# Muon $g$-2 Anomaly & Lattice Results

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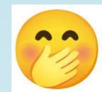
# Summary

Disclaimer: not at all the best person for a technical talk on the subject 

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- The Problem

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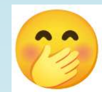
Disclaimer: not at all the best person for a technical talk on the subject 

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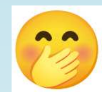
Disclaimer: not at all the best person for a technical talk on the subject 

- The Problem
- The Anomaly
- Recent Lattice Results
- General Aspects of Lattice Calculations

# Why the Muon?

$a = (g-2)/2$ measures the departure of a fermion's $g$ factor from its tree-level value $2$, from Dirac's equation. The value for the electron is the most precise quantity in physics

$$\begin{aligned} a_e &= 0.001\,159\,652\,181\,643(764) \quad \text{from theory} \\ &= 0.001\,159\,652\,180\,73(28) \quad \text{from experiment} \end{aligned}$$

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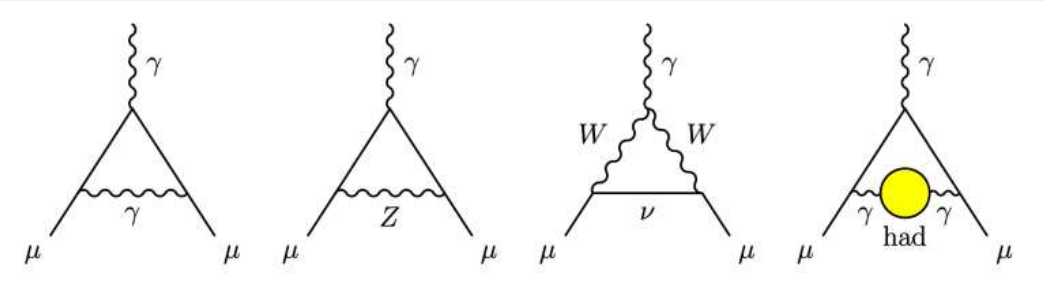
Muon gets other contributions, may couple to BSM particles and lives long enough to yield precise measurements

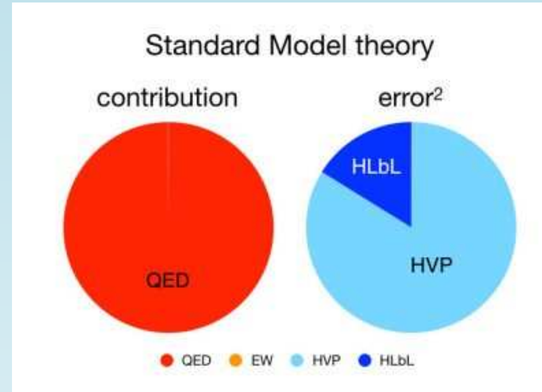
$$a_\mu = 0.001\,165\,920\,61(41)$$



# The Anomaly

Theoretical value ($\times 10^{-11}$)

| | |
|---|---|
| QED | 116 584 718.931(104) |
| Electroweak | 153.6(1.0) |
| HVP ($e^+e^-$, LO + NLO + NNLO) | 6845(40) |
| HLbL (phenomenology + lattice + NLO) | 92(18) |
| Total SM Value | 116 591 810(43) |
| Difference: $\Delta a_\mu := a_\mu^{\text{exp}} - a_\mu^{\text{SM}}$ | 279(76) |

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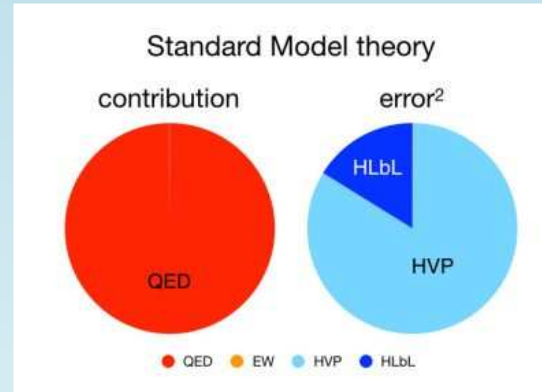


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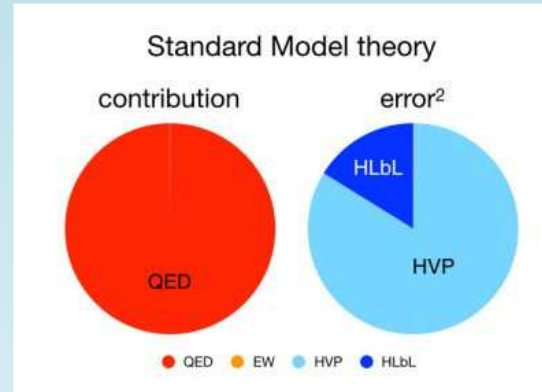


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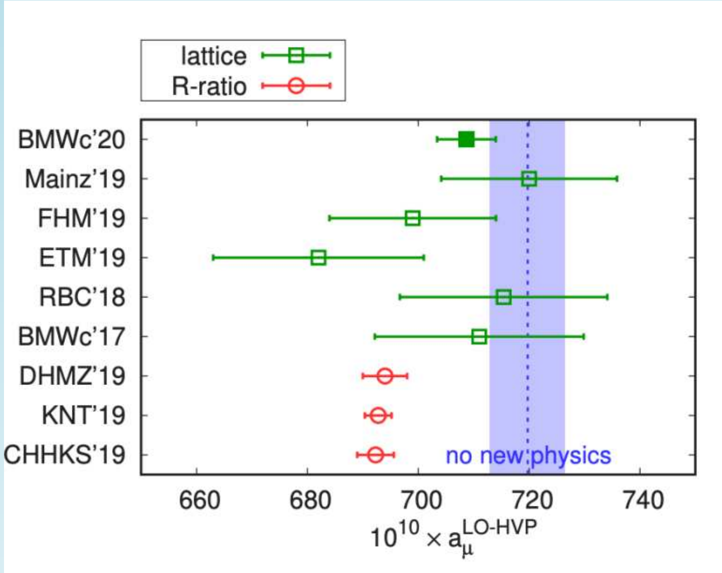
Tension btw theory (SM) and experiment increased from $3.7\sigma$ to $4.2\sigma$

# Latest Lattice Results




BMW Collaboration

Computation of HVP contribution

Nature article (published April 7, same day as Muon $g$-2 PRL paper)

