List of exercises #5 - 7600037

1. **Schrödinger and Heisenberg pictures.** Consider a quantum 1D Harmonic Oscillator of mass m and natural frequency ω which is initially prepared $(t = 0)$ in the state

$$|\psi_S(0)\rangle = \exp\left(-\frac{i}{\hbar} P_S x_0\right) |0\rangle ,$$

where $|0\rangle$ is the ground state, P_S is the momentum operator, and x_0 is a scalar.

   (a) In the Heisenberg picture, compute $\langle X_H(t)\rangle$ for $t > 0$ and relate it with the classical trajectory.

   (b) Relate this result with initial conditions of the classical Harmonic Oscillator?
   Hint: The Baker-Haussdorff identity is $e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \dots$.

2. **Harmonic Oscillator.** Consider a one-dimensional linear harmonic oscillator of mass M and natural frequency ω given by the unperturbed Hamiltonian

$$H_0 = \frac{1}{2M} p^2 + \frac{1}{2} M\omega^2 x^2.$$

   (a) Write the Hamiltonian in terms of raising and lowering operators $a^\dagger$ and a:

$$a = \sqrt{\frac{M\omega}{2\hbar}} x + \frac{i}{\sqrt{2M\hbar\omega}} p \text{ and } a^\dagger = \sqrt{\frac{M\omega}{2\hbar}} x - \frac{i}{\sqrt{2M\hbar\omega}} p.$$

   (b) Compute the matrix element $\left\langle m \left| a^\dagger \right| n \right\rangle$, where $|n\rangle$ and $|m\rangle$ are Eigenstates of H_0.

   (c) Consider the perturbation

$$V(t) = gxe^{-(t/\tau)^2},$$

   with g being a constant (and consider that $\tau \to \infty$). Give a physical interpretation of $V(t)$.

   (d) Consider that at time $t_0 \ll -\tau$ the system is prepared in state $|n\rangle$. In which order of time-dependent perturbation theory the probability of finding the system in state $|n + m\rangle$ (with $m > 0$) is finite?

   (e) In lowest nonvanishing order of perturbation theory, compute the probability amplitude of finding the system in state $|n + m\rangle$ at instants $t \gg \tau$.

   (f) Now, consider a perturbation of type

$$V(t) = \gamma x^2 e^{-(t/\tau)^2},$$

   with γ being a constant. Give a physical interpretation for $V(t)$.

   (g) In which orders of perturbation theory, the transition to state $|n + m\rangle$ is finite?

3. **Fermi's golden rule and adiabaticity.** Given the importance of Fermi's golden rule, it is interesting to derive it in a different fashion. Consider that the transitions between eigenstates of H_0 are due to small perturbations. Thus, first-order perturbation theory should suffice. Assume a perturbation that is adiabatically switched on as $Ve^{\eta t}$ from $-\infty < t < 0$, where V is a time-independent operator and $0 < \eta \ll 1$ is a scalar.

   (a) Compute the transition rate $\Gamma_{m\leftarrow n} = \left.\frac{\mathrm{d}P_{m\leftarrow n}(t)}{\mathrm{d}t}\right|_{t\to 0^-}$, where $P_{m\leftarrow n}(t)$ is the transition probability from the initial state $|n\rangle$ at $t_0 \to -\infty$ to state $|m\rangle$ at t.

   (b) Using your expression for $\Gamma_{m\leftarrow n}$, show the adiabaticity criterion: the transition rate disappear when $\hbar\eta \ll |E_m - E_n|$.

   (c) Why do the disappearance of transition rates mentioned in 3b guarantees an adiabatic evolution of the system?

   (d) In the limit $\eta \to 0$, show that the Fermi's golden rule for discrete levels is recovered.

4. **Hydrogen atom in an electric field.** Consider a hydrogen atom in its ground state subject to an electric field $\mathbf{E} = \mathbf{E}_0 \cos\omega t$ (which, for simplicity, is to be considered as a classical vector).

   (a) What is the minimum frequency of the field in order to have ionization?

   (b) What is the transition rate (probability per unit of time) to an ionized state (assuming the electron can be represented by plane waves)?

   (c) What is the angular distribution of the ejected electron in this process?

   (d) Now consider that the atom is in a certain Eigenstate $|n, l, m\rangle$ and that ω is lower than the corresponding ionization frequency. What can be said about the final Eigenstate $|n', \ell', m'\rangle$, i.e., what are the selection rules?

