# List of exercises #3 - 7600037

1. Calculate the Green function for the Schrödinger equation in one spatial dimension.

2. In the Born approximation, compute the differential and the total cross section for

   (a) $V(r) = -V_0 e^{-r/a}$, with $V_0$ and $a$ being positive constants, and

   (b) $V(r) = Cr^{-n}$, with $C$ and $n$ being real constants.

3. Calculate the cross section for the Yukawa potential in the Born approximation. Discuss about the validity of this approximation.

4. Consider the square spherical well potencital $V(\mathbf{r}) = V(r) = -V_0$ for $r \leq a$, and $V(r) = 0$, for $r > a$, where $V_0$ is a positive constant.

   (a) Obtain the free-particle spectrum (wavefunctions and the corresponding $E \geq 0$ Eigenenergies ).

   (b) What are the bound-state wavefunctions and Eigenenergies ($E < 0$)? What are the corresponding angular momentum? What are the conditions that $V_0$ has to satisfy in order to have those states? (Obtain explicit results for the s-wave case.)

   (c) For the s-wave free particle, obtain the corresponding phase shift $\delta_0$.

   (d) In the low-energy limit ($\sqrt{2mE}a \ll \hbar^2$), what energies produce resonances on $\delta_0$? Interpret your results in terms of the values of potential found in (b)? Show that in these cases, the corresponding s-wave contribution for the total cross section is pratically maximal. Finally, show that when those resonances happen, the weigth of the wavefunction in the scattered part is minimal.

   (e) It is interesting to note that bound states can be viewed as poles of the $S_\ell$ matrix. In this way, exploiting the unitarity and analyicity of $S_\ell$ one obtains information about the spectrum of the system (without the need of solving the Schrödinger equation). Read Sakurai, Sec. 7.7.

   (f) Consider the weak-scattering regime in the low-energy limit $\sqrt{2mE}a \ll \hbar^2$, and $|V_0| \ll E$ (now, admit that $V_0$ can be either positive or negative). Show that the differential cross section is isotropic and that the total cross section is $\left(\frac{16\pi}{9}\right) m^2 V_0^2 a^6 \hbar^{-4}$. Upon increasing the energy, show that the new angular distribution can be written as $\sigma = A + B\cos\theta$ and find an approximate expression for $B/A$.

5. Consider the time-independent Schrödinger equation $(H_0 + V)|\psi\rangle = E|\psi\rangle$.

   (a) Show that $|\psi\rangle = |\phi\rangle + \left(\frac{1}{E-H_0}\right) V|\psi\rangle$, where $H_0|\phi\rangle = E|\phi\rangle$. This is the so-called Lippmann-Schwinger equation.

   (b) Since $E$ is an Eigenvalue of $H_0$, one needs to remove the singularity of the operator $(E - H_0)^{-1}$ in order to this equation become operational. This is often done by slightly making the energy complex, i.e., $|\psi\rangle \to |\psi^{(\pm)}\rangle = |\phi\rangle + (E - H_0 \pm i\eta)^{-1} V|\psi\rangle$. Show that the corresponding Green function is $G^{(\pm)}(\mathbf{r}, \mathbf{r}') = -4\pi \left\langle \mathbf{r} \left| (E - H_0 \pm i\eta)^{-1} \right| \mathbf{r}' \right\rangle$.

   (c) Show that the corresponding transition operator is $T = V + V(E - H_0 \pm i\eta)^{-1} V + V(E - H_0 \pm i\eta)^{-1} V (E - H_0 \pm i\eta)^{-1} V + \ldots$.

   (d) What is the physical meaning of the $\pm$ sign?

6. Using the first 3 partial waves, compute and sketch a polar plot the differential cross section for a hard sphere potential for the case in which the incident particle de Broglie wavelength matches the sphere circunference. Compute the total cross section and estimate the precision of your result. Finally, discuss the case when the particle wavelength is much bigger than the size of the sphere.

7. Consider the scattering by a $\delta$-potential $V(r) = \frac{\hbar^2}{2m} \times \gamma\delta(r - a)$, with $\gamma > 0$.

   (a) Find the equation from which one obtains the s-wave phase shift.

   (b) For $\gamma a \gg 1$ and $\gamma \gg k$, with $\hbar^2 k^2 = 2mE$, show that if $\tan ka \neq 0$ then $\delta_0$ resembles the phase shift of the hard-sphere potential.

(c) For $\gamma a \gg 1$, $\gamma \gg k$, and $\tan ka \ll 1$, the resonance behavior is also possible, i.e., $\cot \delta_0$ becomes zero when $k$ is increased. Find the position of the resonances keeping terms of order $\mathcal{O}(\gamma)^{-1}$, and compare them with the bound-state energies of a particle in a spherical well in which $V(r) = 0$ for $r \leq a$, and $V(r) = \infty$ for $r > a$. Finally, obtain an approximate expression for the width of the resonance defined as

$$\Gamma_q = \frac{-2}{\frac{\mathrm{d}}{\mathrm{d}E} \cot \delta_0 \big|_{E=E_q}},$$

and notice it becomes sharper as $\gamma \to \infty$.

8. (Optional) Consider the Schrödinger equation

$$-\frac{\hbar^2}{2m} \nabla^2 \psi(\mathbf{r}) + \int V(\mathbf{r}, \mathbf{r}') \psi(\mathbf{r}') \mathrm{d}\mathbf{r}' = E\psi(\mathbf{r}),$$

where the nonlocal term $V(\mathbf{r}, \mathbf{r}') = -\frac{\hbar^2}{2m} \lambda u(r) u(r')$.

(a) Show that only the s-wave is affected by the interaction.

(b) Find the corresponding Green-function like integral solution for $\psi(\mathbf{r})$, i.e., the integral equation equivalent to the Schrödinger equation, and the corresponding scattering boundary conditions.

(c) Show that the scattering amplitude for the incident wave of momentum $\hbar\mathbf{k}$ is

$$f(k) = \frac{4\pi\lambda |v(k)|^2}{1 + \frac{2\lambda}{\pi} \int \mathrm{d}\mathbf{q} \frac{|v(q)|^2}{k^2 - q^2 + i\eta}},$$

where $v(k) = \int_0^\infty \frac{\sin(kr)}{kr} u(r) r^2 \mathrm{d}r$.

(d) Compare the expression of the previous item with the Born series. What is the condition imposed on $\lambda$ in order to ensure the series convergency.

(e) Consider from now on that $u(r) = e^{-r/b}/r$. Show that

$$k \cot \delta = \left[ \left(k^2 b^2 + 1\right)^2 + \xi \left(k^2 b^2 - 1\right) \right] / 2\xi b,$$

where $\xi = 2\pi\lambda b^3$.

(f) Determine the condition on $\lambda$ in order to have a bound state. Is it possible to have more than one bound state in this potential?