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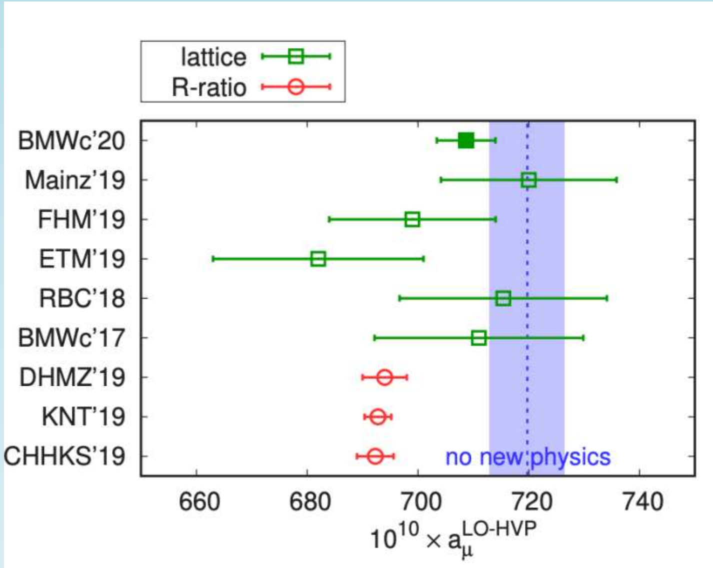


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Very large-scale first principles computation

constructive criticism. The computations were performed on JUQUEEN, JURECA, JUWELS and QPACE at Forschungszentrum Jülich, on SuperMUC and SuperMUC-NG at Leibniz Supercomputing Centre in Munich, on Hazel Hen and HAWK at the High Performance Computing Center in Stuttgart, on Turing and Jean Zay at CNRS IDRIS, on Joliot-Curie at CEA TGCC, on Marconi in Rome and on GPU clusters in Wuppertal and Budapest. We thank the Gauss Centre for Supercomputing, PRACE and GENCI (grant 52275) for awarding us computer time on these machines. This project was partially funded by DFG grant SFB/TR55,

# Two-Point Functions, QED Effects

We compute $a_\mu^{\text{LO-HVP}}$ in the so-called time–momentum representation[8], which relies on the following two-point function with zero three-momentum in Euclidean time $t$:

$$G(t) = \frac{1}{3e^2} \sum_{\mu=1,2,3} \int \mathrm{d}^3x \langle J_\mu(\mathbf{x},t) J_\mu(0)\rangle, \tag{1}$$

where $J_\mu$ is the quark electromagnetic current, with $\frac{J_\mu}{e} = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s + \frac{2}{3}\bar{c}\gamma_\mu c$. $u$, $d$, $s$ and $c$ are the up, down, strange and charm quark fields, respectively, and the angle brackets stand for the QCD + QED expectation value to order $e^2$. It is convenient to decompose $G(t)$ into light, strange, charm and disconnected components, which have very different statistical and systematic uncertainties. Integrating the one-photon-irreducible part of the two-point function (equation (1)), $G_{1\gamma\mathrm{I}}$, yields the LO-HVP contribution to the magnetic moment of the muon[8,9,10,11]:

$$a_\mu^{\text{LO-HVP}} = \alpha^2 \int_0^\infty \mathrm{d}t\, K(t) G_{1\gamma\mathrm{I}}(t), \tag{2}$$

with weight function

$$K(t) = \int_0^\infty \frac{\mathrm{d}Q^2}{m_\mu^2}\, \omega\left(\frac{Q^2}{m_\mu^2}\right) \left[t^2 - \frac{4}{Q^2}\sin^2\left(\frac{Qt}{2}\right)\right], \tag{3}$$

and where $\omega(r) = \left[r + 2 - \sqrt{r(r+4)}\right]^2 / \sqrt{r(r+4)}$, $\alpha$ is the fine-structure constant in the Thomson limit and $m_\mu$ is the muon mass. Because we consider only the LO-HVP contribution, for brevity we drop the superscript and multiply the result by $10^{10}$, that is, $a_\mu$ stands for $a_\mu^{\text{LO-HVP}} \times 10^{10}$ in the following.

# Control over Systematic Effects

Uses staggered quarks; improvements over similar calculations:

- new way to set the lattice scale
- consideration of physical quark-mass differences
- better inversion of fermion matrix using lower modes
- better control of finite-size effects (border effects)

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Thoughts:

*If the anomaly were to disappear — some in the particle physics community fear nothing less than “the end of particle physics”. The Fermilab $g$-2 experiment is our last hope of an experiment proving the existence of BSM physics*

*But also: Tension related to possible break-down of data-driven approach would itself suggest new physics*

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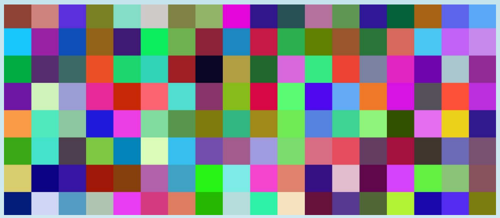

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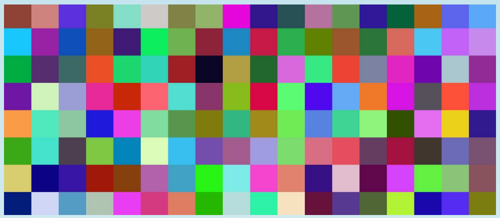

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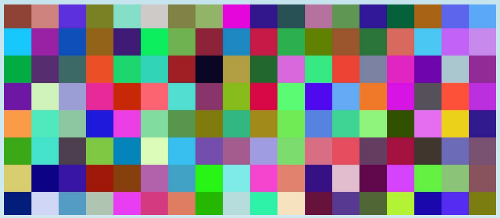

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First principles study of low-energy QCD properties (confinement, chiral-symmetry breaking, dynamical mass generation)

Importance for high-energy physics: Instrumental in precision tests of (strong-sector of) SM, in the search for new physics

# The QCD Lagrangian

$$\mathcal{L} \;=\; -\frac{1}{4}\, F^a_{\mu\nu}\, F_a^{\mu\nu} \;+\; \sum_{f=1}^{6} \bar{\psi}_{f,i}\left(i\,\gamma^\mu D^{ij}_\mu - m_f\,\delta_{ij}\right)\psi_{f,j}$$

