Finite density QCD with a histogram method

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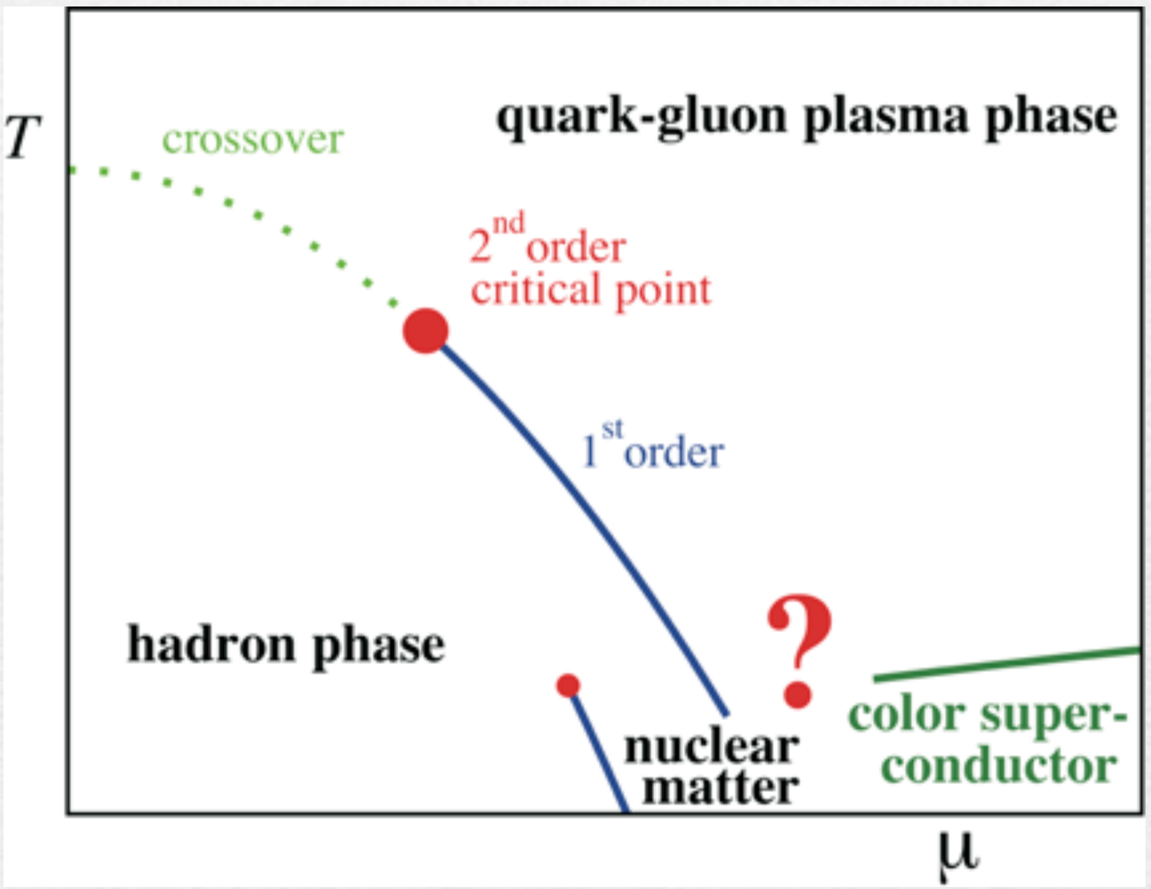
New Horizons in LFT
Natal
2013/3/25

Phase structure of QCD

QCD expected to have a rich phase structure

at finite T's and μ's.



Relevant to the early evolution of the Universe, origin of the matter, structure of neutron stars and supernovae, etc.

We want to know quantitatively

- properties of the matter in each phase
- location of transition lines / critical points / ...

They are now in reach of experiments at RHIC/LHC/...

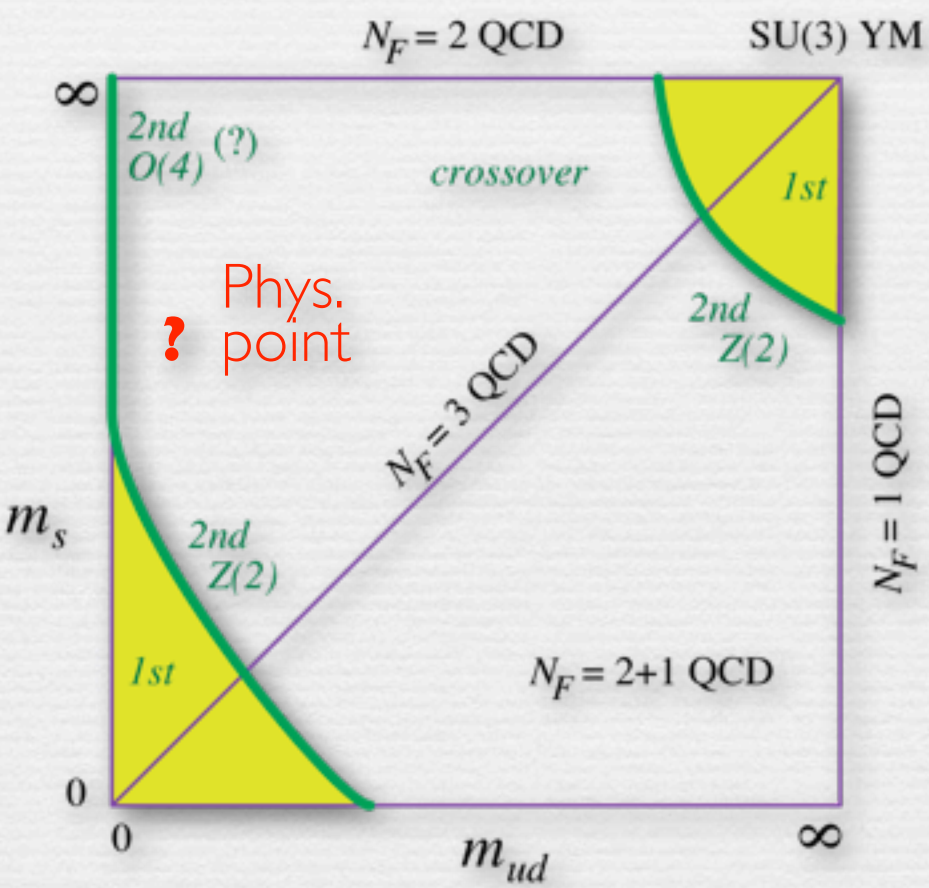
<= theoretical inputs directly from the 1st principles of QCD indispensable.

Phase structure of QCD

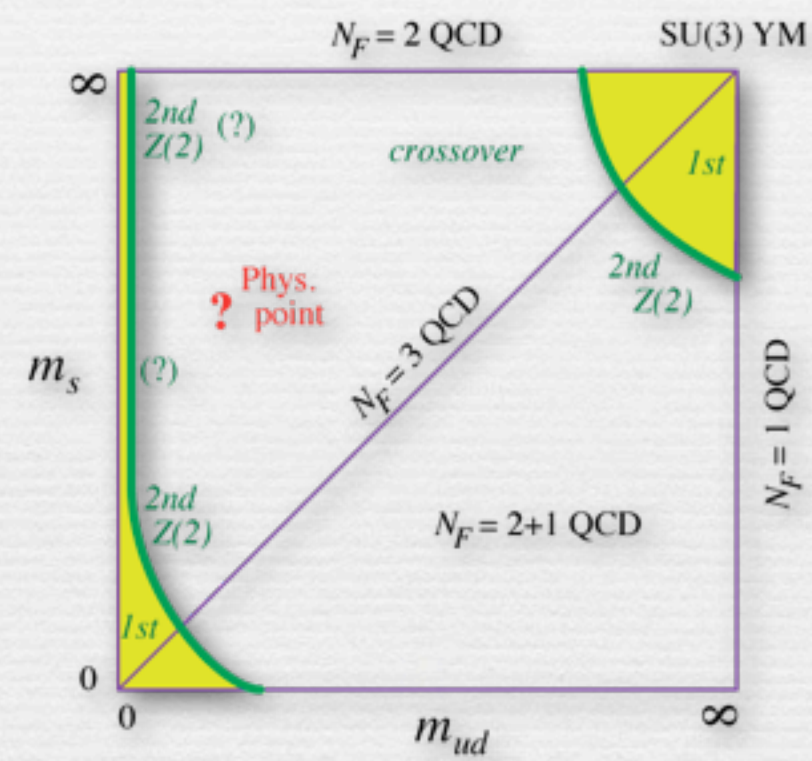
On the lattice, we are free to vary fundamental parameters.

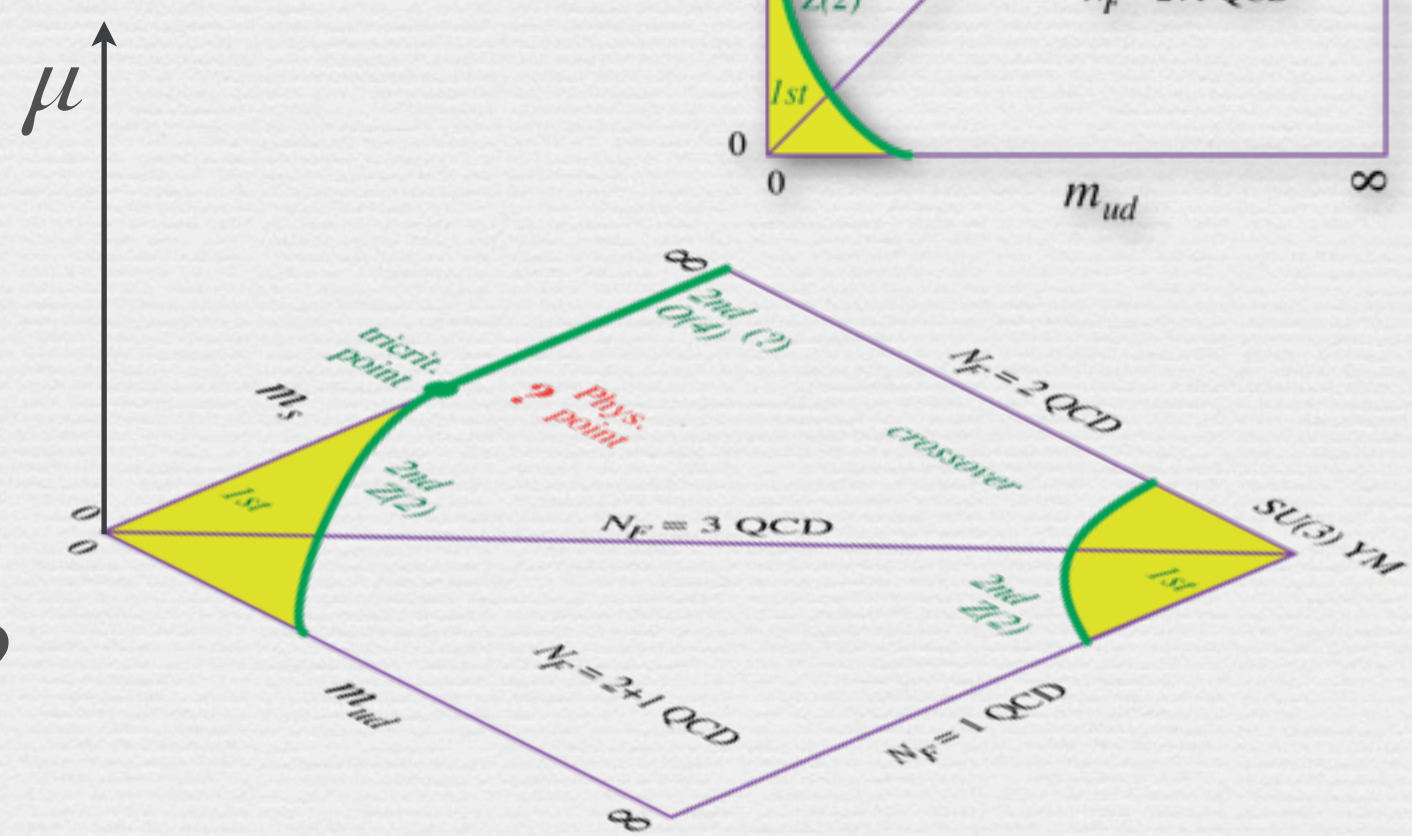
Sensitive dependence on quark masses.

"standard" scenario at μ=0 :



alternative scenario :





What happens at $\mu \neq 0$?

