathcal{J} V^{-1}} = P_3 e^{\oint_{\Omega_2} d\zeta d\tau V \mathcal{J} V^{-1}} P_3 e^{\oint_{\Omega_1} d\zeta d\tau V \mathcal{J} V^{-1}}$$

Path independency

Closed volume (path)



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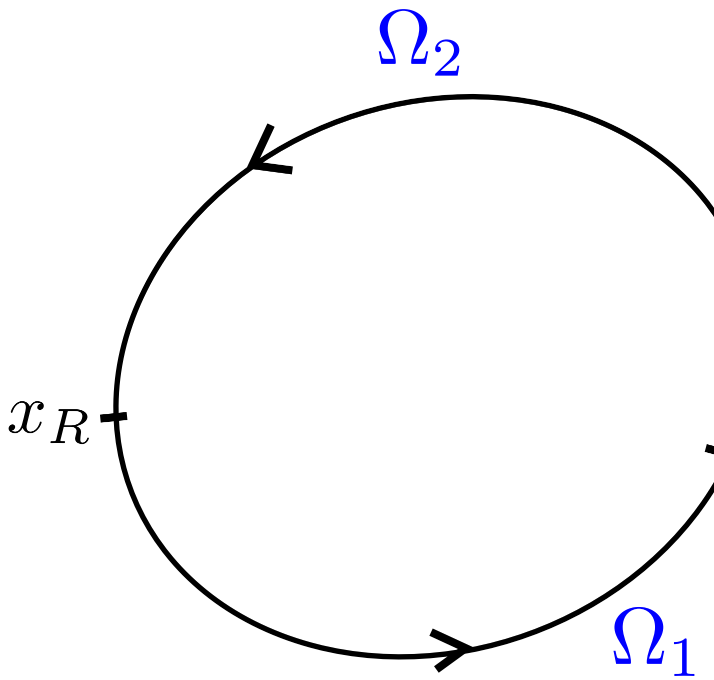
Ω_1 and Ω_2^{-1} are volumes (paths) with the same end points

Path independency

Closed volume (path)



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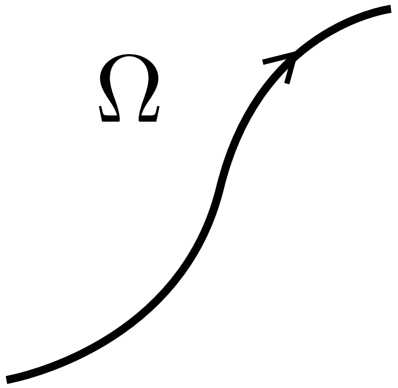
Ω_1 and Ω_2^{-1} are volumes (paths) with the same end points