$a = 1, \ldots, 8$; $i = 1, \ldots, 3$; $T^a_{ij}$ = $SU(3)$ generators

$$F^a_{\mu\nu} \;\equiv\; \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_0\, f_{abc}\, A^b_\mu\, A^c_\nu$$

$$D_\mu \;\equiv\; \partial_\mu - i\, g_0\, A^a_\mu\, T_a$$

Invariant under local gauge transformations $\Omega(x) = \exp\left[-ig_0\Lambda^a(x)T_a\right]$

$$A^\Omega_\mu(x) \;=\; \Omega(x)A_\mu(x)\Omega^{-1}(x) - \frac{i}{g_0}\left[\partial_\mu\Omega(x)\right]\Omega^{-1}(x)$$

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Like QED, but gauge symmetry is $SU(3)$ instead of $U(1)$

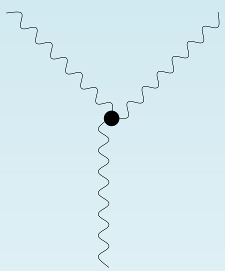
| | | |
|---|---|---|
| quarks (spin-1/2 fermions) | $\Leftrightarrow$ | electrons |
| gluons (vector bosons) / color charge | | photons / electric charge |

# Gluons Have Color

Note: contribution $F^a_{\mu\nu} \sim g_0\, f^{abc}\, A^b_\mu\, A^c_\nu$ means that in addition to quadratic terms (propagators) and the usual vertex

$$\mathcal{L}_{\bar\psi\psi A} \;=\; g_0\,\bar\psi\,\gamma^\mu A_\mu\,\psi$$ (quark-quark-gluon vertex)

Lagrangian contains terms with 3 and 4 gauge fields



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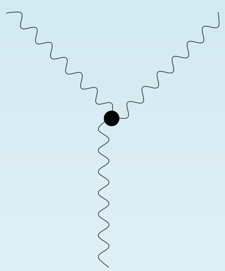

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$\Rightarrow$ nonlinear effects

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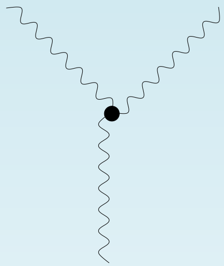

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$\Rightarrow$ Running coupling $\alpha_s(p)$ instead of $\alpha \approx 1/137$

# Confinement vs. Aymptotic Freedom

- At high energies: deep inelastic scattering of electrons reveals proton made of partons: pointlike and free. In this limit $\alpha_s(p) \ll 1$ (asymptotic freedom) and QCD is perturbative

$$\alpha_s(p) = \frac{4\pi}{\beta_0 \log\left(p^2/\Lambda^2\right)}\left[1 - \frac{2\beta_1}{\beta_0^2}\frac{\log\left(\log\left(p^2/\Lambda^2\right)\right)}{\log\left(p^2/\Lambda^2\right)} + \ldots\right]$$

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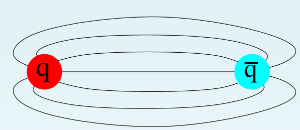
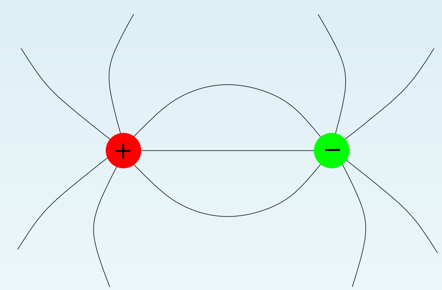
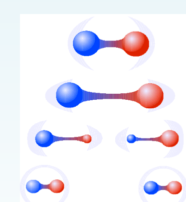
- At low energies: interaction gets stronger, $\alpha_s \approx 1$ and confinement occurs. Color field may form flux tubes



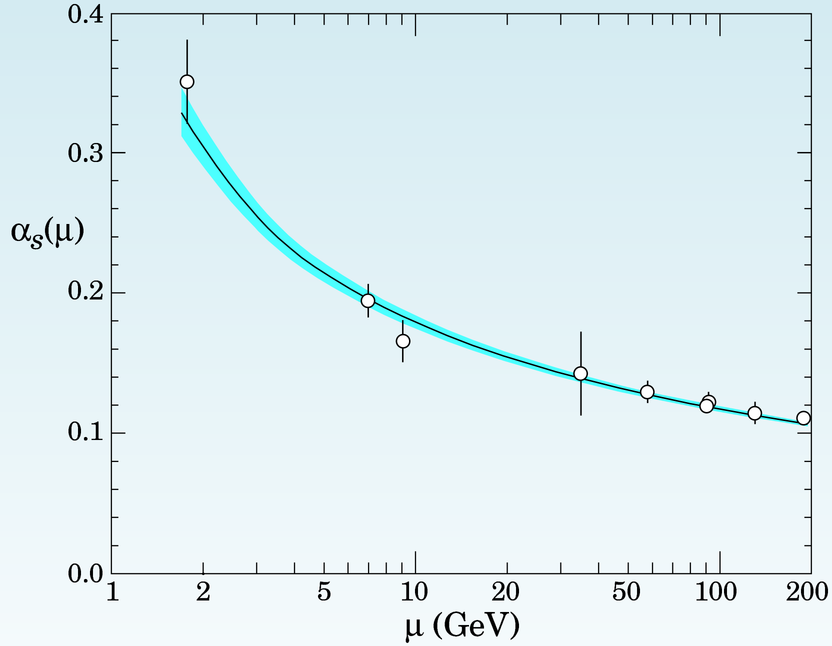

linear increase of inter-quark potential $\rightarrow$ string tension
At large distances $\rightarrow$ string breaks



