**KdV equation: Solitons**

**Note:**
- \( A = \frac{v}{2} \) -- the amplitude is proportional to the velocity of propagation while the “width” is proportional to \( \sqrt{v} \)
  - taller solitary waves are thinner and move faster

- There is another solution: \( u(\theta) = -\frac{v}{2} \text{csch}^2 \left( \frac{\sqrt{v} \theta}{2} \right) \) - singularity at \( x = vt \)

- More general solutions can be found for other choices of \( C_1 \) and \( C_2 \)
  (cnoidal waves, solution is expressed via Jacobi elliptic integrals)

- KdV equation supports multisoliton solutions

- There is **anti-soliton** solution of the another KdV equation obtained by replacing \( u \to -u \): \( u_t - 6uu_x + u_{xxx} = 0 \)

- KdV equation is not Lorentz-invariant (however, there are other symmetries)
- KdV equation is completely integrable
KdV equation as a Hamiltonian System

Note: the KdV equation can be written as

\[ u_t = \frac{\partial}{\partial x} \frac{\delta H}{\delta u} \]

Here

\[ H = \int_{-\infty}^{\infty} T_3 \, dx = \int_{-\infty}^{\infty} (u^3 - \frac{1}{2} u_x^2) \, dx \]

is the “energy” integral and the variational derivative of the functional

\[ H[u] = \int f(u, u_x; x) \, dx \]

is

\[ \frac{\delta H}{\delta u} = \frac{\partial f}{\partial u} - \frac{d}{dx} \left( \frac{\partial f}{\partial u_x} \right) \]

The Poisson bracket:

\[ \{H, G\} = \int_{-\infty}^{\infty} \delta H \, \frac{\partial}{\partial u} \left( \frac{\delta G}{\delta u} \right) \]

Note: integration by parts yields

\[ H = \int_{-\infty}^{\infty} (u^3 + uu_{xx}) \, dx \]

\[ \frac{\delta H}{\delta u} = 3u^2 + 2u_{xx} \]

Question: How to link it to the traditional form of the finite-dimensional Hamiltonian system?

Fourier expansion: \( u(x, t) = \sum u_k e^{ikx} \)

\( \{q_k = u_k/k, \ p_k = u_{-k}, \ \mathcal{H} = \frac{i}{2\pi} H\} \)

We recover the usual Hamiltonian equations

\[ \frac{dq_k}{dt} = \frac{\partial \mathcal{H}}{\partial p_k}, \quad \frac{dp_k}{dt} = -\frac{\partial \mathcal{H}}{\partial q_k} \]

with the Poisson bracket

\[ \{H, G\} = \frac{i}{2\pi} \sum_{k=-\infty}^{\infty} k \frac{\partial H}{\partial u_k} \frac{\partial G}{\partial u_{-k}} \]
KdV equation: Integrability

Existence of the infinite tower of conservation laws → strong indication that we deal with a completely integrable system

\[ u_t + 6uu_x + u_{xxx} = 0 \]

What does it mean?

- **Gardner (1971):** The KdV equation represents an infinite-dimensional Hamiltonian system with an infinite number of integrals of motion in involution.
- **Gardner, Greene, Kruskal & Miura (1967):** Inverse Scattering Transform (IST) method: the method to solve an initial-value problem for the KdV equation within a class of initial conditions.
- **Zakharov, Shabad and other (1971):** Inverse Scattering Transform for the nonlinear Schrödinger equation (NLS), the Sine-Gordon equation and many other completely integrable equations.