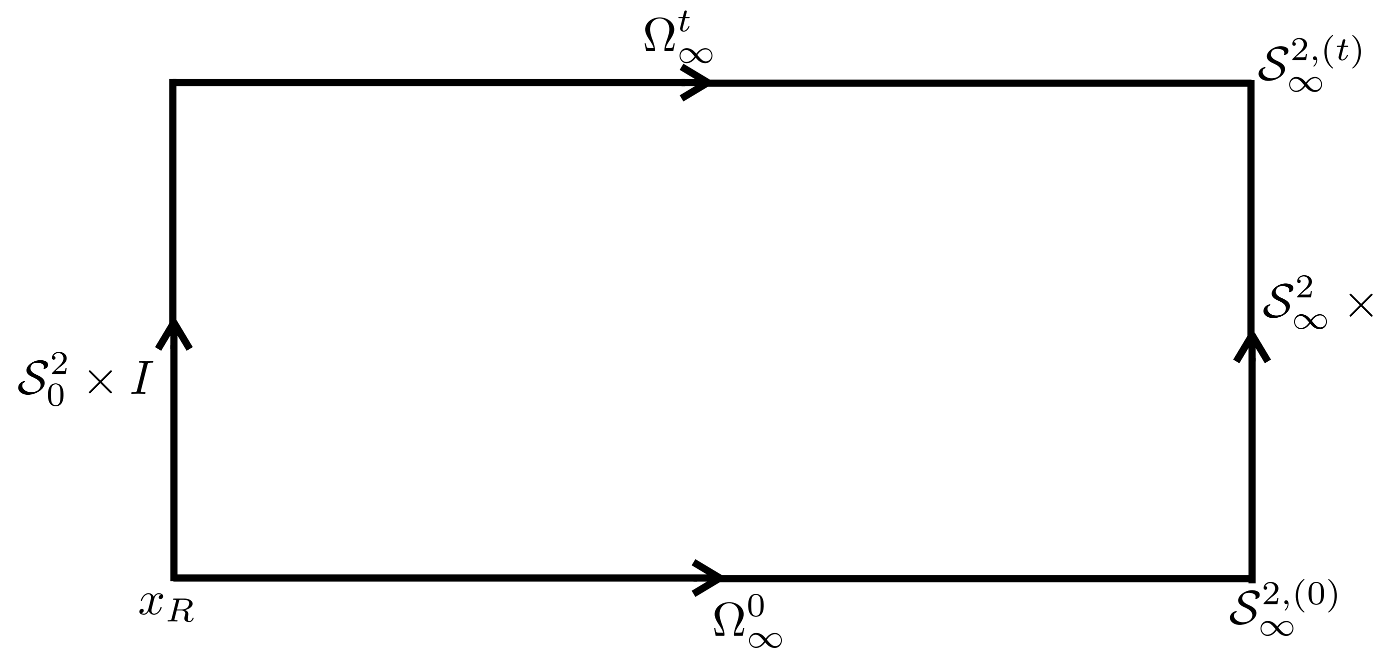
So, $P_3 e^{\oint_{\Omega} d\zeta d\tau V \mathcal{J} V^{-1}}$ is path independent



Conserved charges

Conserved charges

Two paths in loop space
or volumes in space-time



Conserved charges

Two paths in loop space
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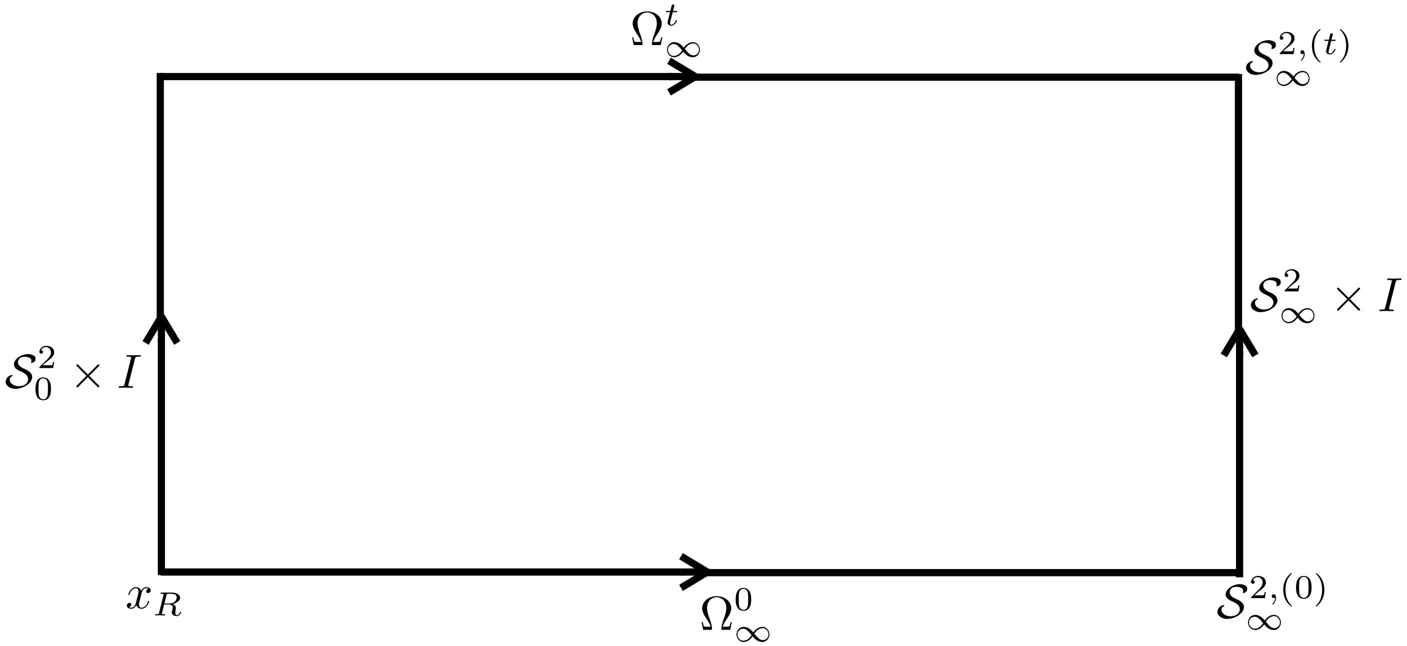