# How do we perform calculations?

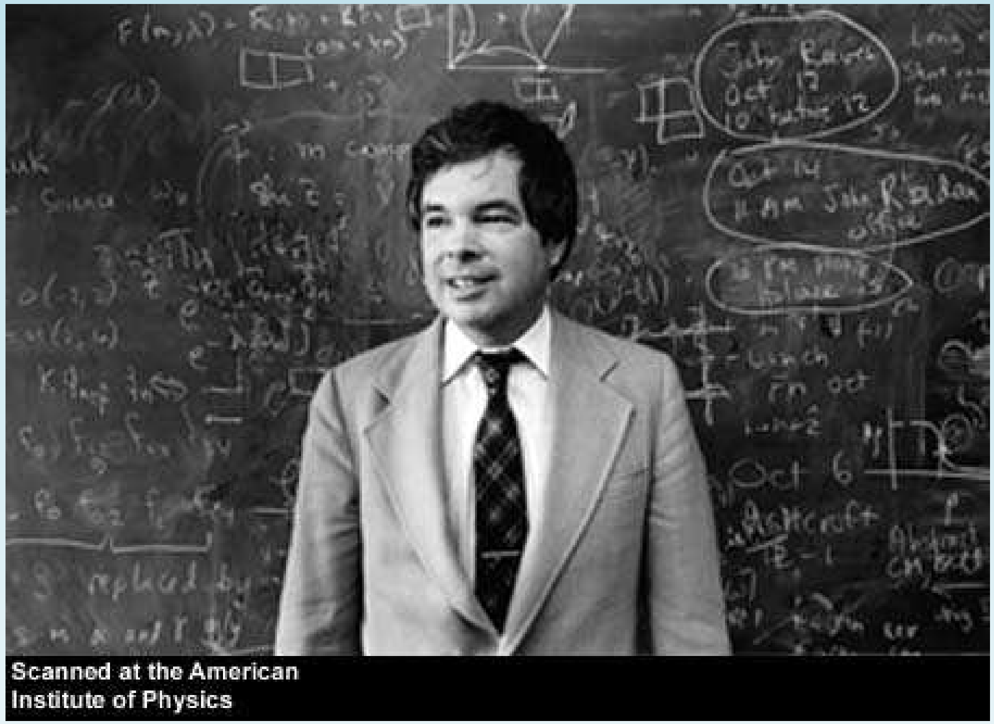
The strength of the interaction $\alpha_s$ increases for larger $r$ (smaller $p$) and vice-versa (asymptotic freedom).

Perturbation theory breaks down in the limit of small energies.



# QCD on a Lattice

Kenneth Geddes Wilson (June 8, 1936 – June 15, 2013)




Lattice used by Wilson in 1974 as a trick to prove confinement in (strong-coupling) QCD

[*Confinement of quarks*, Phys. Rev. D 10, 2445 (1974)]

# QCD on a Lattice (II)

As recalled by Wilson

> *[...] Unfortunately, I found myself lacking the detailed knowledge and skills required to conduct research using renormalized non-Abelian gauge theories. What was I to do, especially as I was eager to jump into this research with as little delay as possible? [...] from my previous work in statistical mechanics I knew a lot about working with lattice theories...*
> *[...] I decided I might find it easier to work with a lattice version of QCD. . .*

*The Origins of Lattice Gauge Theory*, hep-lat/0412043 (Lattice 2004)

# Context

Quark Model, from 1964, was only accepted in 1974, through so-called November Revolution, after discovery of $J/\Psi$ particle, which is a bound state of a charm quark $c$ and its antiquark (later: $b$, $t$)

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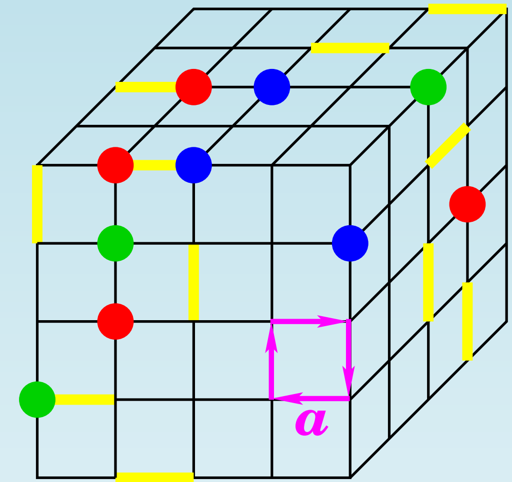
3) Proposal of a method (i.e. the lattice formulation) to study the problem of confinement in QCD (K. Wilson, 1974)

# Lattice QCD Ingredients

Three ingredients

1. Quantization by path integrals $\Rightarrow$ sum over configurations with "weights" $e^{i\,S/\hbar}$
2. Euclidean formulation (analytic continuation to imaginary time) $\Rightarrow$ weight becomes $e^{-S/\hbar}$
3. Discrete space-time $\Rightarrow$ UV cut at momenta $p \lesssim 1/a$ $\Rightarrow$ regularization



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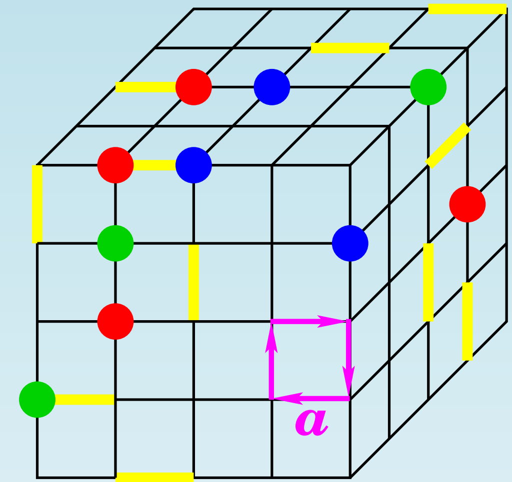


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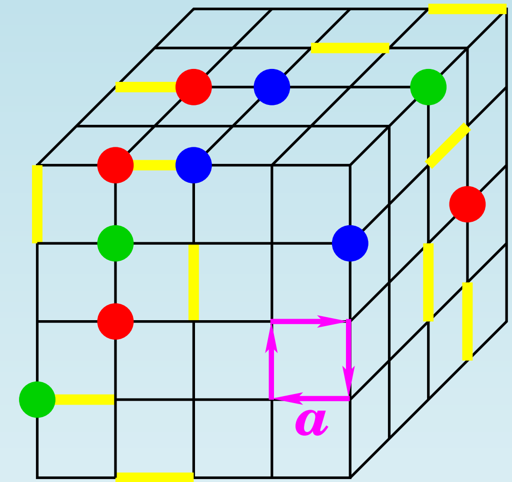


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The Wilson action

- is written for the gauge links $U_{x,\mu} \;\equiv\; e^{i g_0\, a\, A_\mu^b(x) T_b}$
- reduces to the usual action for $a \to 0$
- is gauge-invariant