**Note:** The availability of the travelling wave (and, in particular, soliton) solutions for the KdV equation does not constitute its integrability. Practically the complete integrability means just the ability to integrate the KdV equation for a reasonably broad class of initial or boundary conditions.
KdV equation: Integrability

Existence of the infinite tower of conservation laws → strong indication that we deal with a **completely integrable system**

\[ u_t + 6uu_x + u_{xxx} = 0 \]

**Examples of integrable systems**

- Euler's top, Coulomb potential, harmonic oscillator, KdV equation, Heisenberg spin chain, Nahm equations, BPS Skyrme model, self-dual Yang Mills equations, nonlinear Schrodinger equation, sine-Gordon and sinh-Gordon equations, N=4 SUSY Yang Mills…

- Integrable models = exactly solvable models

**Note:** The availability of the travelling wave (and, in particular, soliton) solutions for the KdV equation **does not** constitute its integrability. Practically the complete integrability means just the ability to integrate the KdV equation for a reasonably broad class of initial or boundary conditions.
**Question:** if the infinite number of conservation laws for KdV means that it is an analogue of a completely integrable Hamiltonian system?

**Recall:** the Burgers equation can be solved exactly through the Hopf-Cole transform:

\[
\psi_t - \gamma \psi_{xx} = 0 \quad \quad \quad u = -2\gamma \frac{\psi_x}{\psi} \quad \quad \quad u_t + uu_x - \gamma u_{xx} = 0
\]

How about KdV?

\[
u_t - 6uu_x + u_{xxx} = 0
\]

**Step I:** Miura transform:

\[
u = v^2 + v_x
\]

**Step II:** Linearization of the modified KdV equation? Should we try

\[
v = \frac{\psi_x}{\psi} \quad \quad \quad v_x = \frac{\psi_{xx}}{\psi} - \frac{\psi_x^2}{\psi^2} \quad \quad \quad v_t - 6v^2v_x + v_{xxx} = 0
\]

Galilean symmetry of KdV: \(u \rightarrow u + E\)

“This Schroedinger” equation:

\[
\psi_{xx} - u\psi = E\psi
\]

This is like quantum mechanics!!
Remark I: the KdV equation can be linearised via the spectral theory of the Schrödinger operator but not by means of an explicit change of variables.

Remark II: the KdV equation $u_t + 6uu_x + u_{xxx} = 0$ can be viewed as a compatibility (integrability) condition for two linear differential equations for the same auxiliary function $\psi(x, t; \lambda)$

Evolution problem: $L\psi \equiv (-\partial_{xx} - u)\psi = \lambda\psi$

Spectral problem: $\psi_t = A\psi \equiv (-4\partial_{xxx} - 6u\partial_x - 3u_x + c)\psi = (u_x + c)\psi + (4\lambda - 2u)\psi_x$

$\lambda$ is a complex parameter, $c(\lambda, t)$ depends on normalization of $\psi$

Compatibility condition + Isospectral evolution: $(\psi_{xx})_t = (\psi_t)_{xx} + \lambda_t = 0 = \text{KdV}$

Homework: Prove it!

The operators $L$ and $A$ are referred to as the Lax pair.

Remark III: the spectral equation $L\psi$ is the Schrödinger equation we discussed!

Remark IV: the KdV equation is isospectral, i.e. $\lambda_t = 0$
Lax Pair

Remark V: the KdV equation can be represented in an operator form as

\[ L_t = AL - LA \equiv [AL] \]

This operator representation provides a route for constructing the **KdV hierarchy** by appropriate choice of the operator \( A \). Indeed, given the \( L \)-operator, the \( A \)-operator in the Lax pair is determined up to an operator commuting with \( L \), which makes it possible to construct an infinite number of equations associated with the same spectral problem but having different evolution properties.

- **KdV hierarchy**

\[
\begin{align*}
L_0[u] &= \frac{1}{2}; \\
\frac{\partial}{\partial x} L_{n+1}[u] &= \left( \frac{\partial^3}{\partial x^3} + 4u \frac{\partial u}{\partial x} + 2 \frac{\partial^2 u}{\partial x^2} \right) L_n[u] \\
t + \frac{\partial}{\partial x} L_{n+1}[u] &= 0
\end{align*}
\]