Path independency $\rightarrow$ $V(\mathcal{S}_{\infty}^{(2)} \times I)V(\Omega_{\infty}^{(0)}) = V(\Omega_{\infty}^{(t)})V(\mathcal{S}_{0}^{2} \times I)$

$\left[V(*) \equiv P_3 e^{\int_* d\zeta d\tau V \mathcal{J} V^{-1}}\right]$

Conserved charges

Two paths in loop space
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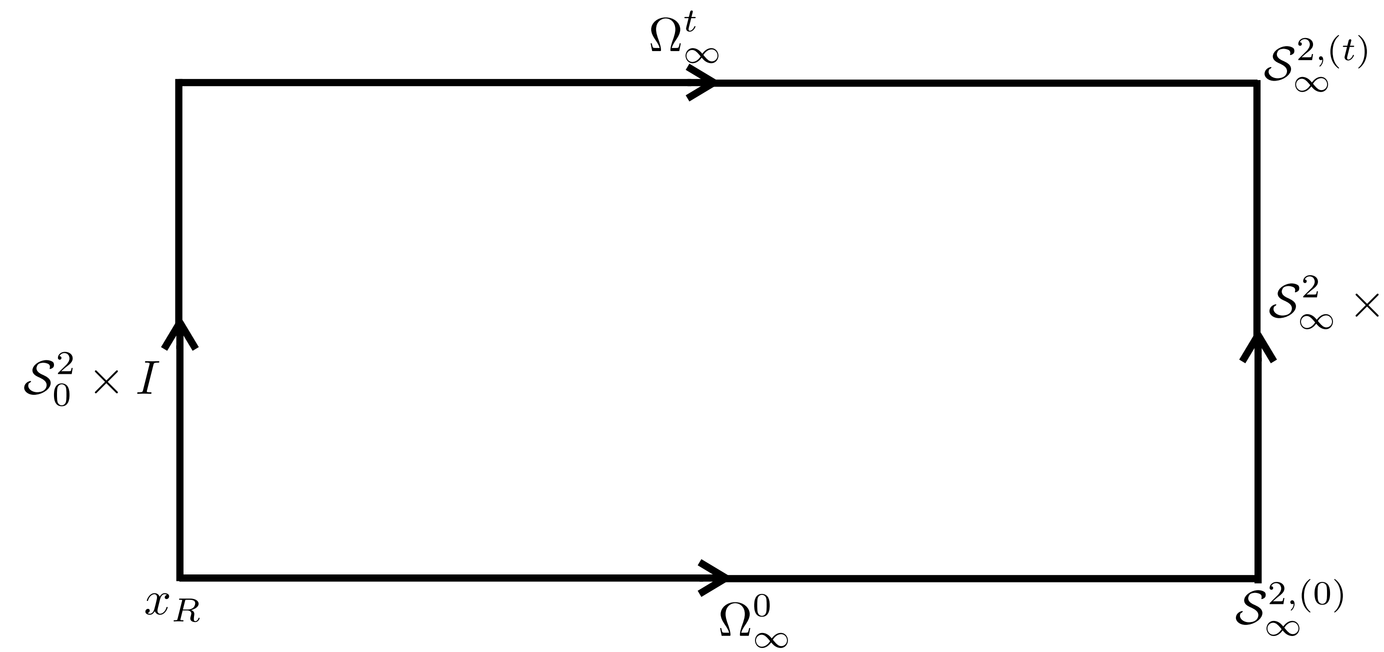