# The Lattice Action

## The Wilson action (1974)

$$S \;=\; -\frac{\beta}{3}\sum_{\Box}\mathrm{Re\,Tr}\,U_{\Box}\,, \quad U_{x,\mu} \;\equiv\; e^{ig_0 a A^b_\mu(x) T_b}\,, \quad \beta \;=\; 6/{g_0}^2$$

- written in terms of oriented plaquettes formed by the link variables $U_{x,\mu}$, which are group elements
- under gauge transformations: $U_{x,\mu} \to g(x)\, U_{x,\mu}\, g^\dagger(x+\mu)$, where $g \in SU(3)$ $\Rightarrow$ closed loops are gauge-invariant quantities
- integration volume is finite: no need for gauge-fixing

At small $\beta$ (i.e. strong coupling) we can perform an expansion analogous to the high-temperature expansion in statistical mechanics. At lowest order, the only surviving terms are represented by diagrams with “double” or “partner” links, i.e. the same link should appear in both orientations, since $\int dU\, U_{x,\mu} = 0$

# Confinement and Area Law

Considering a rectangular loop with sides $R$ and $T$ (the Wilson loop) as our observable, the leading contribution to the observable's expectation value is obtained by "tiling" its inside with plaquettes, yielding the area law

$$\langle W(R,T)\rangle \;\sim\; \beta^{RT}$$

But this observable is related to the interquark potential for a static quark-antiquark pair

$$\langle W(R,T)\rangle \;=\; e^{-V(R)T}$$

We thus have $V(R) \sim \sigma R$, demonstrating confinement at strong coupling (small $\beta$)!

Problem: the physical limit is at large $\beta$...

# (Numerical) Lattice QCD

Classical Statistical-Mechanics model with the partition function

$$Z \;=\; \int \mathcal{D}U \, e^{-S_g} \int \mathcal{D}\psi \, \mathcal{D}\overline{\psi} \;\; e^{-\int d^4x \; \overline{\psi}(x) \, K \, \psi(x)} \;=\; \int \mathcal{D}U \, e^{-S_g} \;\det K(U)$$

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Evaluate expectation values

$$\langle \mathcal{O} \rangle \;=\; \int \mathcal{D}U\, \mathcal{O}(U)\, P(U) \;=\; \frac{1}{N} \sum_i \mathcal{O}(U_i)$$

with the weight

$$P(U) \;=\; \frac{e^{-S_g(U)}\;\det K(U)}{Z}$$

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Very complicated (high-dimensional) integral to compute!

$\Rightarrow$ Monte Carlo simulations: sample representative gauge configurations, then compute $O$ and take average

# Numerical Simulations

Monte Carlo methods (Ulam, 1940's): statistical description of the possible configurations of a system, which is simulated on a computer.

Useful in

- designing/analyzing experiments
- studying the theory of stochastic (statistical) systems
- doing calculations in quantum field theory

# Monte Carlo Simulations

Stochastic systems are simulated on the computer using a random number generator

⇒ theoretical approach, with experimental aspects:

- data, errors
- "measurements" in time

# Monte Carlo Method: Summary

Integral becomes sum of random variables

$$\int f(x)\, d\mu \,, \quad d\mu \;=\; \frac{e^{-\beta\mathcal{H}(x)}}{Z}\, dx \quad \Rightarrow \quad \frac{1}{N}\sum_{i=1}^{N} f(x_i)$$

where $x_i$ have statistical distribution $\mu$

- Static Monte Carlo: independent sampling (error $\sim 1/\sqrt{N}$)
- Dynamic Monte Carlo: Simulation of a Markov chain with equilibrium distribution $\mu$ (error $\sim \sqrt{\tau/N}$). Autocorrelation time $\tau$ related to critical slowing-down. Note: similar to experimental methods, but temporal dynamics was artificially introduced

Errors: either consider only effectively independent samples (via temporal correlation analysis) and error is given by standard deviation, jack-knife, bootstrap or consider all samples and error is estimated taking correlations into account: binning method, self-consistent windowing method

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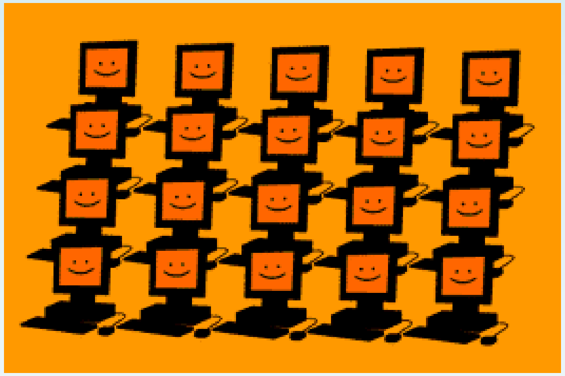
Monte Carlo methods

- pure gauge (quenched):
  Metropolis / Heat Bath $+$ Overrelaxation
- gauge $+$ dynamic quarks (full QCD):
  Hybrid Monte Carlo (HMC)



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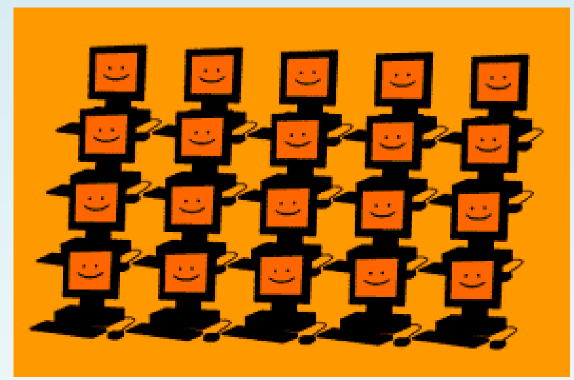

Note: $m = m_{\rm latt}/a$ ; as $a \to 0$ correlation length $\xi_{\rm latt} = 1/m_{\rm latt} \to \infty$

$\Rightarrow$ Continuum limit corresponds to critical point of the lattice theory

# Quark Bound States from Lattice QCD

The recipe for lattice simulations:

1) Evolve gluon fields (link variables) in the Monte Carlo dynamics associated with the partition function

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5) Translate results into physical units: $m = m_{\rm latt}/a$, take $\to 0$.