1. \( L_1[u] = u; \quad u_t + 6u u_x + u_{xxx} = 0 \)
2. \( L_2[u] = u_{xx} + 3u^2; \quad u_t + 6u u_x + u_{xxx} = 0 \)
3. \( L_3[u] = u_{xxxx} + 10uu_{xx} + 5u_x^2 + 10u^3; \quad u_t + 10u u_{xxx} + 30u^2 u_x + 20u_x u_{xx} + u_{xxxxx} = 0 \)
Lax Pair operator formulation

\[ L\psi \equiv (-\partial_{xx}^2 - u)\psi = \lambda \psi \]

\[ \lambda_t = 0 \]

\[ \psi_t = A\psi \equiv u_x \psi + (4\lambda - 2u)\psi_x \]

\[ \frac{d}{dt} (L\psi) = \frac{dL}{dt} \psi + L\psi_t = \frac{dL}{dt} \psi + LA\psi \]

\[ \frac{d}{dt} (\lambda \psi) = \lambda \psi_t = \lambda A\psi = A(\lambda \psi) = AL\psi \]

**Lax equation**

**Reduction of the eigenvalue equation:**

\[ \Psi = \begin{pmatrix} \psi \\ \psi_x \end{pmatrix}, \quad \Psi_x = \begin{pmatrix} 0 & 1 \\ -(u + \lambda) & 0 \end{pmatrix} \Psi \equiv U\Psi \]

\[ U_t = \begin{pmatrix} 0 & 0 \\ -u_t & 0 \end{pmatrix} \]

**The time evolution equation:**

\[ \Psi_t = \begin{pmatrix} u_x \\ -4\lambda^2 = 2\lambda u + 2u^2 + u_{xx} \\ 4\lambda - 2u \\ -u_x \end{pmatrix} \Psi \equiv V\Psi \]
**Lax Pair operator formulation**

\[ U = \begin{pmatrix} 0 & 1 \\ -(u + \lambda) & 0 \end{pmatrix}; \quad V = \begin{pmatrix} u_x & 4\lambda - 2u \\ -4\lambda^2 - 2\lambda u + 2u^2 + u_{xx} & -u_x \end{pmatrix} \]

\[ U_t = \begin{pmatrix} 0 & 0 \\ -u_t & 0 \end{pmatrix}; \quad V_x = \begin{pmatrix} u_{xx} & -2u_x \\ -2\lambda u_x + 4uu_x + u_{xxx} & -u_{xx} \end{pmatrix} \]

**Zero curvature condition:**

\[ U_t - V_x + [U, V] = \begin{pmatrix} 0 & 0 \\ -u_t - 6uu_x - u_{xxx} & 0 \end{pmatrix} = 0 \]

- **The idea** – treat \( U, V \) as it would be gauge fields

Integrals of motion – invariants of the gauge transformation:

\[ \partial_x - U \rightarrow g^{-1}(\partial_x - U)g \]
**KdV equation: scattering problem**

- **Scattering problems:** given a potential $u$, determine the spectrum $\{\psi, E\}$
- **Inverse scattering problem:** given a spectrum $\{\psi, E\}$, determine the potential

Assume $u(\pm\infty) = 0 \rightarrow |\psi|^2$ is integrable over $\mathbb{R}$ and it is normalizable

**The discrete spectrum:** $\psi_n(x) = c_n e^{-\kappa_n x}$; $E = -\kappa_n^2$ as $x \to \pm\infty$

**The continuous spectrum:** $E = k^2$, $k \in \mathbb{R}$

**Question:** What happened if $u = u(x, t)$ such that $u(x, t)$ solves KdV equation?

**Naive answer:** the eigenvalues $E$, which in general depend on $t$ through the parametric dependence in $u$, should change as $t$ varies

**Theorem I:** If $u(x,t)$ solves the KdV equation and it vanishes as $x \to \pm\infty$

the discrete eigenvalues of the Sturm-Liouville problem $\psi_{xx} + (\lambda - u)\psi = 0$

do not depend on $t$
KdV equation: scattering problem

Recall: The soliton solution of the KdV equation is \( u(x) = -\frac{u_0}{\cosh^2(x)} \)

**Sturm-Liouville equation:**

\[
\psi_{xx} + \left( \lambda + \frac{u_0}{\cosh^2(x)} \right) \psi = 0
\]