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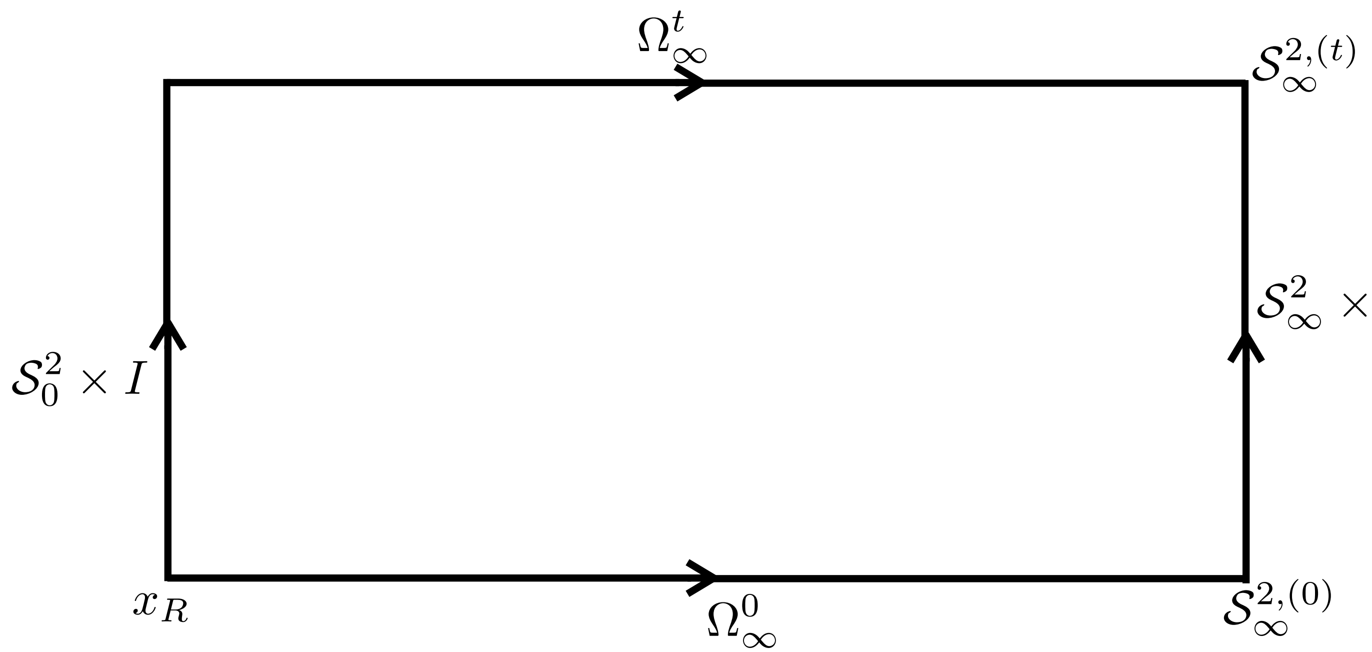
The boundary conditions $$J_\mu \sim \frac{1}{R^{2+\delta}} \qquad F_{\mu\nu} \sim \frac{1}{R^{\frac{3}{2}+\delta'}}$$