Lattice QCD at $\mu \neq 0$

- LQCD at $\mu \neq 0$ $\quad U_4 \longrightarrow \begin{cases} U_4 e^{\mu a} & \cdots \quad \text{positive } t \text{ direction} \\ U_4 e^{-\mu a} & \cdots \quad \text{nagative } t \text{ direction} \end{cases}$

=> $[\det M(\mu)]^* = \det M(-\mu^*) \neq \det M(\mu)$

=> MC based on importance sampling with det$M$ not justified

- Sign problem (complex phase problem)

phase-quenched simulation by det M –> |det M|, and handling the phase in the measurement

"reweighting" $$\langle \mathcal{O} \rangle = \frac{\int d\Phi\, \mathcal{O}\, e^{-S}}{\int d\Phi\, e^{-S}} = \frac{\int d\Phi\, \mathcal{O} e^{-\Delta S}\, e^{-S+\Delta S}}{\int d\Phi\, e^{-\Delta S}\, e^{-S+\Delta S}} = \frac{\langle \mathcal{O} e^{-\Delta S} \rangle_{S-\Delta S}}{\langle e^{-\Delta S} \rangle_{S-\Delta S}}$$

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int DU\, \mathcal{O}\, [\det M]^{N_F} e^{-S_g} = \frac{\langle \mathcal{O} e^{iN_F\theta} \rangle_{\text{p.q.}}}{\langle e^{iN_F\theta} \rangle_{\text{p.q.}}}, \qquad \det M = |\det M|\, e^{i\theta}$$

=> Exponentially high statistic required when θ fluctuates a lot (<= large μ).

Lattice QCD at $\mu \neq 0$

- Techniques for small μ/T
  - ✦ Taylor expansion around $\mu = 0$
  - ✦ multi-parameter reweighting
  - ✦ imaginary μ (analytic continuation to real μ)
  - ✦ canonical ensemble
  - ✦ complex Langevin
  - ✦ Lefschetz thimble etc. etc.
  - ✦ combination of them & other techniques to extend the range of applicability

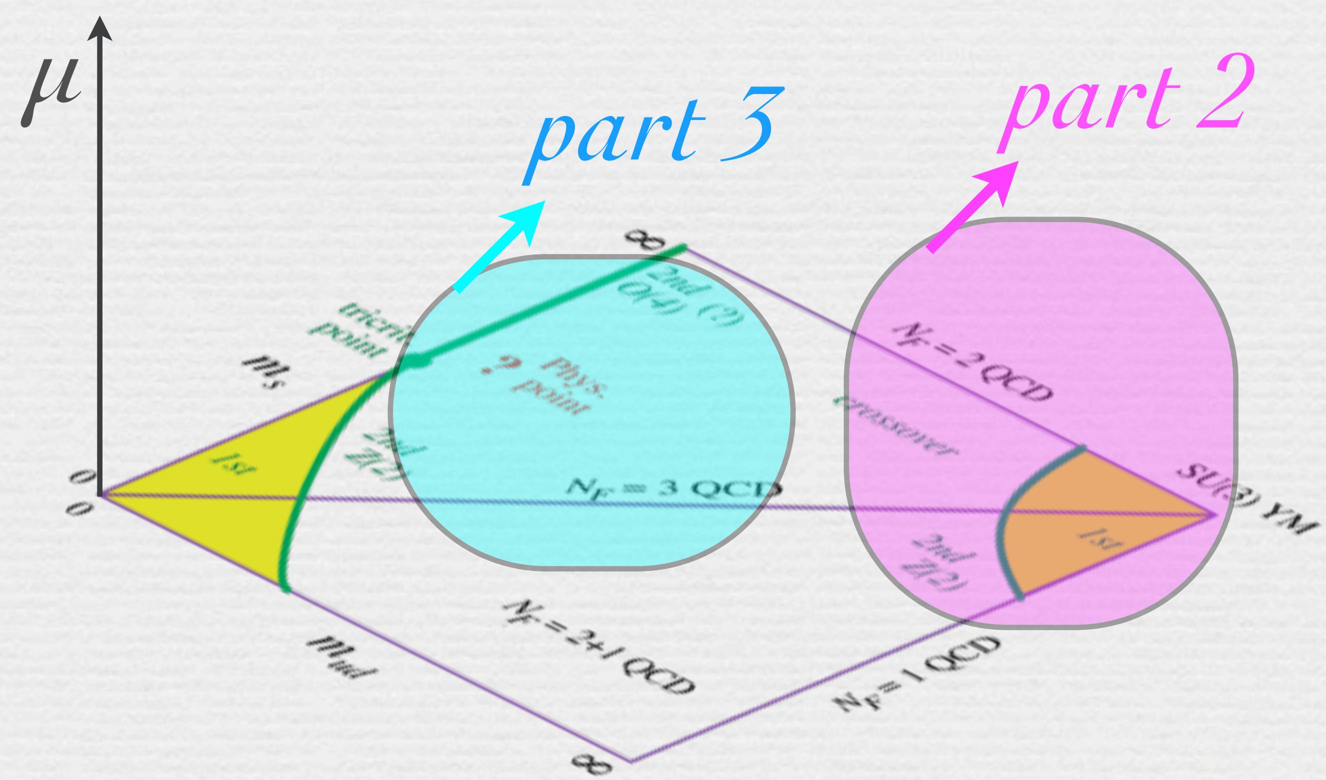
Histogram method: → *part 1*

spectral density

with a judicious choice of operator(s)

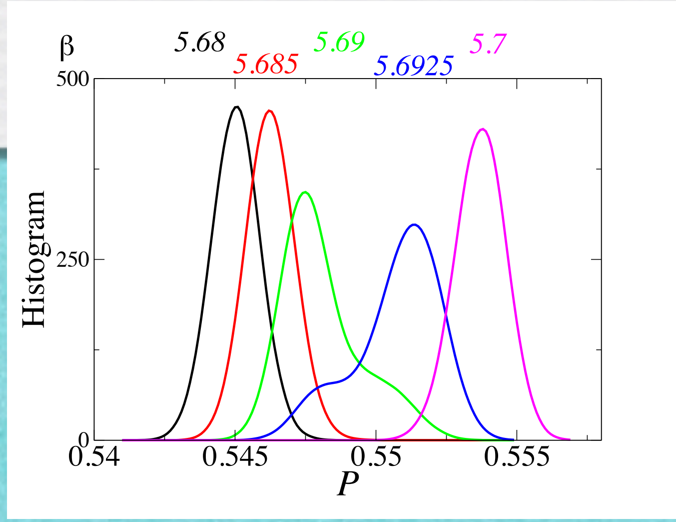
\+ reweighting

\+ cumulant expansion



part 1

Histogram method



Histogram method

Histogram of observables
("spectral density" ∝ probability distribution)

For simplicity, let us consider N_F-flavor QCD with the gauge action S_g.
We also assume that det$M$ is indep. of β (see discussions later).

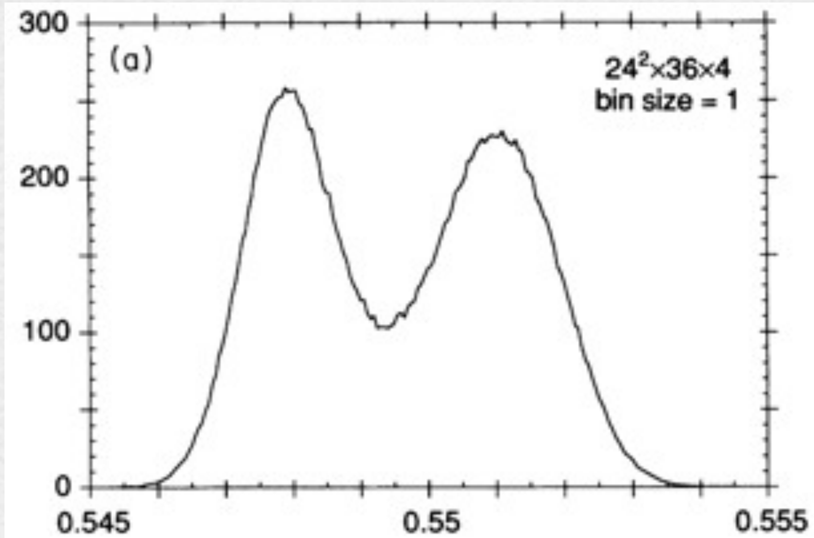
$$w(\mathcal{O}_1, \mathcal{O}_2, \cdots; \beta, m, \mu) \stackrel{\text{def.}}{=} \int DU \prod_i \delta(\hat{\mathcal{O}}_i[U] - \mathcal{O}_i) \, [\det M(m,\mu)]^{N_F} \, e^{-S_g(\beta)}$$

$$Z_{(\beta,m,\mu)} = \int w(\mathcal{O}_1, \cdots; \beta, m, \mu) \prod_i d\mathcal{O}_i$$

$$\left\langle f(\hat{\mathcal{O}}_1, \cdots) \right\rangle_{(\beta,m,\mu)} = \frac{1}{Z_{(\beta,m,\mu)}} \int f(\mathcal{O}_1, \cdots) \, w(\mathcal{O}_1, \cdots; \beta, m, \mu) \prod_i d\mathcal{O}_i$$

Choosing $\mathcal{O}$ which is sensitive to the phase
(e.g. order parameter, energy density, ...),

we can detect the phase transition
through the shape of w.



Plaquette in SU(3) YM at T>0.
Iwasaki et al., PR D46('92)4657

Histogram method

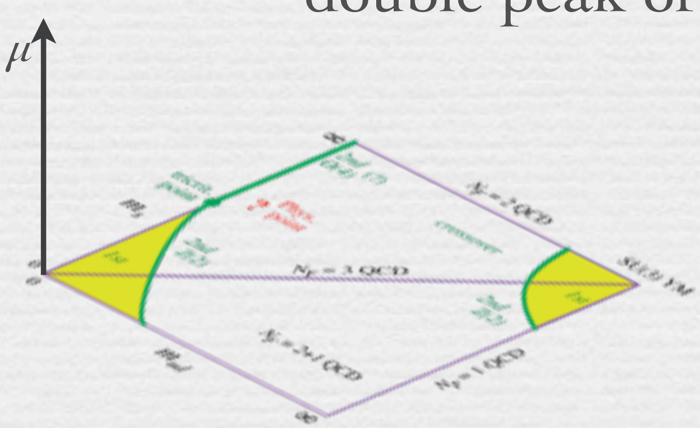
Effective potential

$$V_{\rm eff}(\hat{\mathcal{O}}_1,\cdots;\beta,m,\mu) \stackrel{\rm def.}{=} -\ln w(\mathcal{O}_1,\cdots;\beta,m,\mu)$$

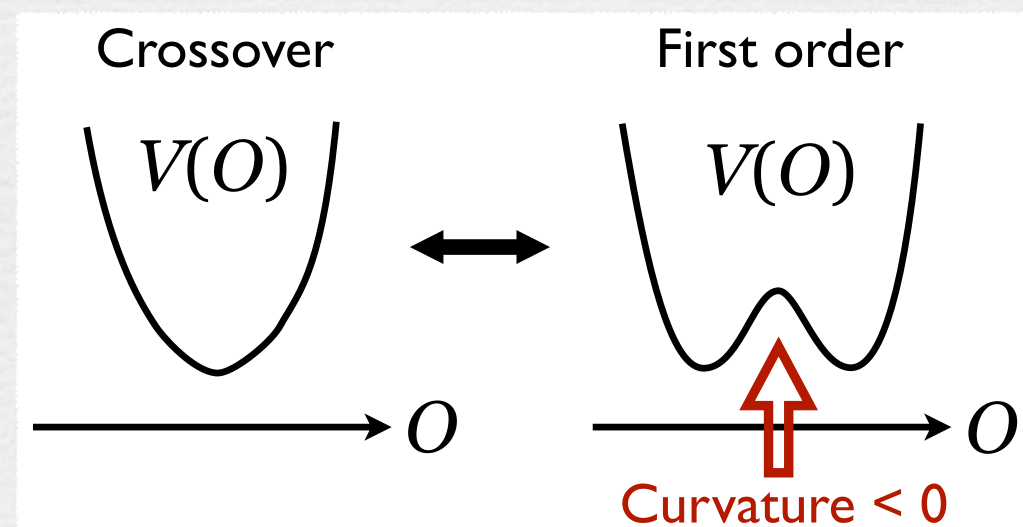
$$w(\mathcal{O}_1,\cdots;\beta,m,\mu) = e^{-V_{\rm eff}(\hat{\mathcal{O}}_1,\cdots;\beta,m,\mu)}$$

peak of w ~ min. of $V_{\rm eff}$

double peak of w ~ double well of $V_{\rm eff}$



Critical boundary of 1st order region can be detected by $V_{\rm eff}$.