**Change of variable:** \( \eta = \tanh x \)

\[
\frac{\partial}{\partial \eta} \left( (1-\eta^2) \frac{\partial \psi}{\partial \eta} \right) + \left( u_0 + \frac{\lambda}{1-\eta^2} \right) \psi = 0
\]

Discrete spectrum: \( u_0 = l(l+1); \quad \lambda = -k^2 < 0 \) - Legendre polynomials

\[
\psi = P^{(k)}_l = (-1)^k (1-\eta^2)^{k/2} \frac{d^k}{d\eta^k} P_l(\eta); \quad P_l(\eta) = \frac{1}{2^l l!} \frac{d^l}{d\eta^l} (\eta^2 - 1)^l
\]

**Examples:**

\[
P^{(1)}_0 = \sqrt{1-\eta^2} = \frac{1}{\cosh^2 \frac{\eta}{2}}; \quad P^{(0)}_0 = \eta = \tanh x
\]

**Remark:** There are certain eigenvalues for those the potential is reflectionless

**Continuum:** \( \psi_k(x) = A \frac{2^{ik}}{(\cosh x)^{ik}} F_{2,1}(a, b, c; z); \quad z = \frac{1+\eta}{2}, \)

\[
a = \frac{1}{2} - ik + \sqrt{l(l+1) + \frac{1}{4}}, \quad b = \frac{1}{2} - ik - \sqrt{l(l+1) + \frac{1}{4}}, \quad c = 1 - ik
\]

Asymptotically:

\[
\psi_k(x) \sim A \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} e^{-ikx} + A \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} e^{ikx}
\]

Reflection coefficient
Consider linearized KdV equation:

\[ u_t + u_{xxx} = 0; \quad x \in \mathbb{R}, \quad u(x, 0) = u_0(x) \]

**Fourier transform:**

\[ u(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} u(x)e^{-ikx} \, dx; \quad u(x, t) = \int_{-\infty}^{+\infty} u(k, t)e^{ikx} \, dk \]

\[ u_x(x, t) = \frac{\partial}{\partial x} \int u(k, t)e^{ikx} \, dk = (ik) \int u(k, t)e^{ikx} \, dk \equiv \int u_x(k, t)e^{ikx} \, dk \]

\[ u_x(k, t) = iku(k, t), \quad u_{xxx}(k, t) = -ik^3 u(k, t) \]

**Fourier transform of the linearized KdV:**

\[ u_t(k, t) - ik^3 u(k, t) = 0 \]

\[ u(k, t) = u_0(k)e^{-ik^3 t} \quad \Rightarrow \quad u(x, t) = \int_{-\infty}^{+\infty} e^{-ik^3 t}e^{-ikx}u_0(k) \, dk \]

\[ \mathcal{F} \]

**PDE**

\[ u(x, 0) \quad \mathcal{F} \quad u(k, 0) \]

**Time evolution**

\[ u(x, t) \quad \mathcal{F}^{-1} \quad u(k, 0)e^{-ik^3 t} \]
**Scattering problem and the Fourier transform**

**Question:** What is the analogue of the Fourier transform for KdV?

This is the Sturm-Liouville equation!

\[ \psi_{xx} + (\lambda - u)\psi = 0 \]

\[ \psi(x; k) = \begin{cases} 
  e^{ikx} + R(k)e^{-ikx} & \text{as } x \to -\infty \\
  T(k)e^{ikx} & \text{as } x \to +\infty 
\end{cases} \]

### 3 steps for solving the KdV equation:

1. Given the initial condition \( u(x, 0) \) consider \(-u\) as a potential in the Schrödinger equation and calculate the discrete spectrum \( E = -\kappa^2 \), the norming constant \( c_n = c_n(0) \) and reflection coefficient \( R(k) = R(k; 0) \).
2. Introduce time dependence of these spectral data, the eigenvalues \( E = -\kappa^2 \) are fixed.
3. Carry out the procedure of the inverse scattering problem to recover \( u(x, t) \).
KdV equation: Direct scattering problem