Conserved charges

Two paths in loop space or volumes in space-time



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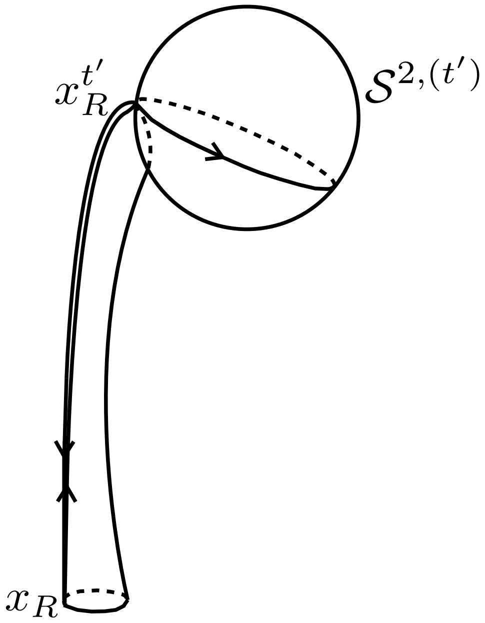
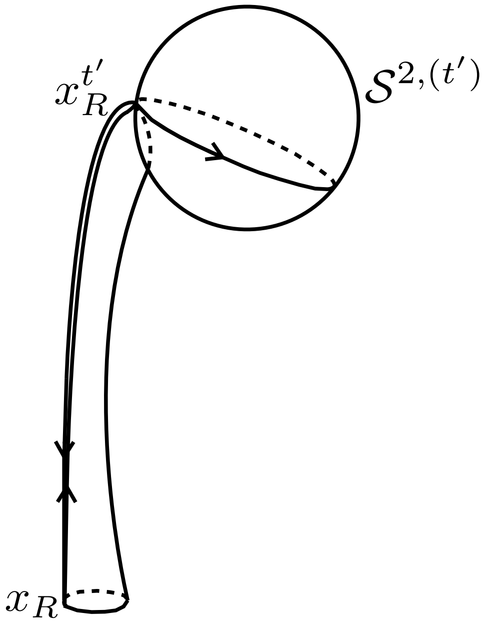
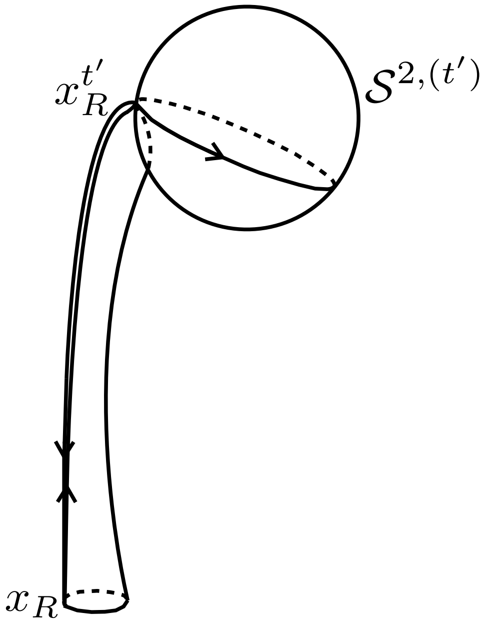
imply $$V(\mathcal{S}_\infty^{(2)} \times I) = V(\mathcal{S}_0^2 \times I)$$



$x_R^{t'}$
$\mathcal{S}^{2,(t')}$
x_R



$$V_{x_R}(\Omega_\infty^{(t)}) = W^{-1}(x_R^{(t)}, x_R) V_{x_R^t}(\Omega_\infty^{(t)}) W(x_R^{(t)}, x_R)$$



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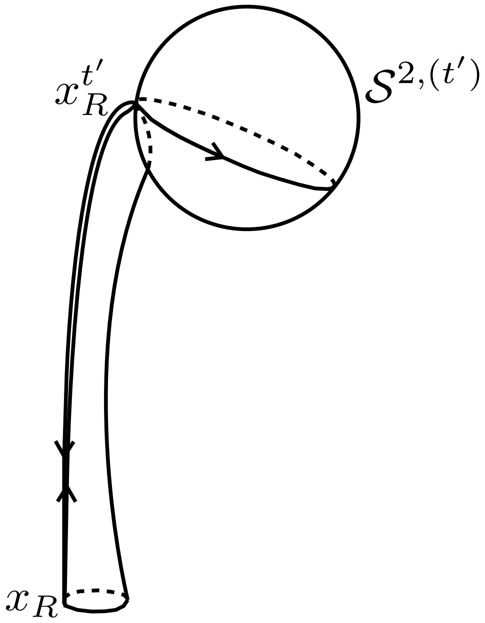
Iso-spectral evolution

$$V_{x_R^t}(\Omega_\infty^{(t)}) = U(t) \cdot V(\Omega_\infty^{(0)}) \cdot U^{-1}(t)$$



$$V_{x_R}(\Omega_\infty^{(t)}) = W^{-1}(x_R^{(t)}, x_R) V_{x_R^t}(\Omega_\infty^{(t)}) W(x_R^{(t)}, x_R)$$