To identify 1st order transition from $V_{\rm eff}$,
we need precise $V_{\rm eff}$ in a wide range of $\mathcal{O}$ covering both phases.

<= It is expensive to achieve by a single simulation: Very high statistics required due to small probability to flip.

Histogram method

To remedy it, we take

$\hat{\mathcal{O}}_1 = \hat{P}[U] \equiv -S_g/(6N_{\rm site}\beta)$:(generalized) plaquette

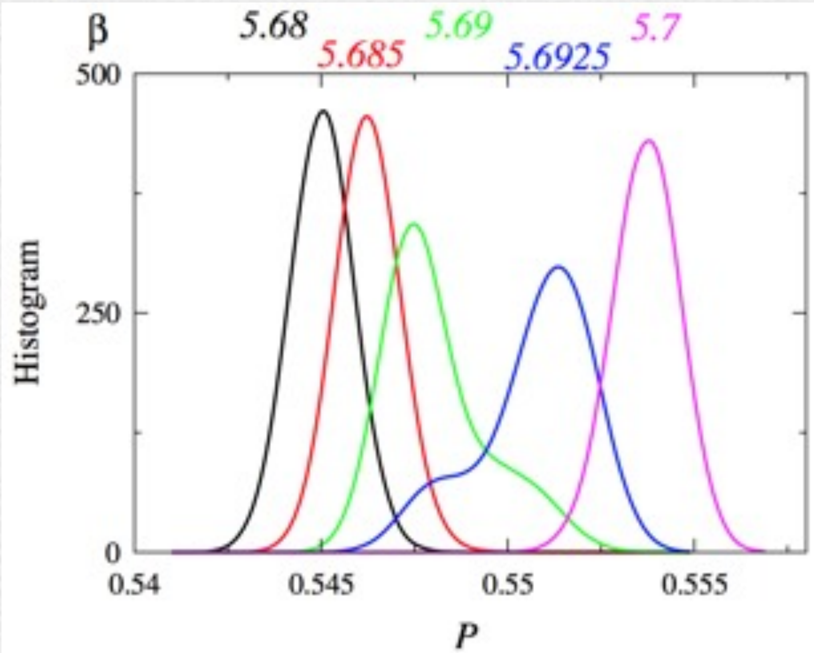
With P fixed, reweighting in β is simple:

$$w(P,\cdots;\beta,m,\mu) = \int DU\,\delta(\hat{P}[U]-P)\cdots[\det M(m,\mu)]^{N_F}\,e^{6\beta N_{\rm site}P}$$

$$w(P,\cdots;\beta',m,\mu) = w(P,\cdots;\beta,m,\mu)\,\boxed{e^{6(\beta'-\beta)N_{\rm site}P}}$$

Assumed: $\det M$ is indep. of β.

When c_{SW} has dependency on β, its effect should be taken into account. Alternatively, we may take a scheme in which c_{SW} is kept fixed. Final physics should be scheme-independent.



Each β has different support of $\mathcal{O}$'s.

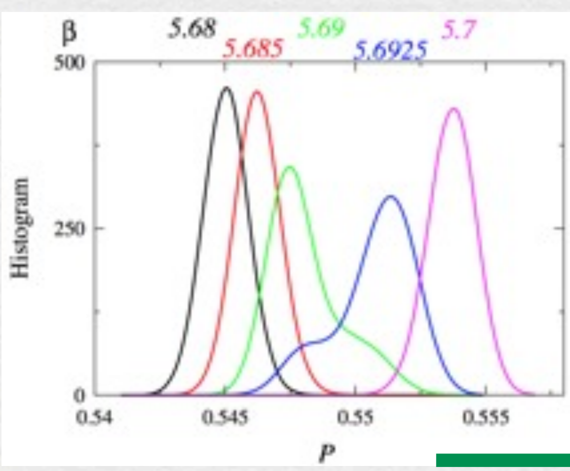
We can combine data at different β's to cover a wide range of $\mathcal{O}$'s.

H. Saito et al., PRD 84, 054502 ('11)

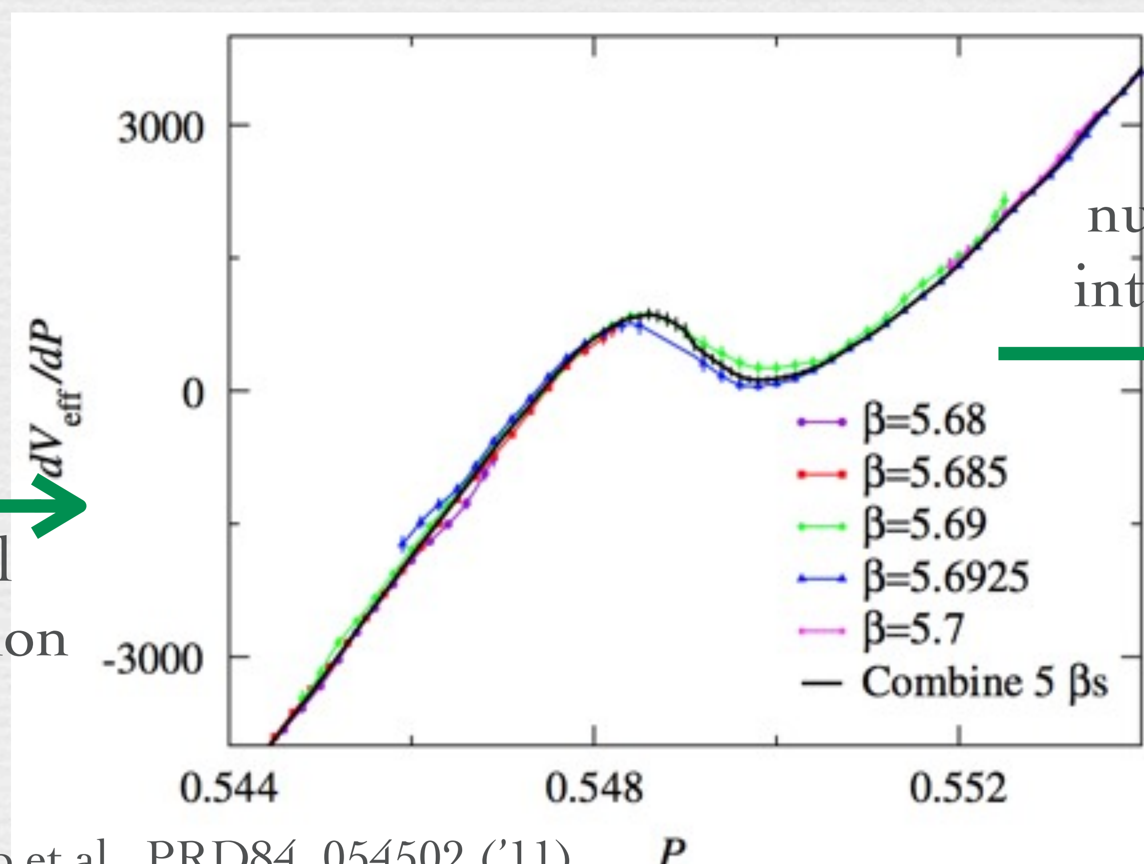
Histogram method

$$w(P,\cdots;\beta',m,\mu) = w(P,\cdots;\beta,m,\mu)\, e^{6(\beta'-\beta)N_{\rm site}P}$$

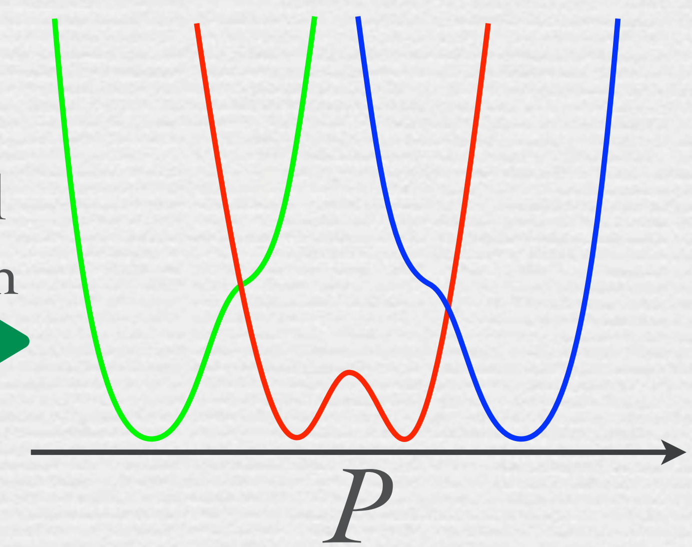
$$\begin{cases} V_{\rm eff}(P,\cdots;\beta',m,\mu) = V_{\rm eff}(P,\cdots;\beta,m,\mu) - 6(\beta'-\beta)N_{\rm site}P \\ \dfrac{\partial}{\partial P}V_{\rm eff}(P,\cdots;\beta',m,\mu) = \dfrac{\partial}{\partial P}V_{\rm eff}(P,\cdots;\beta,m,\mu) - \underbrace{6(\beta'-\beta)N_{\rm site}}_{\text{constant, known shift}} \\ \dfrac{\partial^2}{\partial P^2}V_{\rm eff}(P,\cdots;\beta',m,\mu) = \dfrac{\partial^2}{\partial P^2}V_{\rm eff}(P,\cdots;\beta,m,\mu) \end{cases}$$



numerical
differentiation



numerical
integration



We obtain $V_{\rm eff}$ covering a wide range of P.

quenched QCD: H. Saito et al., PRD84, 054502 ('11)

Histogram method

We also reweight in *m* and *μ* to further extend the ranges etc.

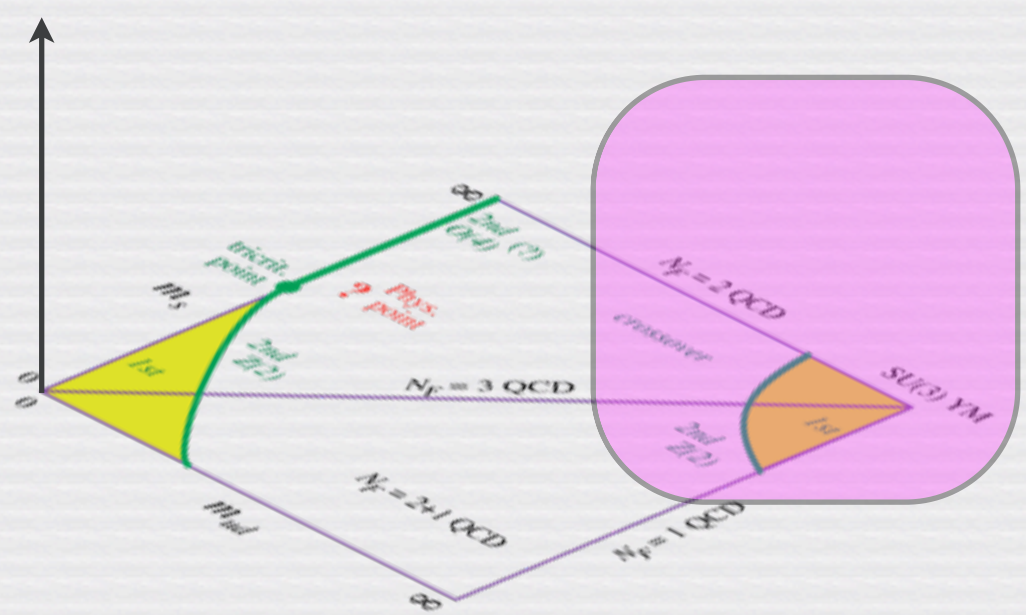
$$w(P;\beta,m,\mu) = \int DU\,\delta(\hat{P}[U]-P)\,[\det M(m,\mu)]^{N_F}\,e^{-S_g(\beta)}$$
$$= w(P;\beta_0,m_0,0)\times R(P)$$

$$R(P) = \frac{w(P;\beta,m,\mu)}{w(P;\beta_0,m_0,0)} = e^{6N_{\rm site}(\beta-\beta_0)\,P}\,\frac{\left\langle \delta(\hat{P}-P)\left[\frac{\det M(m,\mu)}{\det M(m_0,0)}\right]^{N_F}\right\rangle_{(\beta_0,m_0,0)}}{\left\langle \delta(\hat{P}-P)\right\rangle_{(\beta_0,m_0,0)}}$$
$$= e^{6N_{\rm site}(\beta-\beta_0)\,P}\,\underbrace{\left\langle \left[\frac{\det M(m,\mu)}{\det M(m_0,0)}\right]^{N_F}\right\rangle_{P;m_0}}$$

P-constrained average: indep. of β_0 !