5. **Interacting spin-1/2 particles.** Consider two spin-1/2 particles interacting as

$$V(t) = \frac{E(t)}{\hbar^2}\mathbf{S}_1 \cdot \mathbf{S}_2,$$

   where $E(t)$ vanishes when $t \to \pm\infty$ and approaches to a nonzero value of order $\bar{E}$ on the time interval of length τ. (You may think on a gaussian, for instance.)

   (a) At $t \to -\infty$, the system is in the state $|+-\rangle$. Compute exactly the state of the system at time t. With this, show that the probability of finding the system in the state $|-+\rangle$ for $t \to +\infty$ depends only on the integral $I = \int_{-\infty}^{\infty} E(t)\mathrm{d}t$.

   (b) Compute the same probability in first-order of time-dependent perturbation theory. By comparing your results with those of item 5a, discuss the validity of this calculation.

   (c) Make some estimations about the contribution to this probability in second-order of perturbation theory in the limits of $\tau \to 0$ and $\tau \to \infty$ and discuss your results with the validity of the approximation conclude in item 5b.

   (d) Now consider that both spins are subjected to a static magnetic field $\mathbf{B} = B_0\hat{z}$. The corresponding Zeeman Hamiltonian is

$$H_0 = -\frac{\mu_B}{\hbar}B_0\left(g_1 S_1^z + g_2 S_2^z\right),$$

   where $g_{1,2}$ are the gyromagnetic ratios (assume them distinct from each other) and μ_B is the Bohr magneton. Consider also that $E(t) = \bar{E}\exp\left(-(t/\tau)^2\right)$. Compute the same probability of the previous itens in first-order of perturbation theory, and discuss its dependence on B_0 and on τ.

   (e) Like in item 5c, compute the second-order contribution to the transition amplitude $c_{f\leftarrow i}^{(2)}(\infty)$ in the limits $\tau \to 0$ and $\tau \to \infty$. *Hint*: In the limits $\tau \to 0$ and $\tau \to \infty$, one can use the approximations $E(t) = \bar{E}\tau\delta(t)$ and $E(t) = \bar{E}\theta(\tau/2 - |t|)$, respectively.

6. **Transition probability per unit time under the effect of a random perturbation. Simple relaxation model.** (Cohen-Tannoudji - complement E-XIII, problem 9.) A physical system, subjected to a perturbation $W(t)$, is at time $t = 0$ in the Eigenstate $|\varphi_i\rangle$ of its Hamiltonian H_0. Let $P_{f\leftarrow i}(t)$ be the probability of finding the system at time t in another Eigenstate of H_0, $|\varphi_j\rangle$. The transition probability per unit time $w_{f\leftarrow i}(t)$ is defined by $w_{f\leftarrow i}(t) = \frac{\mathrm{d}}{\mathrm{d}t}P_{f\leftarrow i}(t)$.

   (a) Show that, to first order in perturbation theory, we have

$$w_{f\leftarrow i}(t) = \frac{1}{\hbar^2}\int_0^t \mathrm{d}\tau W_{fi}(\tau)W_{fi}^*(t-\tau)e^{i\omega_{fi}\tau} + \text{c.c.} \tag{1}$$

   with $\hbar\omega_{fi} = E_f - E_i$.

   (b) Consider $\mathcal{N} \gg 1$ identical and independent copies of the system (labeled by k). Each of them has a different microscopic environment and, consequently, "sees" a different perturbation $W^{(k)}(t)$. It is also impossible to know each of the individual perturbation $W^{(k)}$, and only statistical averages can be specified such as:

$$\overline{W_{fi}(t)} = \lim_{\mathcal{N}\to\infty} \frac{1}{\mathcal{N}} \sum_{k=1}^{\mathcal{N}} W_{fi}^{(k)}(t),$$

$$\overline{W_{fi}(t)W_{fi}^*(t-\tau)} = \lim_{\mathcal{N}\to\infty} \frac{1}{\mathcal{N}} \sum_{k=1}^{\mathcal{N}} W_{fi}^{(k)}(t)W_{fi}^{(k)*}(t-\tau).$$