Iso-spectral evolution

$$V_{x_R^t}(\Omega_\infty^{(t)}) = U(t) \cdot V(\Omega_\infty^{(0)}) \cdot U^{-1}(t)$$

with $\qquad U(t) = W(x_R^{(t)}, x_R) \cdot V(\mathcal{S}_0^2 \times I)$

Conserved charges are eigenvalues of the operator

$$V_{x_R^{(t)}}(\Omega_\infty^{(t)}) = P_2\, e^{ie\int_{\mathcal{S}_\infty^{2,(t)}} d\tau\; d\sigma\; \left(\alpha F^W_{\mu\nu} + \beta \widetilde{F}^W_{\mu\nu}\right)\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}} = P_3\, e^{\int_{\Omega_\infty^{(t)}} d\zeta\; d\tau\; V \mathcal{J} V^{-1}}$$

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The self-dual sector

$$F_{\mu\nu} = \kappa\, \widetilde{F}_{\mu\nu} \qquad \widetilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} \qquad \kappa = \pm 1$$

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Integral form is given by

$$P_1\, e^{-ie\oint_{\partial\Sigma} d\sigma\, A_\mu \frac{dx^\mu}{d\sigma}} = P_2\, e^{ie\int_\Sigma d\sigma d\tau\, W^{-1}\left[\alpha\, F_{\mu\nu}+\kappa\,(1-\alpha)\,\widetilde{F}_{\mu\nu}\right]W\, \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}}$$

with Σ being any two-dimensional surface on the space-time, and α being an arbitrary parameter.

Non-Abelian Stokes Theorem

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$$P_1\, e^{-\oint_{\partial\Sigma} d\sigma\, C_\mu \frac{dx^\mu}{d\sigma}}\, W_R = W_R\, P_2\, e^{\int_\Sigma d\sigma d\tau\, W^{-1}\, H_{\mu\nu}\, W \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}}$$

with $H_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu + [\, C_\mu\,,\, C_\nu\,]$

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The charges are the eigenvalues of the operator

$$V_{x_R^{(t)}}\left(\mathcal{D}_\infty^{(t)}\right) = P_2\, e^{ie\int_{\mathcal{D}_\infty^{(t)}} d\sigma d\tau\, W^{-1}\left[\alpha\, F_{\mu\nu}+\kappa\,(1-\alpha)\,\widetilde{F}_{\mu\nu}\right]W\,\frac{dx^\mu}{d\sigma}\,\frac{dx^\nu}{d\tau}}$$

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if the boundary conditions are satisfied

$$F_{\rho\sigma} = \kappa\,\widetilde{F}_{\rho\sigma} \sim \frac{1}{r^{2+\delta}}\,T\left(\hat{r}\right) \qquad \text{for} \qquad r \to \infty$$

Instantons

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$$A_\mu = -\frac{2}{e}\,\sigma_{\mu\nu}\,\frac{x^\nu - a^\nu}{\left(x^\rho - a^\rho\right)^2 + \lambda^2}$$

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One gets

$$P_1\,e^{-ie\oint_{S^{1,(t)}_\infty} d\sigma\,A_\mu\,\frac{dx^\mu}{d\sigma}} = e^{-i\,2\int_0^{2\pi} d\theta\,\sigma_{\mu\nu}} = e^{-i\,4\,\pi\,\sigma_{\mu\nu}}$$

Are integrable theories gauge theories?