Reweighting formula for $V_{\rm eff}$

$$V_{\rm eff}(P;\beta,m,\mu) = V_{\rm eff}(P,\beta_0,m_0,0) \underbrace{- 6N_{\rm site}(\beta-\beta_0)\,P - \ln\left\langle\left[\frac{\det M(m,\mu)}{\det M(m_0,0)}\right]^{N_F}\right\rangle_{P;m_0}}_{-\ln R}$$



Heavy quark QCD

Heavy quark QCD

To test the method, we first study the case of heavy quark QCD.

H. Saito et al., PRD 84, 054502 ('11); paper in preparation.

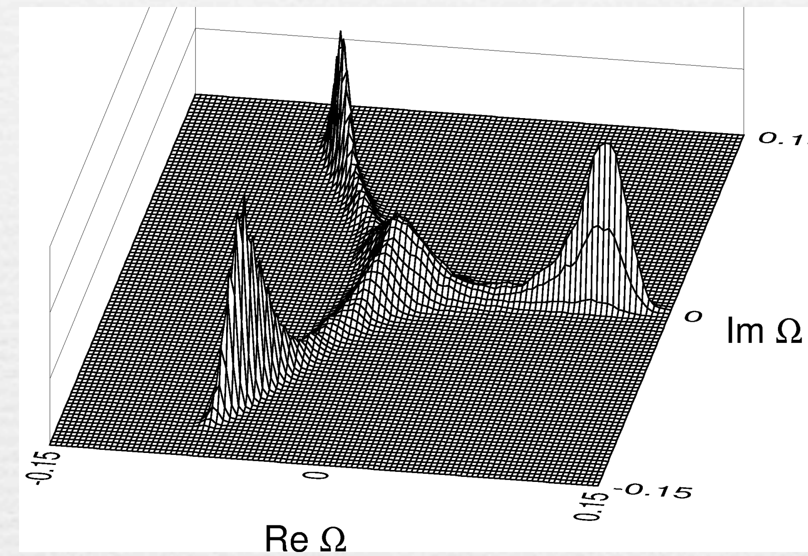
- Plaquette gauge action + Wilson quark action
- Simulation at $m_0 = \infty$ ($\kappa_0 = 0$): quenched QCD

  $24^3 \times 4$, β=5.68-5.70 (5 points around βc)

  1st order deconfining transition
- Reweight from $m_0 = \infty$

  The 1st order trans. expected to turn into crossover at some m.

  <= **hopping parameter expansion** $\kappa \sim 1/m_q a$



Polyakov loop histogram at $m_0 = \infty$

Iwasaki et al., PR D46('92)4657

$$\left[\frac{\det M(\kappa,\mu)}{\det M(0,0)}\right]^{N_F} = \exp\left[N_F\left\{288 N_{\rm site}\kappa^4 \hat{P} + 12\cdot 2^{N_t} N_s^3 \kappa^{N_t}\left(\cosh(\mu/T)\,\hat{\Omega}_{\rm R} + \underline{i\sinh(\mu/T)\,\hat{\Omega}_{\rm I}}\right) + \cdots\right\}\right]$$

$\Omega = \Omega_{\rm R} + i\Omega_{\rm I}$: Polyakov loop

complex phase at $\mu \neq 0$

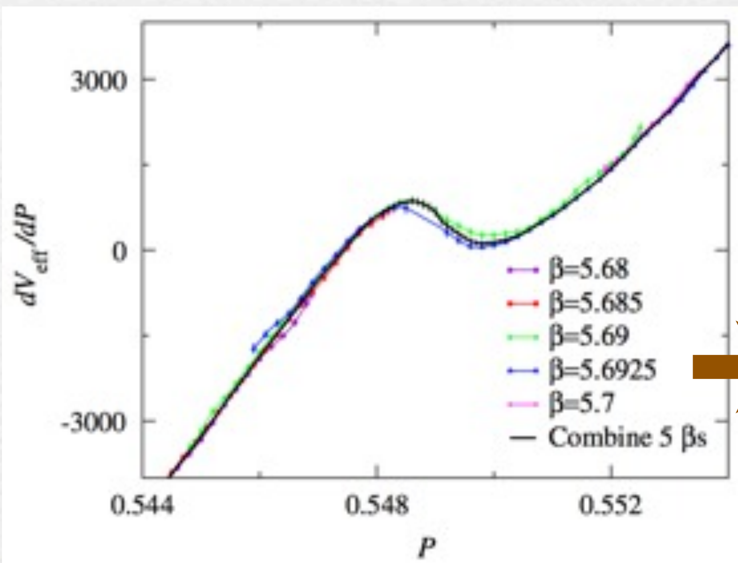
$\det M(0,0) = 1$

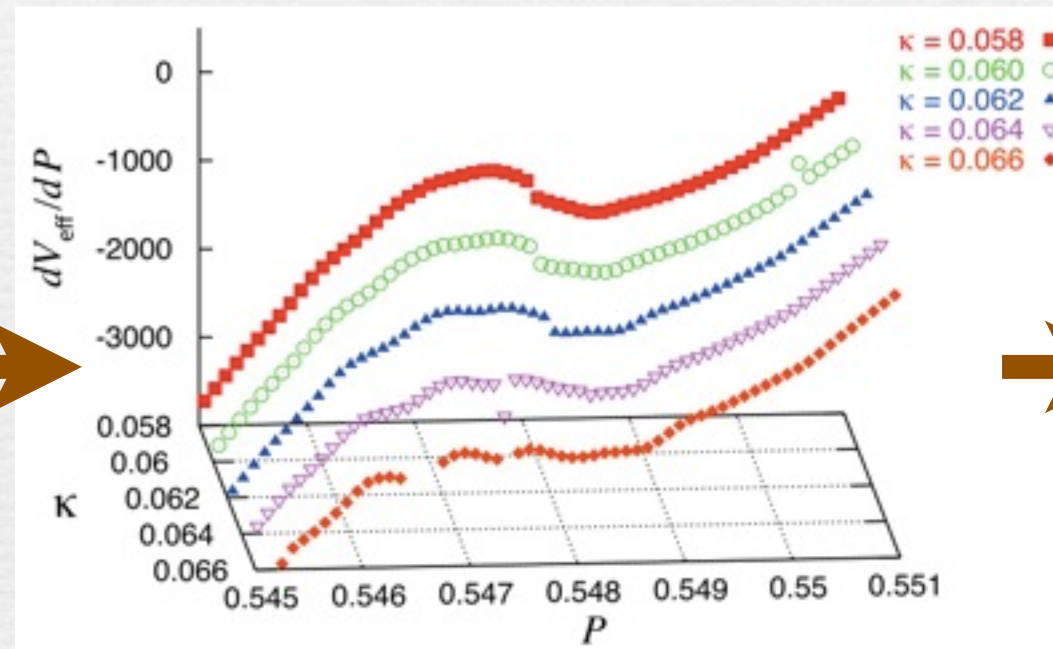
We study the fate of the 1st order transition in the lowest order at N_t=4.

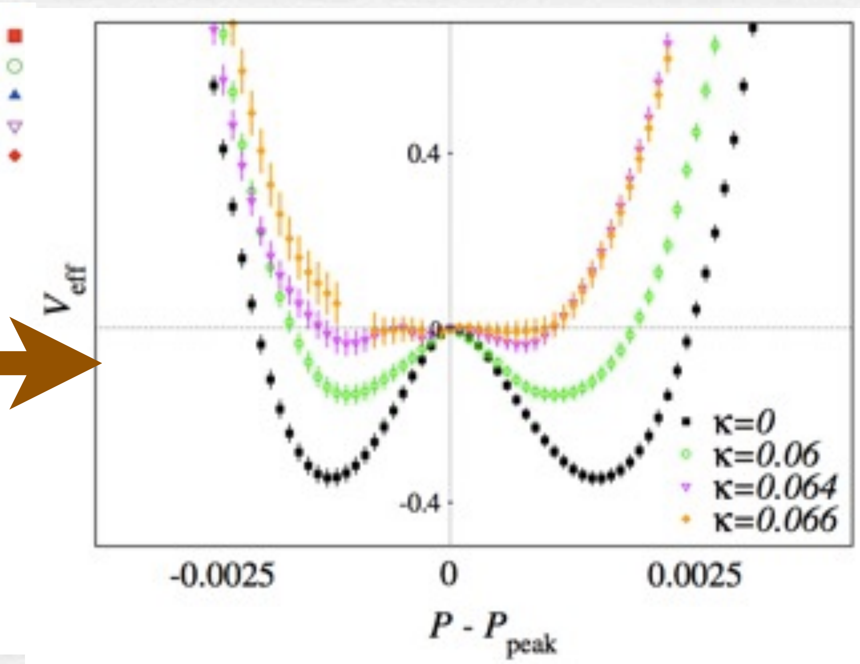
Heavy quark QCD at $\mu = 0$

H. Saito et al., PRD 84, 054502 ('11)

$$\left[\frac{\det M(\kappa,0)}{\det M(0,0)}\right]^{N_F} = \exp\left[N_F\left\{288 N_{\rm site}\kappa^4 \hat{P} + 12\cdot 2^{N_t} N_s^3 \kappa^{N_t}\,\hat{\Omega}_{\rm R} + \cdots\right\}\right]$$





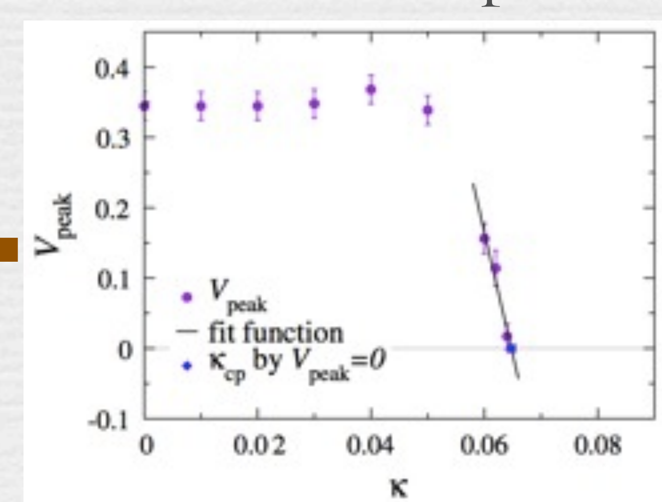


$\kappa_0 = 0$

With decreasing m_q from ∞, the 1st order deconf. transition weakens and turns into crossover.
Critical point can be identified by the disappearance of the double-well shape of $V_{\rm eff}$.