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⇒ “Valence” (considered for the inversion of $K$) and “sea” quarks (considered for the Monte Carlo dynamics) may not have the same masses ⇒ quenched/partially quenched approximations

⇒ Fermion operator $K$ depends on the choice of lattice formulation for the fermions. Most common choices are

- Wilson fermions: break chiral symmetry at finite $a$
- Staggered (Kogut-Susskind) fermions: good chiral properties, but produce 4 flavors of quarks; fewer-flavor case obtained by taking roots of $\det K$

nowadays: chiral symmetry (at zero quark mass) and locality are satisfied by so-called chiral fermions

# Context

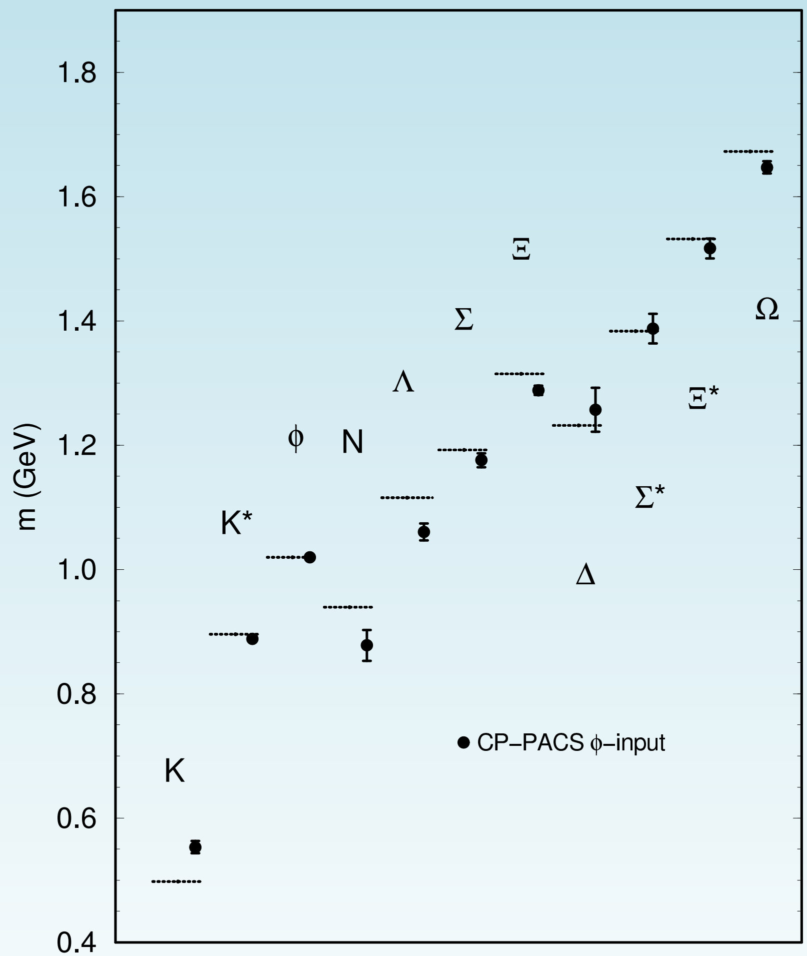
The approach had a “marvelous side effect”, as Michael Creutz calls it

> *By discreetly making the system discrete, it becomes sufficiently well defined to be placed on a computer. This was fairly straightforward, and came at the same time that computers were growing rapidly in power. Indeed, numerical simulations and computer capabilities have continued to grow together, making these efforts the mainstay of lattice gauge theory.*

*The Early days of lattice gauge theory*,
AIP Conf. Proc. 690, 52 (2003)

# Spectroscopy via Lattice QCD

Light hadron spectrum - 20th century computation (quenched)




CP-PACS Collaboration, Phys. Rev. Lett. (2000).

# Lattice QCD Then...

Won't you admit it's a trifle hysterical
To disbelieve *every* result that's numerical?

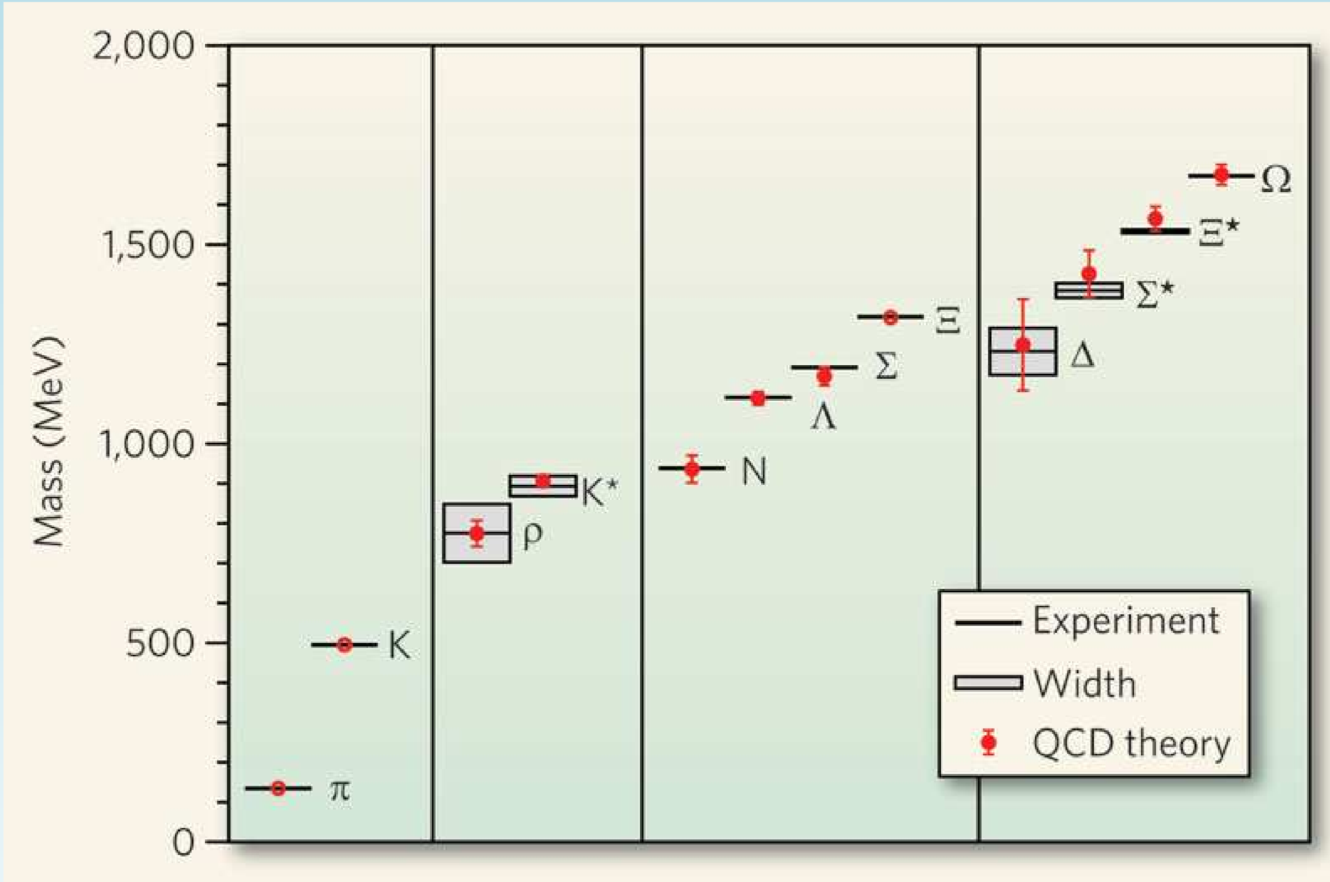
How, then, could you use modern aviation?
For the planes are designed by simulation.
And are experiments at accelerators all unsound,
Because they *simulate* the QCD background?