**Scattering data:** \{\kappa_n, c_n(0), R(k, 0), T(k, 0)\}

- **Discrete spectrum** \(\psi_n(x, 0) \sim c_n(0)e^{-\kappa_n x}\) as \(x \to \pm\infty\)

- **Continuous spectrum** \(\psi(x; k) =\begin{cases} e^{ikx} + R(k)e^{-ikx} & \text{as } x \to -\infty \\ T(k)e^{ikx} & \text{as } x \to +\infty \end{cases}\)

Substitution into the second Lax equation yields for spectral data of continuum:

\[ \psi_t = A \psi \equiv (-4\partial_{xxx} - 6u\partial_x - 3u_x + c)\psi \quad \Rightarrow \quad c(\lambda, t) = -4i\kappa^3; \quad \frac{dR}{dt} = -8i\kappa^3 R; \quad \frac{dT}{dt} = 0 \]

Hence

\[ R(k, t) = R(k, 0)e^{-8i\kappa^3 t}; \quad T(k, t) = T(k, 0) \]

Time evolution of the spectral data of the discrete spectrum:

\[ c = c_n = 4\kappa_n^3 \quad \Rightarrow \quad \frac{dc_n}{dt} = 4c_n\kappa_n^3 \quad \Rightarrow \quad c_n(t) = c_n(0)e^{4\kappa_n^3 t} \]

**Remark:** the bound state problem can be viewed as an analytic continuation of the scattering problem defined on the real \(k\)-axis, to the upper half of the complex \(k\)-plane. Then the discrete points of the spectrum are found as *simple* poles \(k = i\kappa\) of the reflection coefficient \(R(k)\).
It is well known from 1950s that the potential of the Schrödinger equation can be completely recovered from the scattering data – *Gelfand-Levitan-Marchenko equation*

We define the function of the scattering data

\[
F(x, t) = \sum_{n=1}^{N} c_n^2 e^{-\kappa_n x} + \frac{1}{2\pi} \int_{-\infty}^{\infty} R(k, t) e^{ikx} dk
\]

Then the potential \( u(x, t) \) can be restored from the equation

\[
u(x, t) = 2 \frac{\partial}{\partial x} K(x, x, t)\]

where function \( K(x, y, t) \) can be found from the linear integral-differential GLM equation

\[
K(x, y) + F(x + y) + \int_{x}^{\infty} K(x, z) F(y + z) dz = 0
\]

**Note:** at each step of solving of the KdV equation we consider a linear problem
KdV inverse scattering problem: examples

- One bound state: $N=1$, reflectionless potential: $R(k)=0$, $c_n(t) = c_n(0)e^{4\kappa_n^3 t}$

$$F(x, t) = \sum_{n=1}^{N} c_n^2 e^{-\kappa_n x} + \frac{1}{2\pi} \int_{-\infty}^{\infty} R(k, t) e^{ikx} dk$$

$$F(x, t) = c^2(0)e^{-\kappa x + 8\kappa^3 t}$$

- $u(x) = -\frac{l(l+1)}{\cosh^2(x)}$; $\lambda = -\kappa^2$, $\kappa = \kappa_1 = 1$

Sturm-Liouville equation ($l=1$):
$$\psi_{xx} = \left(\lambda + \frac{2}{\cosh^2 x}\right) \psi = 0$$

Single normal discrete mode: $\psi_1(x) = \frac{1}{\sqrt{2}} \frac{1}{\cosh x} \rightarrow \frac{1}{\sqrt{2}} 2e^{-x}$ as $x \rightarrow \infty$

The scattering data $c(t) = \sqrt{2}e^{4t}$

$$F(x, t) = 2e^{8t-x}$$

- $c(0) = \sqrt{2}$

GLM equation:
$$K(x, y; t) + 2e^{8t-(x+y)} + 2 \int K(x, z; t)e^{8t-(y+z)} dz = 0$$