$$\left.\begin{aligned} \kappa_{\rm cp} &= 0.0658(03)\left({}^{+04}{}_{-10}\right) \\ T_{\rm c}/m_\pi{}^{\rm cp} &\approx 0.023 \end{aligned}\right\}\ \text{for } N_F = 2$$

i.e., $m_\pi{}^{\rm cp}$ ~ 7–9 GeV using Tc ~ 160–200 MeV

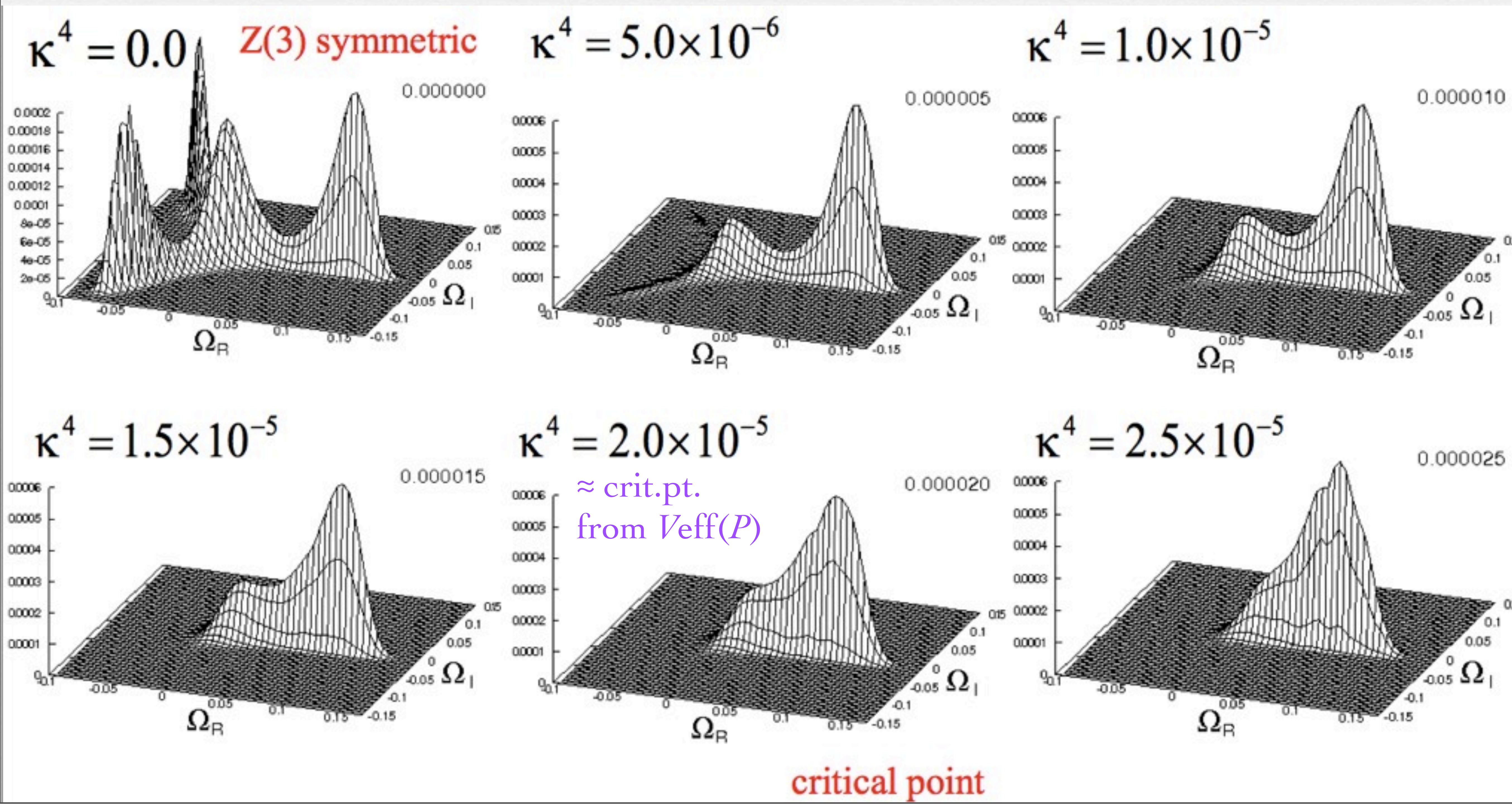


$\kappa_{\rm cp}$ from double-well height

etc.

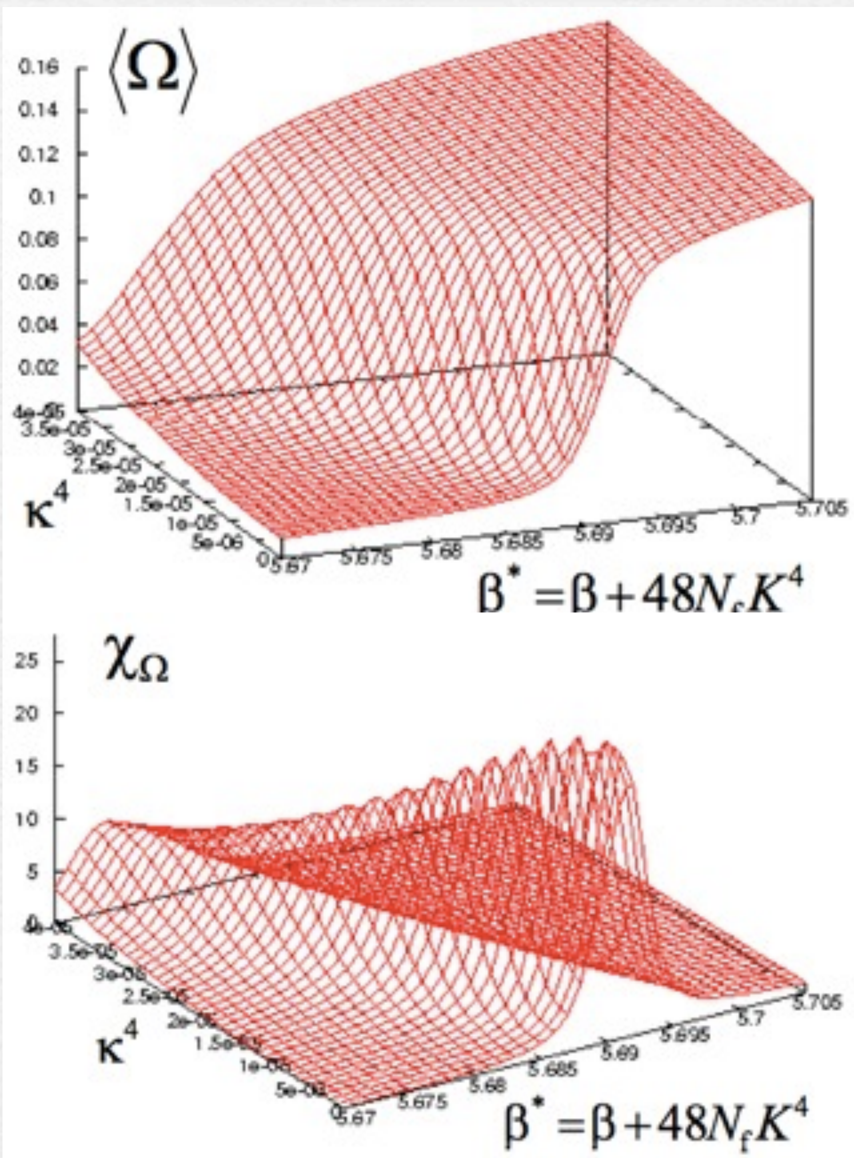
Heavy quark QCD at $\mu = 0$

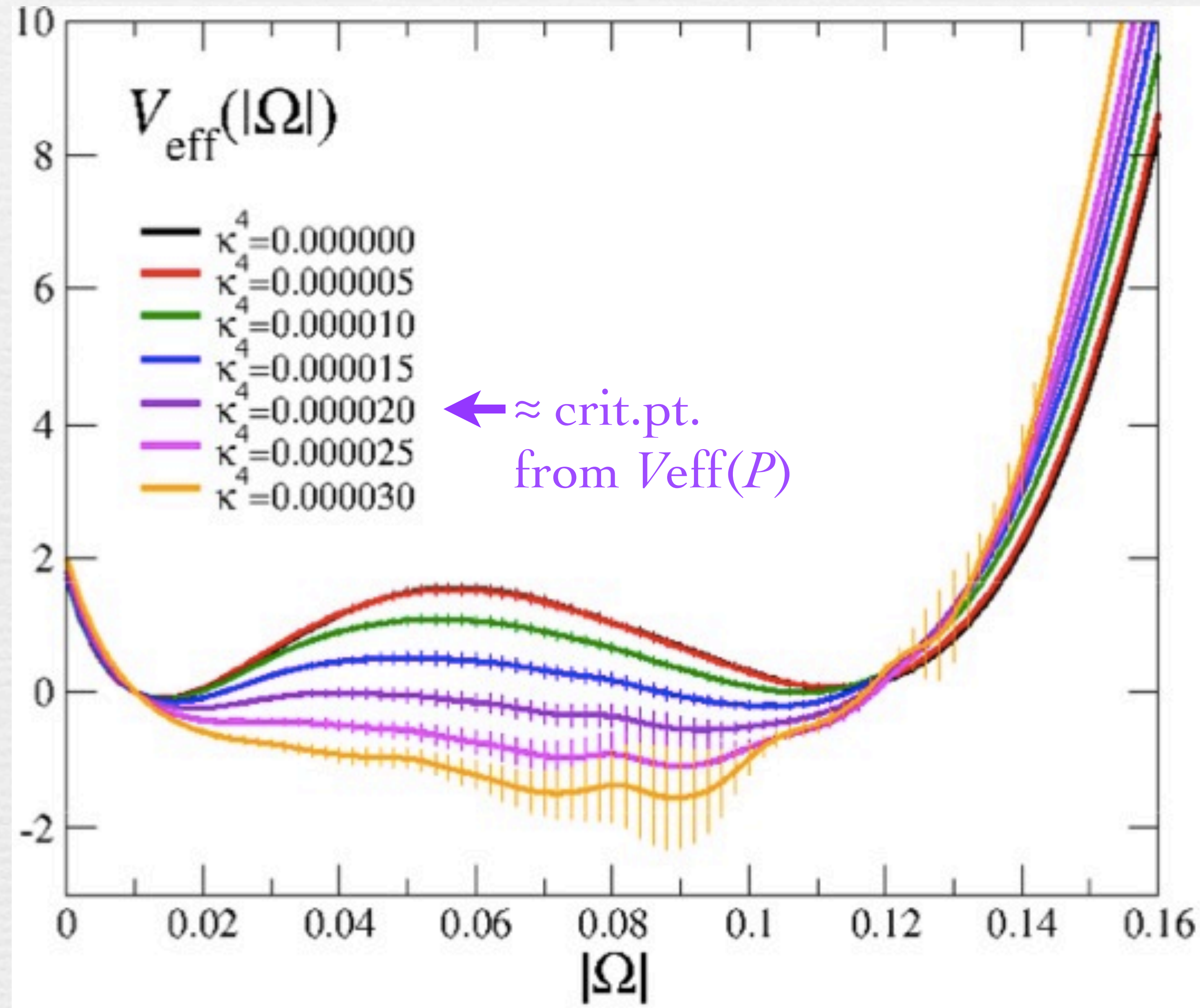
Polyakov loop (by multi-point reweighting using 5 ß's and adjusted to $ß_c(\kappa)$ from χ_Ω)



Heavy quark QCD at μ = 0

Polyakov loop





=> 1st order transition line

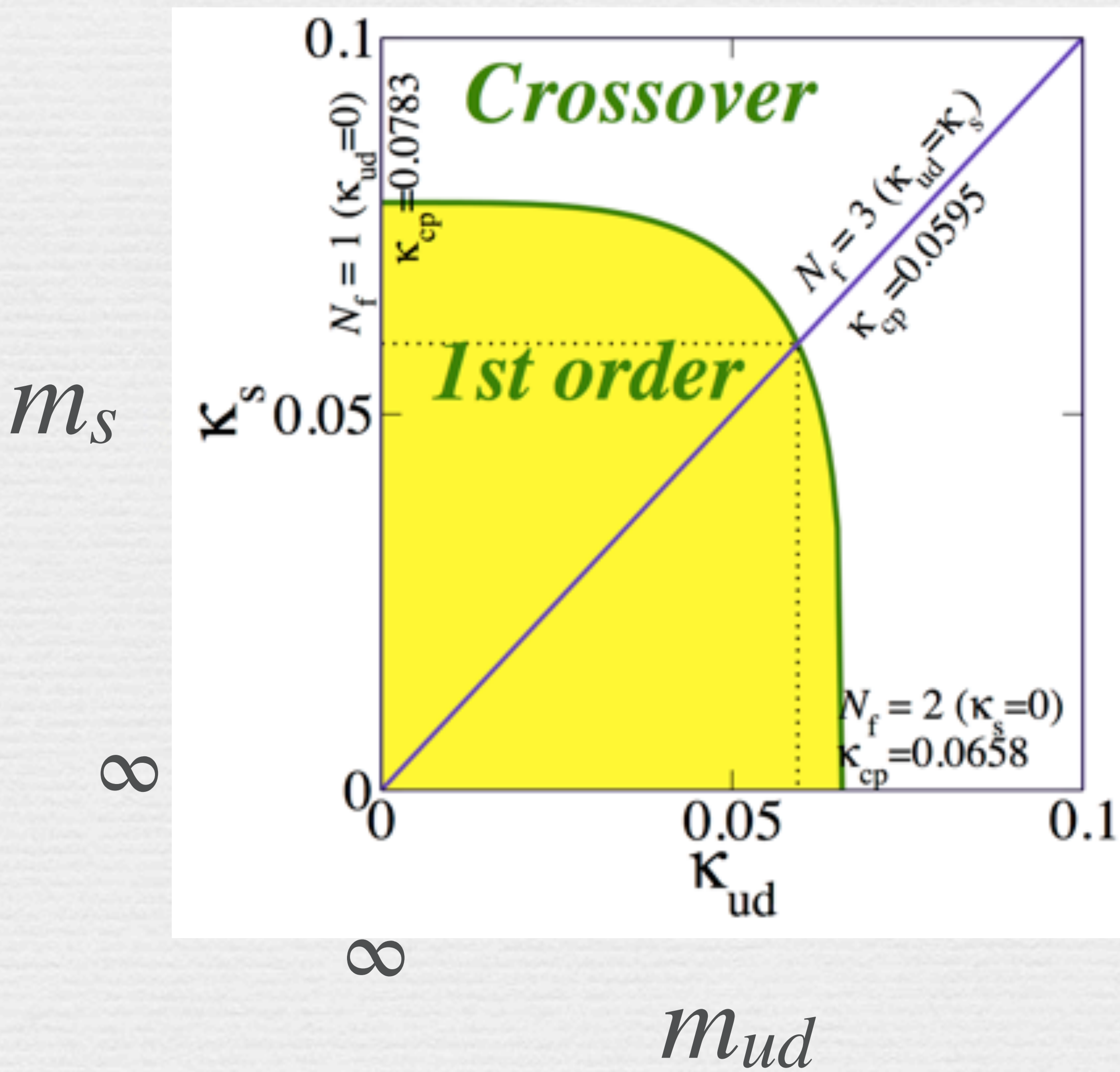
cf. κ_{cp} = 0.0658(03)($^{+04}{}_{-10}$), $\kappa_{cp}{}^4 \approx 1.9\text{x}10^{-5}$ from $V_{\text{eff}}(P)$