O why do you recoil in terror
From calculations that control their error?
Give it up! The symmetry's surely broken,
The order parameter (its token)
Refuses, by 20 $\sigma$, to go away.
What's that, *a coincidence*? No way!

No offense, but it's silly to avert your eyes
After $10^{18}$ floating point multiplies.

Frank Wilczek; *Physics Today*, March 1999

# ...and Now




Light hadron masses computed by S. Dürr et al. (Science, 2008) & experimental values. Note: $\pi$, $K$ and $\Xi$ used as inputs

Cited by F. Wilczek in Nature 456, 449 (November 2008)

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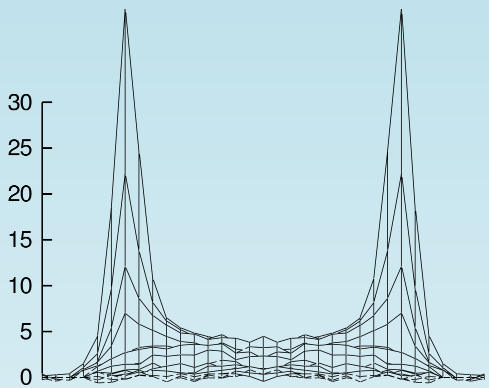
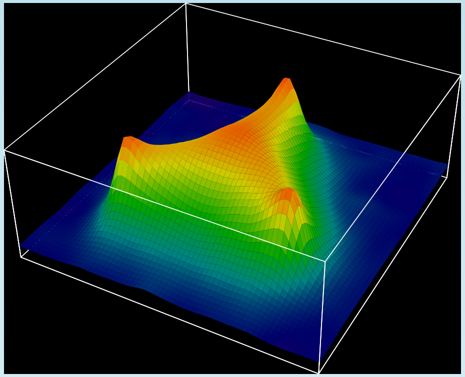
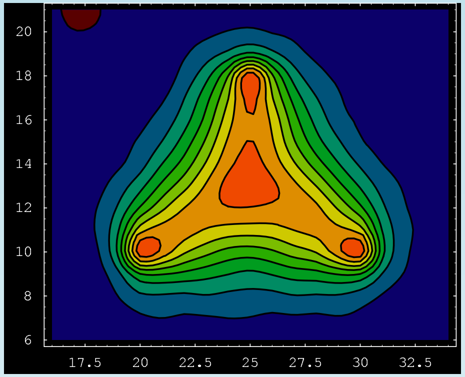
High-precision tests of the Standard Model are formidable technical and conceptual challenge; spectrum calculations provide confirmation of QCD as the theory of strong interactions $\Rightarrow$ first step towards understanding of fundamental QCD questions , e.g. confinement

# Confinement from Simulations

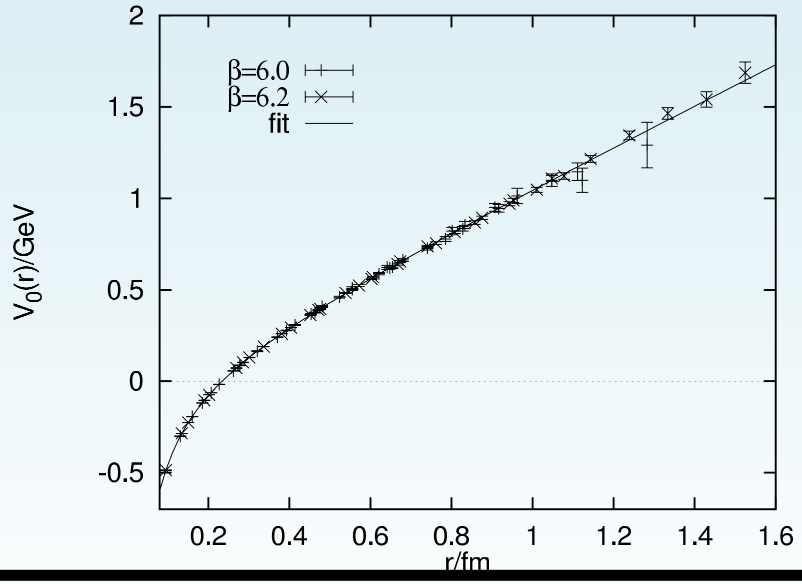
May observe formation of flux tubes

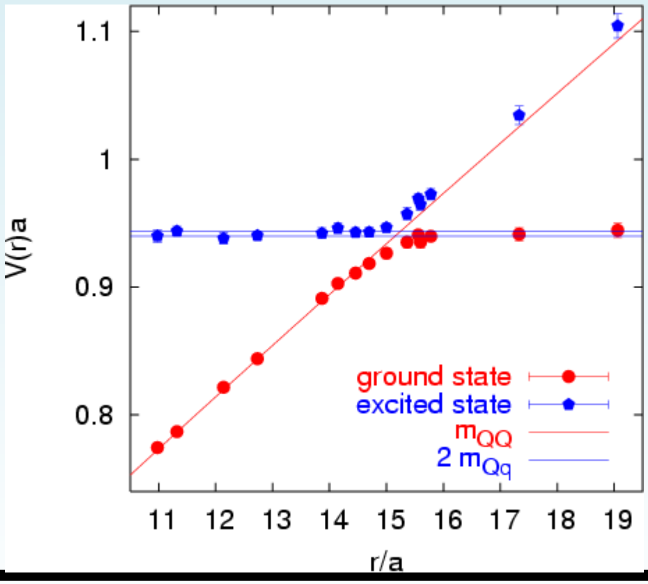





Linear Growth of potential between quarks, string breaking



# Confinement: the Elephant in the Room

Do we understand confinement?