Ansatz: $K(x, y, t) = M(x, t)e^{-y}$

$$M(x, t) + 2e^{8t-x} + 2M(x, t)e^{8t} \int e^{-2z} dz = 0$$

$$M(x, t) = -\frac{2e^{8t-x}}{1 + e^{8t-2x}}$$

$$u(x, t) = 2 \frac{\partial}{\partial x} K(x, x, t) = 2 \text{sech}^2(x - 4t)$$

One-soliton solution propagating to the right
KdV inverse scattering problem: examples

- Two bound states: \( N=2 \), reflectionless potential: \( R(k) = 0 \),

\[
\begin{align*}
  l(l+1) = 6; \quad \lambda = -k^2, \quad \kappa_1 = 1; \quad \kappa_2 = 2
\end{align*}
\]

\[
\psi_{xx} + \left( \lambda + \frac{6}{\cosh^2 x} \right) \psi = 0
\]

Two normal discrete modes:

\[
\begin{align*}
  \psi_1 &= \sqrt{\frac{3}{2}} \frac{\tanh x}{\cosh x} \to \sqrt{6}e^{-x} \\
  \psi_2 &= \sqrt{\frac{3}{2}} \frac{1}{\cosh^2 x} \to 2\sqrt{3}e^{-x}
\end{align*}
\]

\[
\begin{align*}
  c_1(0) = \sqrt{6} \implies c_1(t) = \sqrt{6}e^{4t} \\
  c_2(0) = 2\sqrt{3} \implies c_2(t) = 2\sqrt{3}e^{32t}
\end{align*}
\]

GLM equation:

\[
K(x, y; t) + 6e^{8t-(x+y)} + 12e^{64t-2(x-y)} + \int K(x, z; t) \left\{ 6e^{8t-(y+z)} + 12e^{64t-2(y+z)} \right\} dz = 0
\]

Ansatz: \( K(x, y, t) = M_1(x, t)e^{-y} + M_2(x, t)e^{-2y} \)

Collecting the coefficients at \( e^{-y} \) and \( e^{-2y} \)

\[
\begin{align*}
  M_1 + 6e^{8t-x} + 6e^{8t} \left\{ M_1 \int_x e^{-2z} dz + M_2 \int_x e^{-3z} dz \right\} &= 0 \\
  M_2 + 12e^{64t-2x} + 12e^{64t} \left\{ M_1 \int_x e^{-3z} dz + M_2 \int_x e^{-4z} dz \right\} &= 0
\end{align*}
\]
2-soliton solution of the GLM equation:

\[ K(x, x, t) = M_1(x, t)e^{-x} + M_2(x, t)e^{-2x} \]

\[
\begin{align*}
M_1(x, t) &= \frac{6(e^{72t-5x} - e^{8t-x})}{1 + 3e^{8t-2x} + 3e^{64t-4x} + e^{72t-6x}} \\
M_2(x, t) &= -\frac{12(e^{64t-2x} + e^{72t-4x})}{1 + 3e^{8t-2x} + 3e^{64t-4x} + e^{72t-6x}}
\end{align*}
\]

\[
u_2 = 2 \frac{\partial}{\partial x} (M_1 e^{-x} + M_2 e^{-2x}) = -12 \frac{3 + 4 \cosh(2x - 8t) + \cosh(4x - 64t)}{[3 \cosh(x - 28t) + \cosh(3x - 36t)]^2}
\]

Most general case: \(N\)-soliton solution

- \(N\) bound states, reflectionless potential: \(R(k)=0\)

\[ F(x, t) = \sum_{n=1}^{N} c_n^2(t)e^{-\kappa_n x} \]

The ansatz for the solution of the GLM equation:

\[ K(x, y, t) = \sum_{n=1}^{N} M_n(x, t)e^{-\kappa_n y} \]
KdV inverse scattering problem: N-soliton solution

The solution of the GLM equation is given by

\[ u_N = 2 \frac{\partial^2}{\partial x^2} \ln \det A(x, t) \]

Here the \( N \times N \) matrix \( A \) is defined as

\[ A_{mn} = \delta_{mn} + \frac{c_n^2(0)}{\kappa_n + \kappa_m} e^{-(\kappa_n - \kappa_m)x + 8\kappa_n^3 t} \]