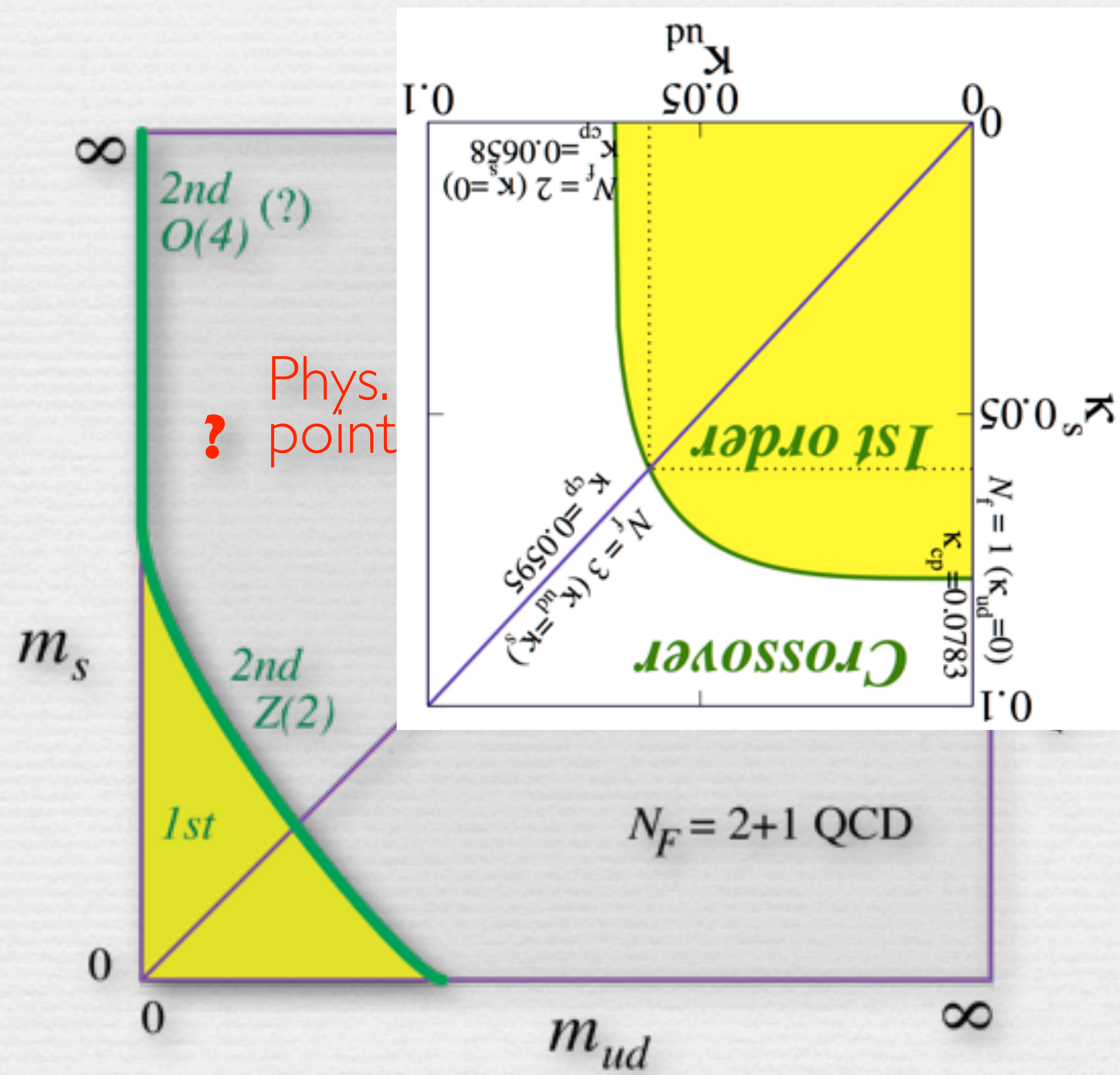
Heavy quark QCD at $\mu = 0$

For N_F = 2+1 $\left[\dfrac{\det M(\kappa,0)}{\det M(0,0)}\right]^{N_F} = \exp\left[N_F\left\{288 N_{\rm site}\kappa^4 \hat{P} + 12\cdot 2^{N_t} N_s^3 \kappa^{N_t}\, \hat{\Omega}_{\rm R} + \cdots\right\}\right]$

We just replace $N_F \kappa^{N_t} \implies 2\kappa_{ud}^{N_t} + \kappa_s^{N_t}$, besides the shift of β to β^*.

$$2(\kappa_{ud})^{N_t} + (\kappa_s)^{N_t} = 2(\kappa_{\rm cp}^{N_F=2})^{N_t}$$





Heavy quark QCD at $\mu \neq 0$

H. Saito et al., PoS Lattice2011; WHOT-QCD, in preparation.

$$V_{\rm eff}(\mathcal{O};\beta,\kappa,\mu) = V_{\rm eff}(\mathcal{O};\beta,0,0) - \ln\left\langle \left[\frac{\det M(\kappa,\mu)}{\det M(0,0)}\right]^{N_F}\right\rangle_{\mathcal{O};\beta,0,0}$$

$$\left[\frac{\det M(\kappa,\mu)}{\det M(0,0)}\right]^{N_F} = \exp\left[N_F\left\{\underline{288 N_{\rm site}\kappa^4 \hat{P}} + 12\cdot 2^{N_t} N_s^3 \kappa^{N_t}\left(\cosh(\mu/T)\,\hat{\Omega}_{\rm R} + \underline{i\sinh(\mu/T)\,\hat{\Omega}_{\rm I}}\right) + \cdots\right\}\right]$$

complex phase at μ≠0

$$\beta^* = \beta + 48 N_F \kappa^4$$

Choose $\Omega_{\rm R}$ as an $\mathcal{O}$:

$$V_{\rm eff}(\Omega_{\rm R};\beta,\kappa,\mu) = \boxed{V_{\rm eff}(\Omega_{\rm R};\beta^*,0,0) - 12\cdot 2^{N_t} N_F N_s^3 \underline{\kappa^{N_t}\cosh(\mu/T)}\,\Omega_{\rm R}} - \boxed{\ln\left\langle e^{i\hat{\theta}}\right\rangle_{\Omega_{\rm R};\beta,0,0}}$$

Phase-quenched QCD

$$\hat{\theta} = 12\cdot 2^{N_t} N_F N_s^3 \kappa^{N_t}\sinh(\mu/T)\,\hat{\Omega}_{\rm I}$$

$$V_{\rm eff}^{\rm phase\,quenched}(\Omega_{\rm R};\beta,\kappa,\mu) = V_{\rm eff}\left(\Omega_{\rm R};\beta,\underline{\kappa[\cosh(\mu/T)]^{1/N_t}},0\right)$$

The phase quenched part is just $V_{\rm eff}$ at μ=0 with shifting $\kappa \rightarrow \kappa\,[\cosh(\mu/T)]^{1/N_t}$

=> Critical line in the phase-quenched QCD is just given by

$$\kappa_{\rm cp}^{\rm p.q.}(\mu) \;=\; \kappa_{\rm cp}(\mu{=}0)\cdot[\cosh(\mu/T)]^{-1/N_t}$$

Heavy quark QCD at $\mu \neq 0$

$\langle e^{i\hat{\theta}} \rangle$ may cause the sign problem when θ fluctuates largely <= large μ

Cumulant expansion S. Ejiri, PRD *77*, 014508 ('08); WHOT, PRD 82, 014508 ('10)

$$\langle e^{i\hat{\theta}} \rangle = \exp\left[i\langle\hat{\theta}\rangle_c - \frac{1}{2!}\langle\hat{\theta}^2\rangle_c - \frac{i}{3!}\langle\hat{\theta}^3\rangle_c + \frac{1}{4!}\langle\hat{\theta}^4\rangle_c + \cdots \right]$$

$$\langle\hat{\theta}\rangle_c = \langle\hat{\theta}\rangle, \quad \langle\hat{\theta}^2\rangle_c = \langle\hat{\theta}^2\rangle - \langle\hat{\theta}\rangle^2, \quad \langle\hat{\theta}^3\rangle_c = \langle\hat{\theta}^3\rangle - 3\langle\hat{\theta}^2\rangle\langle\hat{\theta}\rangle + 2\langle\hat{\theta}\rangle^3, \ \cdots$$

Remarks:

- Odd terms are the origin of the complex phase.
- Odd terms vanish due to the symmetry under $\mu \leftrightarrow -\mu$.

=>

- W/o odd terms, $\langle e^{i\hat{\theta}} \rangle$ is positive definite!
- Sign problem resolved *if* the expansion converges.
  The sign problem is transformed into a convergence problem of the cumulant expansion.

Heavy quark QCD at $\mu \neq 0$

$$\langle e^{i\hat{\theta}} \rangle = \exp\left[-\frac{1}{2!}\langle \hat{\theta}^2 \rangle_c + \frac{1}{4!}\langle \hat{\theta}^4 \rangle_c + \cdots \right]$$

$$\hat{\theta} = 12 \cdot 2^{N_t} N_F N_s^3 \kappa^{N_t} \sinh(\mu/T)\, \hat{\Omega}_{\rm I}$$

Convergence of the cumulant expansion

When a cumulant expansion converges?

The most convergent case: $\langle e^{i\hat{\theta}} \rangle = \exp\left[-\frac{1}{2!}\langle \hat{\theta}^2 \rangle_c \right]$ i.e., Gaussian distribution

We note that $\hat{\theta} = \sum_{\mathbf{x}} \hat{\theta}_{\mathbf{x}}$ => distribution of $\theta \approx$ Gaussian for $V >> v$ (correlated range).

Therefore, if the correlation length is finite as in the case of massive QCD we do expect Gaussian distribution of θ on large lattices.

A delicate point is the V-dependence, because θ is $O(V)$, θ^n may diverge as $O(V^n)$.

However, because $\langle e^{i\hat{\theta}} \rangle = \langle \prod_{\mathbf{x}} e^{i\hat{\theta}_{\mathbf{x}}} \rangle \approx \langle \prod_{\mathbf{x}\in v} e^{i\hat{\theta}_{\mathbf{x}}} \rangle^{V/v}$, we find $\langle \hat{\theta}^n \rangle_c = O(V)$ for $\forall n$

i.e., convergence is independent of V !