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Look for theories in $d+1$ dimensions where the equations of motion take the integral form

$$P_{d-1} e^{\int_{\partial\Omega} \mathcal{F}} = P_d \, e^{\int_{\Omega} \mathcal{J}}$$

with Ω a d-dimensional hyper volume

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2) Chern-Simons theories in $2+1$ dimensions
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L. A. Ferreira and G. Luchini,
An integral formulation of Yang-Mills on loop space, [arXiv:1109.2120 [hep-th]]
Gauge and Integrable Theories in Loop Spaces, [arXiv:1109.2606 [hep-th]],
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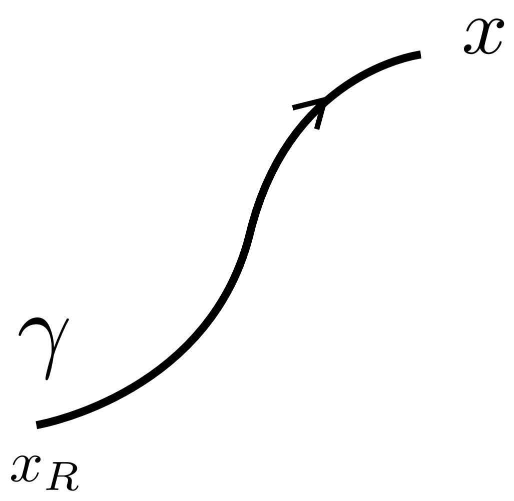

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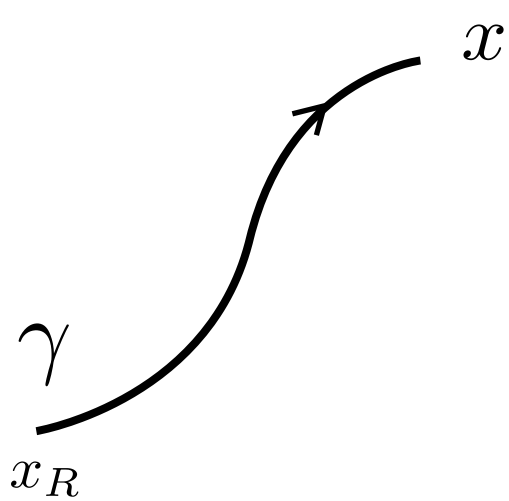

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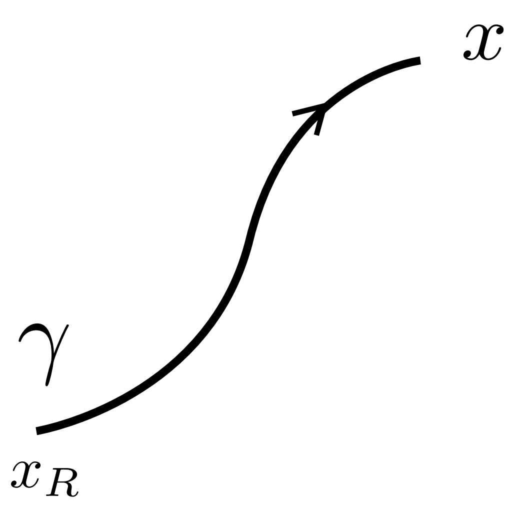

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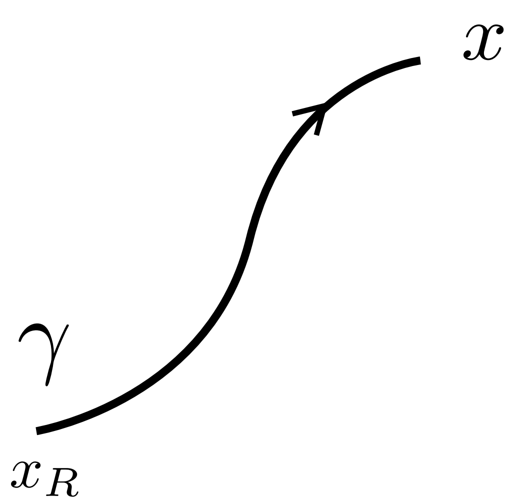

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The fact that it must be true for any curve joining x_R to x, it implies A_μ is flat

$$\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu\,,\,A_\nu] = 0 \qquad\qquad A_\mu = -\partial_\mu g\,g^{-1}$$

Thank You