Asymptotically, as \( t \to \pm \infty \) this solution of the KdV equation represents a superposition of \( N \) single-soliton solutions propagating to the right and ordered in space by their speeds (amplitudes):

\[ u_N(x, t) \sim \sum_{n=1}^{N} 2\kappa_n^2 \text{sech}^2[\kappa_n(x - 4\kappa_n^2 t - x_n^{\pm})] \]

The position of the \( n \)-th soliton is given by:

\[ x_n^{\pm} = \frac{1}{2\kappa_n} \ln \frac{c_n^2(0)}{2\kappa_n} \pm \frac{1}{2\kappa_n} \left\{ \sum_{m=1}^{n-1} \ln \left| \frac{\kappa_n - \kappa_m}{\kappa_n + \kappa_m} \right| - \sum_{m=n+1}^{N} \ln \left| \frac{\kappa_n - \kappa_m}{\kappa_n + \kappa_m} \right| \right\} \]

The \( N \)-soliton solution is characterised by \( 2N \) parameters: \( \kappa_1 \ldots \kappa_N, c_1(0), \ldots c_N(0) \)

The evolution is isospectral, i.e. \( \kappa_n = \text{const} \) - the solitons preserve their amplitudes (and velocities) in the interactions; the only change they undergo is an additional phase shift \( \delta_n = x_n^+ - x_n^- \) due to collisions.
KdV solitons: 2-soliton solution

\[ A(x, t) = 1 + \exp \{ \kappa_1(x - x_1) - 4\kappa_1^3 t \} + \exp \{ \kappa_2(x - x_2) - 4\kappa_2^3 t \} + \left( \frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2} \right)^2 \exp \{ \kappa_1(x - x_1) - 4\kappa_1^3 t + \kappa_2(x - x_2) - 4\kappa_2^3 t \} \]  

\[ u_2 = 2\frac{\partial^2}{\partial x^2} \ln A(x, t) \]

Asymptotically, as \( t \to \pm \infty \)

\[ u_2(x, t) \sim 2\kappa_1^2 \operatorname{sech}^2[\kappa_1(x - 4\kappa_1^2 t - x_1) + 2\kappa_2^2 \operatorname{sech}^2[\kappa_2(x - 4\kappa_2^2 t - x_2)] \]

For a two-soliton collision the outcome is the phase shift \( \kappa_1 > \kappa_2 \)

\[ \delta_1 = 2x_1 = \frac{1}{\kappa_1} \ln \left( \frac{\kappa_1 + \kappa_2}{\kappa_1 - \kappa_2} \right), \quad \delta_2 = 2x_2 = \frac{1}{\kappa_2} \ln \left( \frac{\kappa_1 + \kappa_2}{\kappa_1 - \kappa_2} \right) \]

As a result of the interaction, the taller soliton gets an additional shift forward by the distance \( \delta_1 \) while the shorter soliton is shifted backwards by the distance \( -\delta_2 \).
KdV solitons: 2-soliton collision

\[ \kappa_1 = 1; \quad \kappa_2 = 2 \]
Given the initial condition \( u(x, 0) \) consider \(-u\) as a potential in the Schrödinger equation and calculate the discrete spectrum \( E = -\kappa^2 \), the norming constant \( c_n = c_n(0) \) and reflection coefficient \( R(k) = R(k; 0) \) (scattering data).

Introduce time dependence of these spectral data, the eigenvalues \( E = -\kappa^2 \) are fixed.

Carry out the procedure of the inverse scattering problem to recover \( u(x, t) \) making use of the GLM equation:

\[
K(x, y) + F(x + y) + \int_x^\infty K(x, z) F(y + z) dz = 0
\]

\[
F(x, t) = \sum_{n=1}^{N} c_n^2 e^{-\kappa_n x} + \frac{1}{2\pi} \int_{-\infty}^{\infty} R(k, t) e^{ikx} dk \quad \quad u(x, t) = 2 \frac{\partial}{\partial x} K(x, x, t)
\]