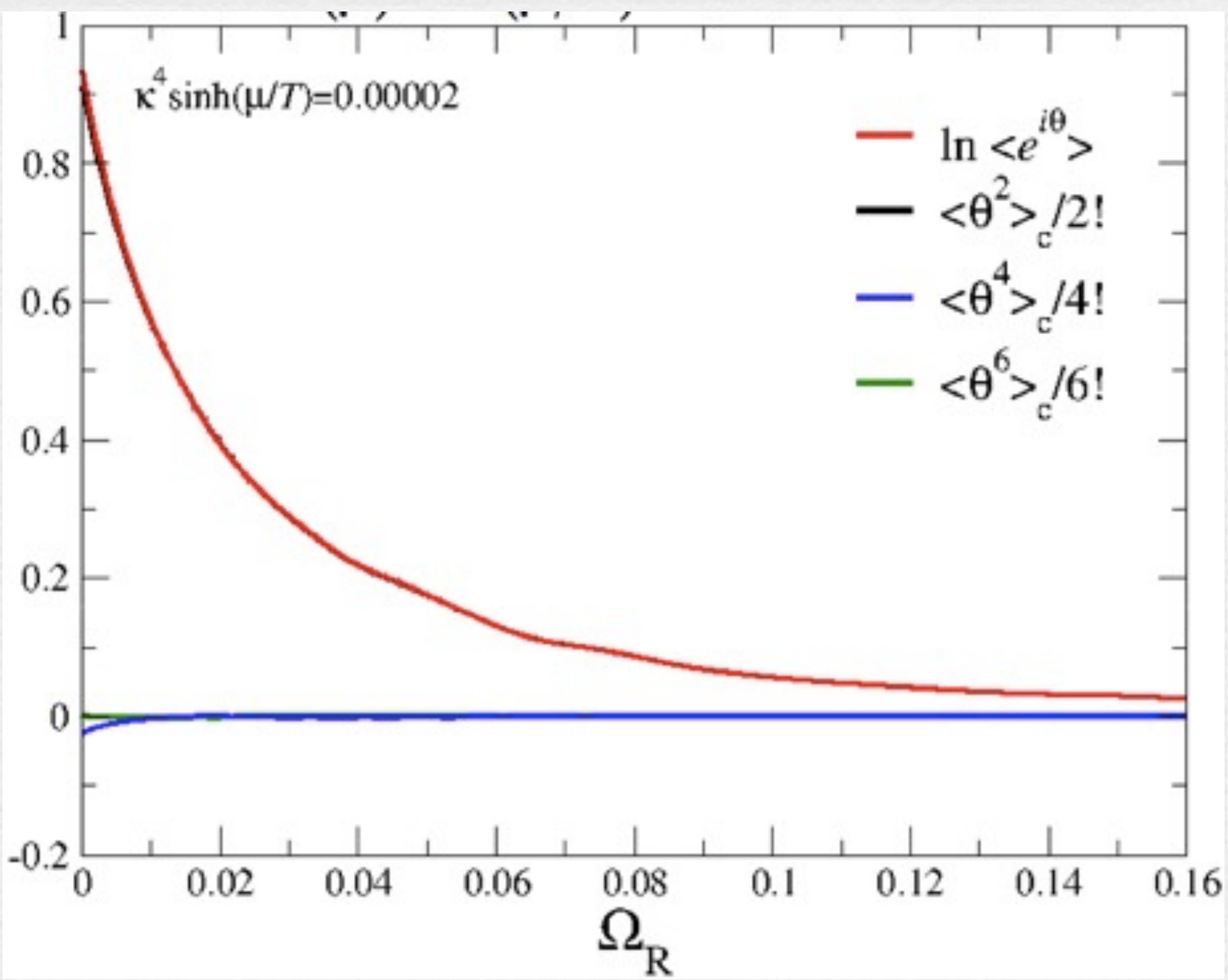
Heavy quark QCD at $\mu \neq 0$

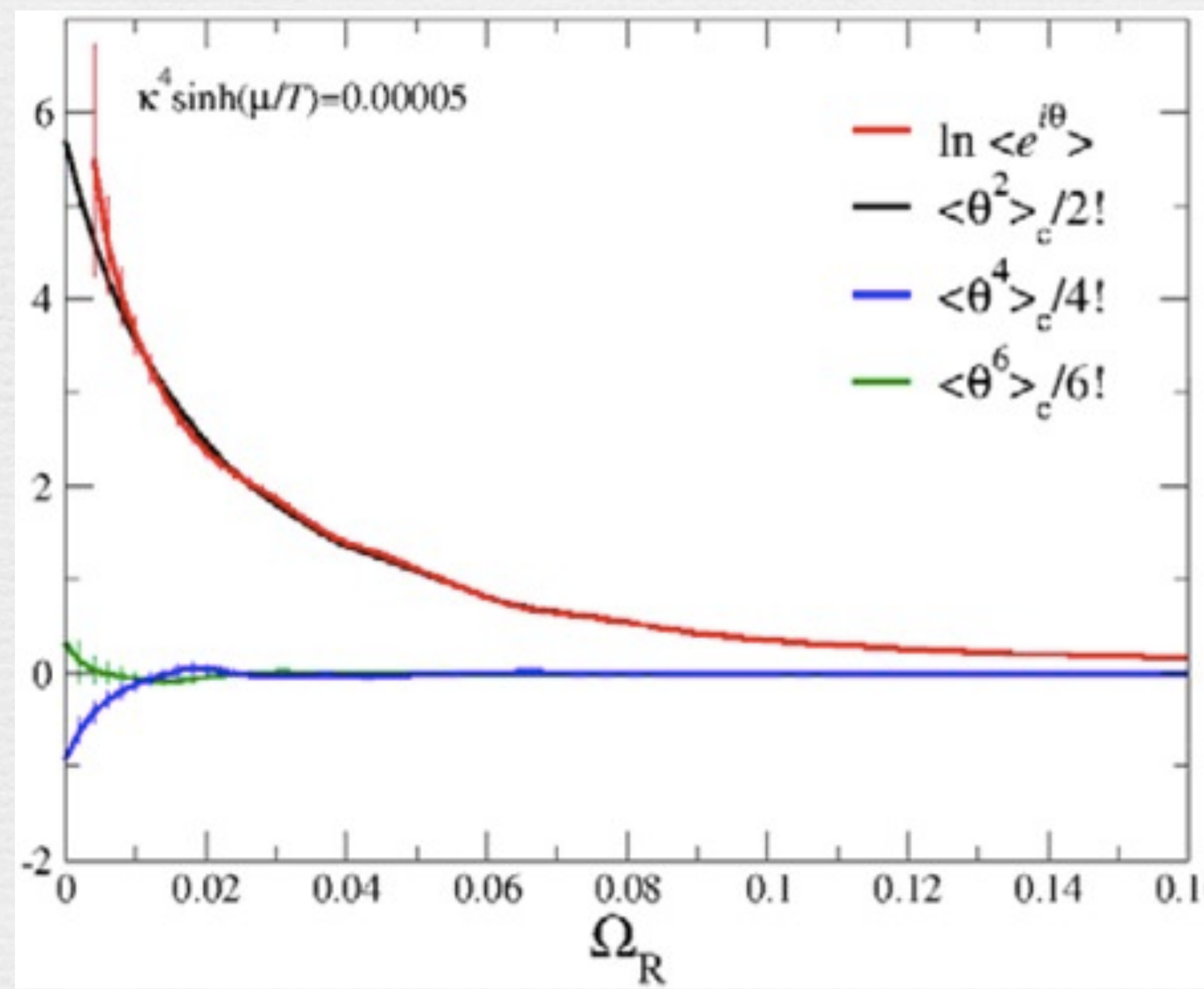
$$\langle e^{i\hat{\theta}} \rangle = \exp\left[-\frac{1}{2!}\langle \hat{\theta}^2 \rangle_c + \frac{1}{4!}\langle \hat{\theta}^4 \rangle_c + \cdots\right]$$

$$\hat{\theta} = 12 \cdot 2^{N_t} N_F N_s^3 \kappa^{N_t} \sinh(\mu/T)\,\hat{\Omega}_{\mathrm{I}}$$

Dominance of the Gaussian term around $\kappa_{\mathbf{cp}}^{\mathrm{p.q.}}(\mu)$

(important in the discussions below)

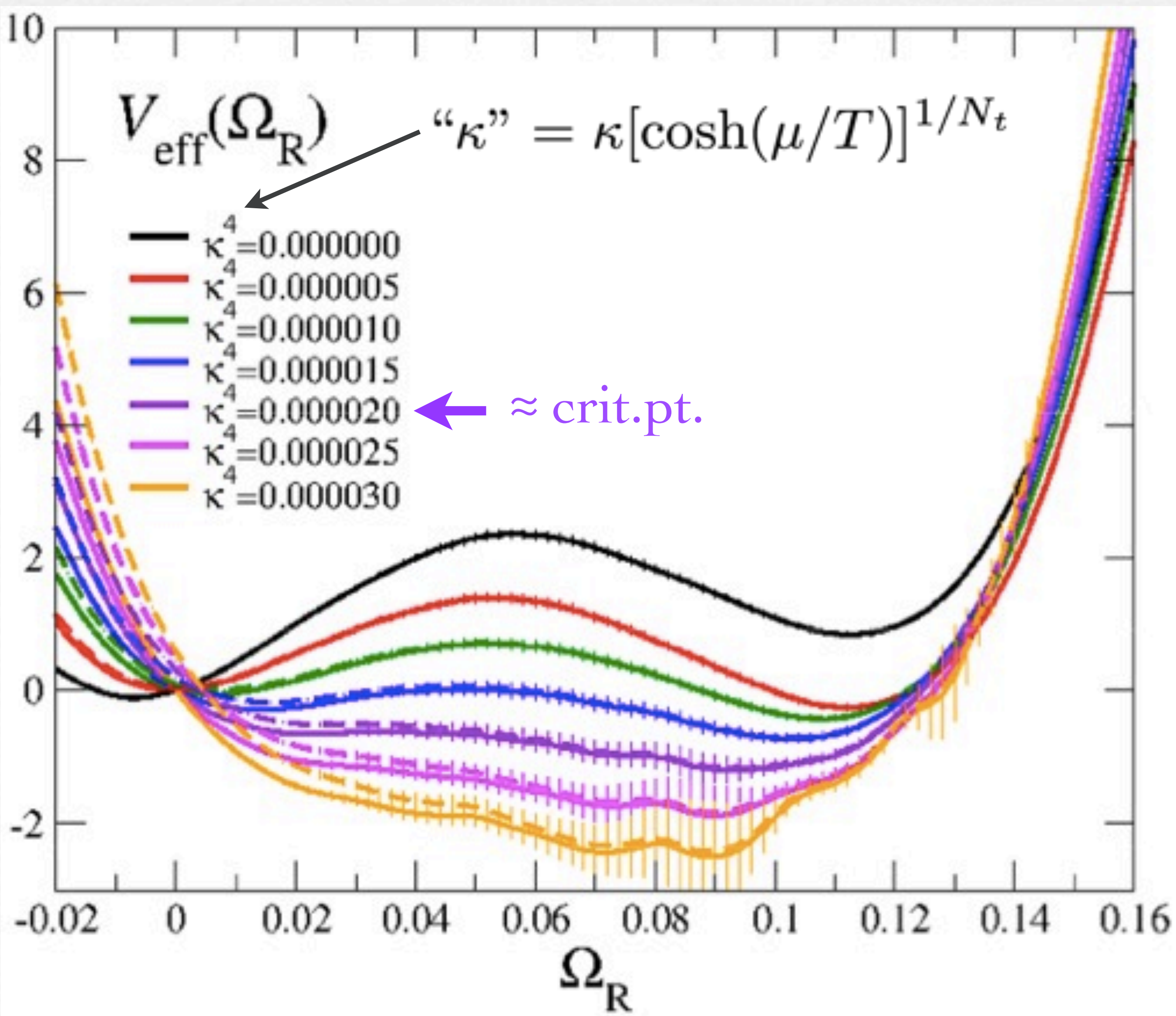




Heavy quark QCD at $\mu \neq 0$

We can now reliably evaluate $\langle e^{i\hat{\theta}} \rangle$ by the cumulant expansion

around $\kappa_{\rm cp}^{\rm p.q.}(\mu)$



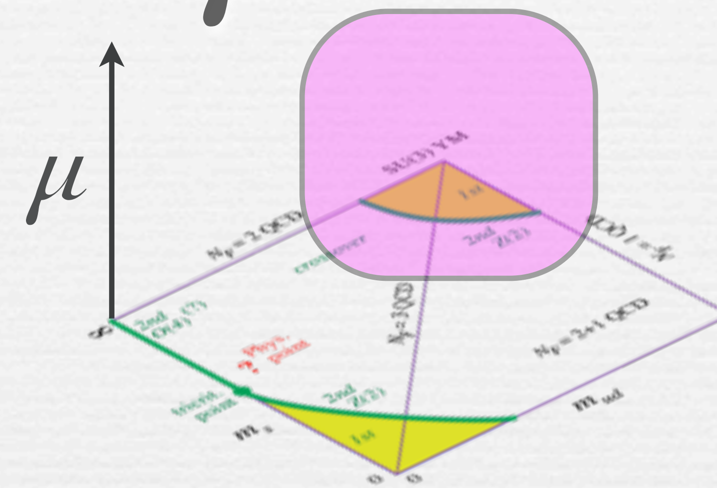
Solid line: phase-quenched
Dashed line: with maximal μ
[sinh(μ/T)=cosh(μ/T)]

=> The effects of the phase is quite small in the range of Ω_R relevant to the transition.