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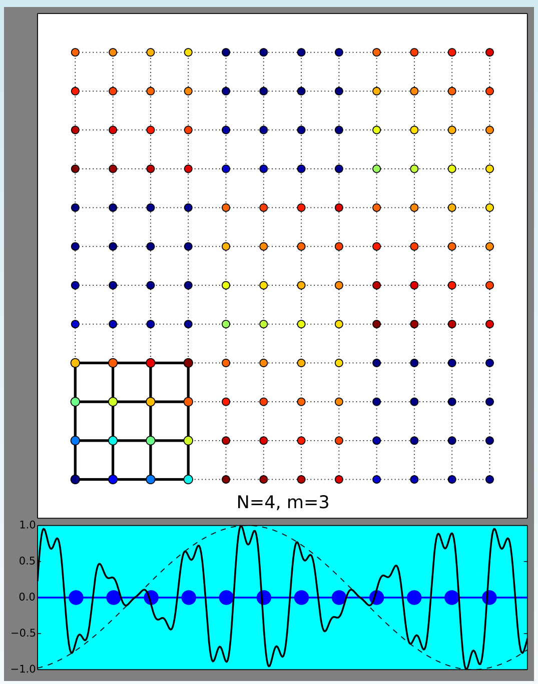
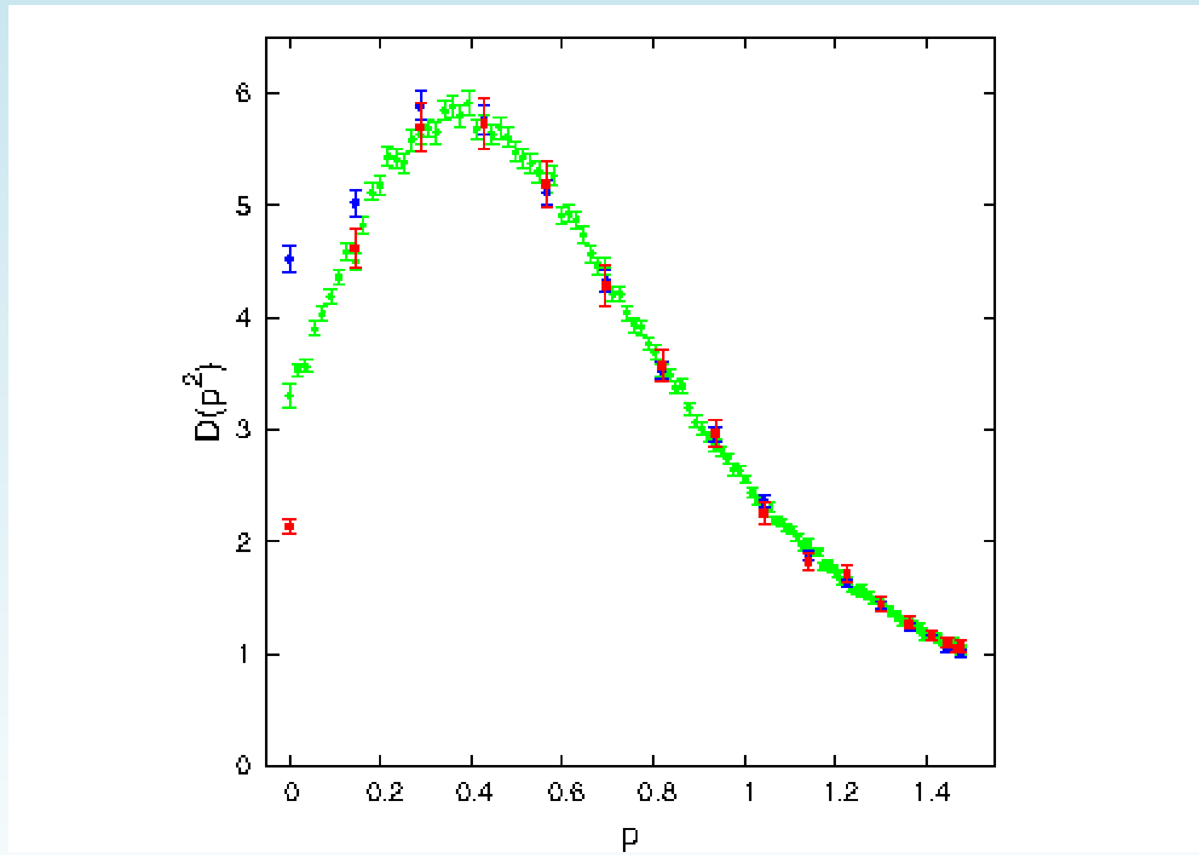
Millenium Prize Problems (Clay Mathematics Institute, USA/UK)

Yang-Mills and Mass Gap: Experiment and computer simulations suggest the existence of a mass gap in the solution to the quantum versions of the Yang-Mills equations. But no proof of this property is known.

# Large Lattices via Bloch's Theorem

Perform thermalization step on small lattice, then replicate it and use Bloch's theorem from condensed-matter physics to obtain gauge-fixing step for much larger lattice (A. Cucchieri, TM, PRL 2017)



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15 Oct 2019

**Exploring the tension between nature and the Standard Model: the muon g-2**

**M Krstić Marinković[1] and N Cardoso[2]**

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[2] CeFEMA, Departamento de Física, Instituto Superior Técnico (Universidade Teécnica de Lisboa), Av. Rovisco Pais, 1049-001 Lisboa, Portugal

E-mail: mmarina@maths.tcd.ie, nuno.cardoso@tecnico.ulisboa.pt

**Abstract.** Anomalous magnetic moment of the muon (muon g-2) is one of the most precisely measured quantities in particle physics. At the same time, it can be evaluated in the Standard Model with an unprecedented accuracy. The Muon g-2 experiment at Fermilab has started the