This perturbation is said to be “random”.
This random perturbation is said to be stationary if the preceding averages are time independent. In this case, we can redifine H_0 in order to make $\overline{W_{fi}} = 0$ and set:

$$g_{fi}(\tau) = \overline{W_{fi}(t)W_{fi}^*(t-\tau)},$$

which is called the “correlation function” of the perturbation. Usually, $g_{fi}(\tau)$ goes to zero for $|\tau| \gg \tau_c$, a characteristic time scale, called correlation time of the perturbation, i.e., the perturbation has a “memory” which extend into the past (or future) only to an interval of order of τ_c.

i. At $t = 0$, the $\mathcal{N}$ systems are prepared in the state $|\varphi_i\rangle$ and are subject to a random stationary perturbation. Its correlation function is $g_{fi}(\tau)$ with correlation time τ_c. Calculate the fraction $\pi_{fi}(t)$ of systems which transition to state $|\varphi_j\rangle$ per unit time. Show that for $t \gg t_1$, to be specified, $\pi_{fi}(t)$ no longer depends on t.

ii. For fixed τ_c, how does π_{fi} vary with ω_{fi}? Consider the case in which $g_{fi}(\tau) = |v_{fi}|^2 e^{-|\tau|/\tau_c}$, with v_{fi} constant.

iii. The preceding theory is valid only for $t \ll t_2$ [since Eq. (1) results from perturbation theory]. What is the order of magnitude of t_2? Taking $t_2 \gg t_1$, find the condition for introducing a transition probability per unit time which is independent of t [use the form of $g_{fi}(\tau)$ given in the preceding question]. Would it be possible to extend the preceding theory beyond t_2?

(c) Application to a system. The $\mathcal{N}$ systems under consideration are spin-1/2 particles, with gyromagnetic ratio γ, placed in a static magnetic field $\mathbf{B}_0$ (set $\omega_0 = \gamma B_0$). These particles are enclosed in a spherical shell of radius R. Each of them bounces constantly back and forth between the walls. The mean time between the collisions of the same particle with the wall is called “time of flight” τ_v. During this time, the particle sees only the magnetic field $\mathbf{B}_0$. In a collision with the wall, each particle remains adsorbed on the surface during a mean time τ_a ($\tau_a \ll \tau_v$), during which it seems, in addition to $\mathbf{B}_0$, a constant microscopic field $\mathbf{b}$ due to paramagnetic impurities contained in the wall. The direction of $\mathbf{b}$ varies randomly from one collision to another; the mean amplitude of $\mathbf{b}$ is b_0.

i. What is the correlation time of the perturbation seen by the spins? Give the physical justification for the following form, to be chosen for the correlation function of the components of the microscopic magnetic field $\mathbf{b}$:

$$\overline{b_x(t)b_x(t-\tau)} = \frac{1}{3}b_0^2\left(\frac{\tau_a}{\tau_b}\right)e^{-|\tau|/\tau_a},$$

and analogous expressions for the y- and z-components, and all the cross terms $\overline{b_x(t)b_y(t-\tau)} = \overline{b_x(t)b_z(t-\tau)} = \cdots = 0$.

ii. Let M_z be the z-component of the total magnetization. (Consider $\mathbf{B} = B_0\hat{z}$.) Show that, under the effect of the collisions with the walls, M_z “relaxes”, with a time constant T_1:

$$\frac{\mathrm{d}M_z}{\mathrm{d}t} = -\frac{M_z}{T_1}$$

(T_1 is called the longitudinal relaxation time). Calculate T_1 in terms of γ, B_0, τ_v, τ_a, b_0.

iii. Show that studying the variation of T_1 with B_0 permites the experimental determination of the mean adsorption time τ_a.

iv. We have at our disposition several cells, of different radii R, constructed of the same material. By measuring T_1, how can we determine experimentally the mean amplitude b_0 of the microscopic field in the wall.

7. **Emission and absorption.**
   (a) Consider a structureless free quantum particle in the infinity space. Show that this particle cannot spontaneously emit a single photon. Physically, why this is the case? *Hint*: Use that the initial and final states of the free particle have well define momenta and that the dispersion relation for the particle is quadratic while for the photon it is linear.
   (b) Consider the spontaneous decay of the Hydrogen atom (fixed in space) in state $|2,1,1\rangle$. Compute the amplitude of the decay using plane waves for photons, and explain the angular dependence of the amplitude for each helicity ± 1 of the final-state photon in terms of the angular momentum conservation. Show that the rate is the same as the decay rate of the $|2,1,0\rangle$ state.
   (c) (Optional) Compare the previous decay rate with the case of a free Hydrogen atom, *i.e.*, for the case of a finite-mass proton. Without doing any calculation, in which case do you expect the transition rate to be larger? Justify.
   (d) (Optional) How can the 2s state decay to the 1s state? There is no need in computing it, but discuss in detail. Discuss about the electric and magnetic dipolar transitions. Discuss about the decay route 2s $\rightarrow$2p$\rightarrow$1s. (Recall that due to Lamb shift splitting, 2s and 2p are not degenerate.) Compute this amplitude transition (see *Advanced Quantum Mechanics*, J. J. Sakurai, problem 2.6).

8. **Single-mode resonant cavity.** Consider the Jaynes-Cummings Hamiltonian given by

$$H = \hbar\omega a^\dagger a + \frac{1}{2}\hbar\omega_0\sigma^z + \frac{1}{4}\hbar\Omega\left[a^\dagger\left(\sigma^x - i\sigma^y\right) + a\left(\sigma^x + i\sigma^y\right)\right].$$

   The creation and anihillation operators $a^\dagger$ and a act on the radiation field while the Pauli matrices $\sigma^{x,y,z}$ act on the matter. ω, ω_0 and Ω are constants (frequencies).
   (a) Give a detailed physical interpretation of each term in the Hamiltonian.
   (b) Compute all the Eigenenergies and Eigenvectors of H. (They are called dressed states of the matter.)
   (c) Consider now that the system is prepared in the state $|\psi_0\rangle = \sum_n C_n\, |n\text{-photons}\rangle_{\text{radiation}} \otimes |\text{ground}\rangle_{\text{matter}}$, with $C_1 = C_2$ and all others $C_i = 0$. Compute the probability of finding the two-level system in the excited state as a function of time.

9. **Scattering.** We are interested in the scattering process in which the initial and final states are

$$|I\rangle = |i\rangle \otimes |n_{\mathbf{k},\lambda}, 0_{\mathbf{k}',\lambda'}\rangle\,, \text{ and } |F\rangle = |f\rangle \otimes \left|(n-1)_{\mathbf{k},\lambda}\,, 1_{\mathbf{k}',\lambda'}\right\rangle,$$

   i.e., in the beginning, there are n photons of momentum $\hbar\mathbf{k}$ and polarization λ while, in the end, there is one less photon in such state which was scattered into a photon of momentum $\hbar\mathbf{k}'$ and polarization λ'. Such process involves two photons and have contribution in second order of perturbation theory from the term $\frac{e}{m}\sum_i \mathbf{p}_i\cdot\mathbf{A}(\mathbf{r}_i)$ (where $\mathbf{p}_i$ are the momentum of the i-th electron in the system), and contribution in first order in perturbation theory from the diamagnetic term $V = \frac{e^2}{2m}\sum_i \mathbf{A}(\mathbf{r}_i)\cdot\mathbf{A}(\mathbf{r}_i)$. Here, consider only the effects of this latter term.
   (a) Rewrite V in terms of the density operator $\rho(\mathbf{r})$.
   (b) Compute the matrix element $\langle F|V|I\rangle$.
   (c) Compute the differential cross section and show that

$$\frac{\mathrm{d}\sigma_{I\to F}}{\mathrm{d}\Omega} = r_0^2\frac{\omega_{k'}}{\omega_k}\left|\hat{e}_{\mathbf{k},\lambda}\cdot\hat{e}^*_{\mathbf{k}',\lambda'}\right|^2 |\langle f|\tilde{\rho}(\mathbf{k}-\mathbf{k}')|i\rangle|^2,$$

   where $r_0 = \frac{e^2}{4\pi\epsilon_0 mc^2}$ is the classical radius of the electron, and $\tilde{\rho}(\mathbf{k})$ is the Fourier transform of $\rho(\mathbf{r})$.
   (d) Consider the simplest case of the scattering by a single free electron in which $|i\rangle = |\hbar\mathbf{q}_i\rangle$ and $|f\rangle = |\hbar\mathbf{q}_f\rangle$ and compute the corresponding differential cross section (dubbed the Thomson cross section). Explain why this process is allowed.