=> The critical point well close to the phase-quenched critical point.

$$\kappa_{\rm cp}(\mu) \approx \kappa_{\rm cp}^{\rm p.q.}(\mu) = \kappa_{\rm cp}(\mu=0) \cdot [\cosh(\mu/T)]^{-1/N_t}$$

Heavy quark QCD at $\mu \neq 0$



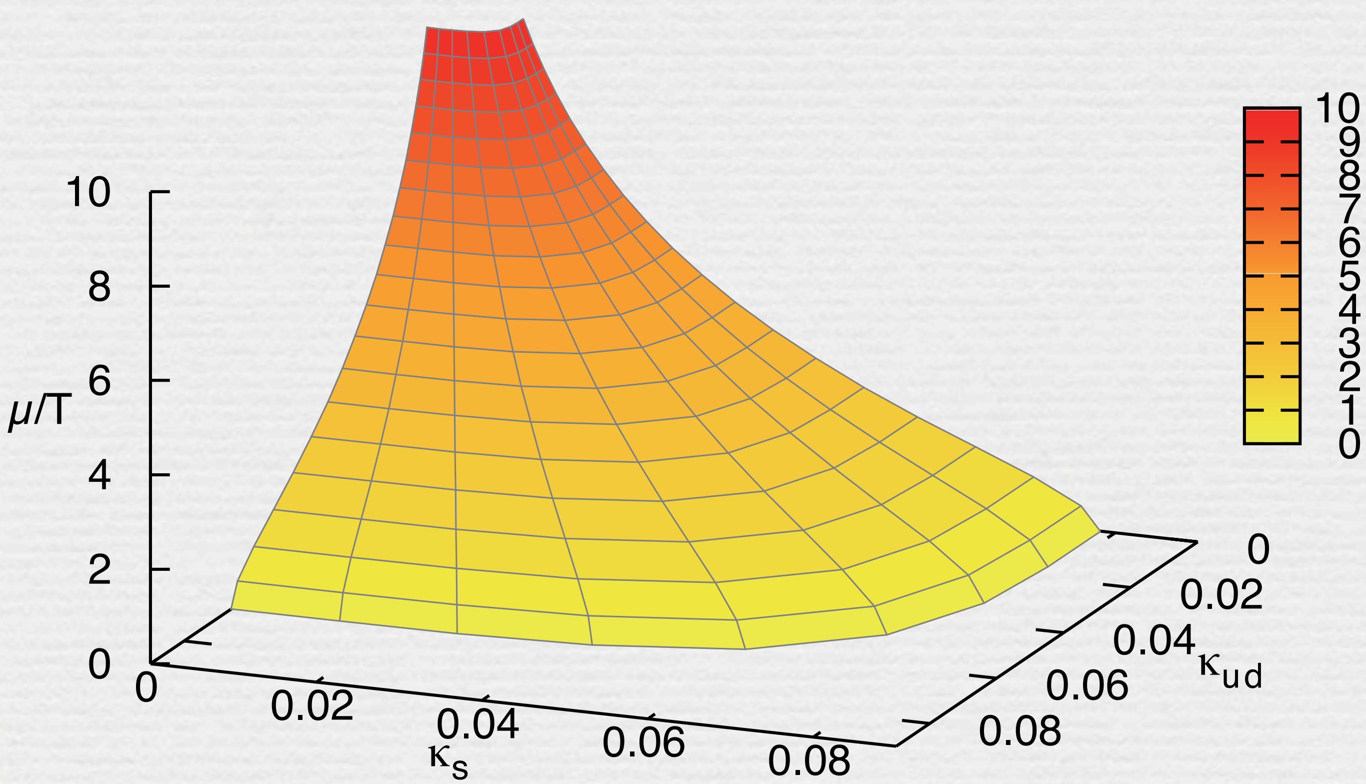
For $N_F = 2+1$, using $\kappa_{\mathrm{cp}}(\mu) \approx \kappa_{\mathrm{cp}}^{\mathrm{p.q.}}(\mu)$

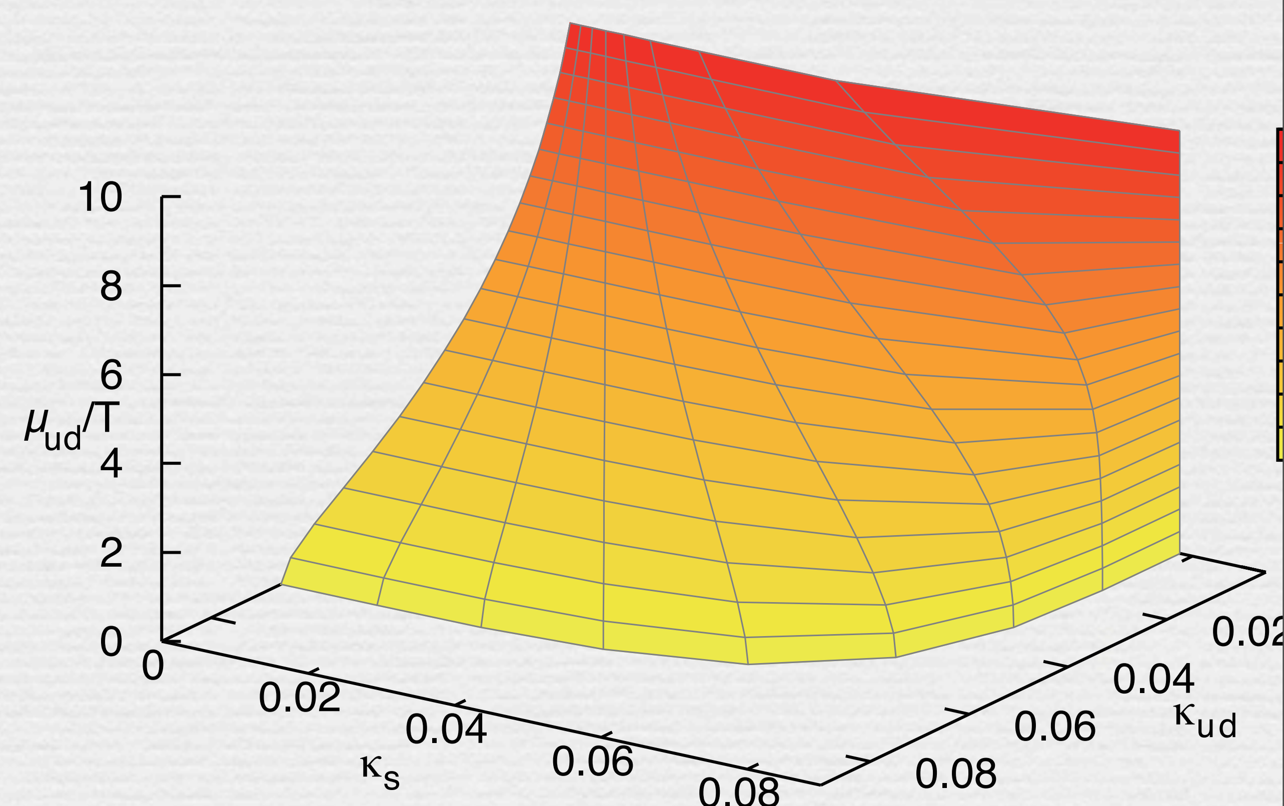
the critical surface is given by

$$2\left(\kappa_{ud}(\mu)\right)^{N_t}\cosh(\mu_{ud}/T) + \left(\kappa_s(\mu)\right)^{N_t}\cosh(\mu_s/T) = 2\left(\kappa_{\mathrm{cp}}^{N_F=2}(0)\right)^{N_t}$$

$\mu_u = \mu_d = \mu_s \equiv \mu$

$\mu_u = \mu_d \equiv \mu_{ud}, \quad \mu_s = 0$



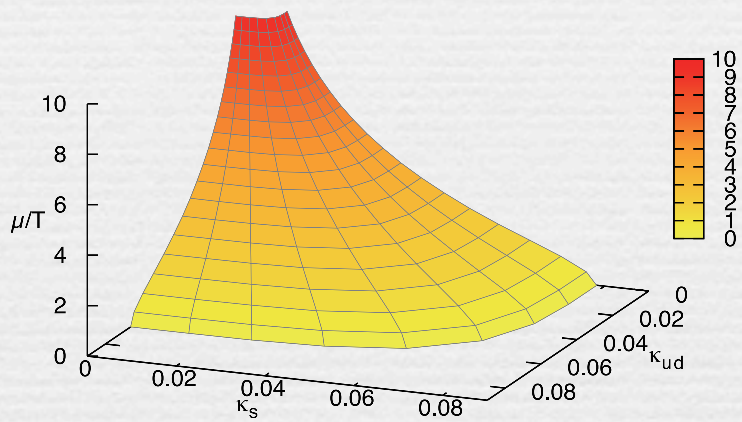


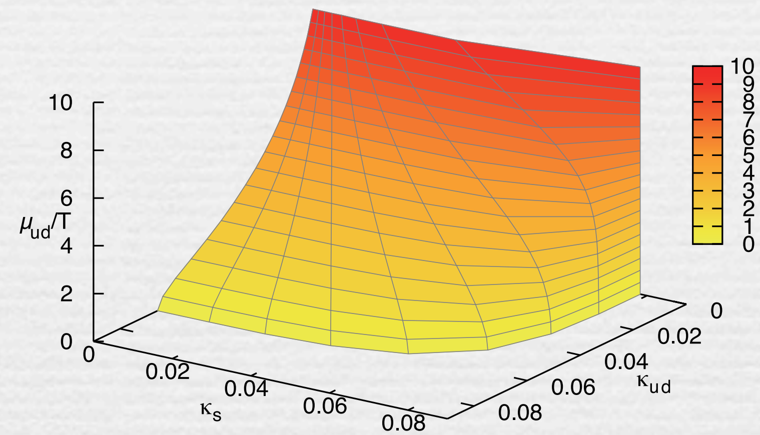
Heavy: summary & notes

- Histogram method

  = spectral density + reweighting technique + cumulant expansion

- useful to determine the phase diagram





- leading order of the hopping parameter expansion
  - effects of the next leading order small around κ_{cp}. WHOT-QCD in preparation

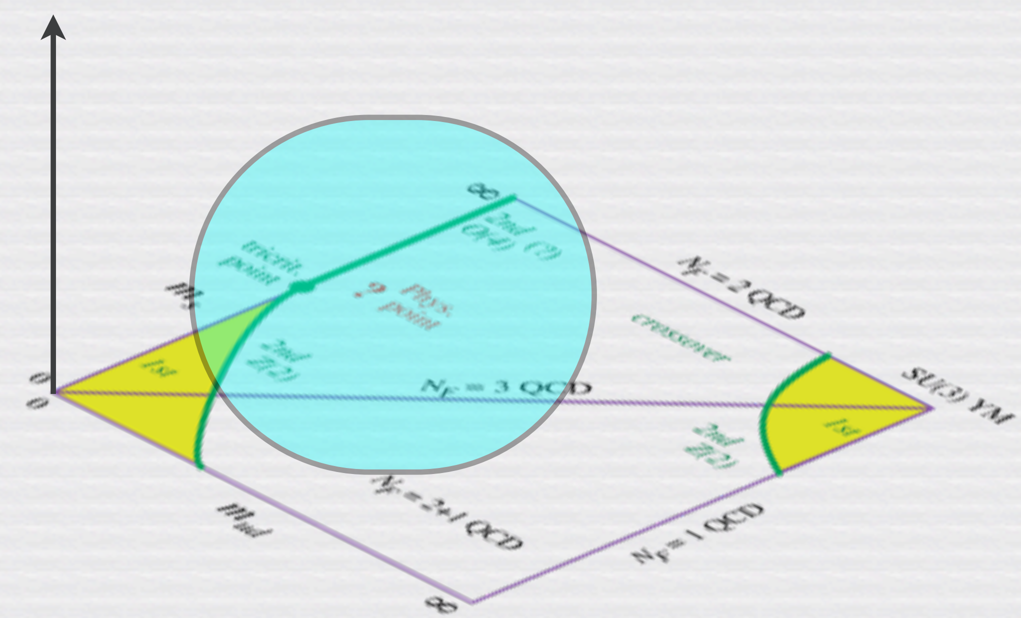
    ★ κ^6 loops ≈> renormalization of ß

    ★ κ^{Nt+2} eared Polyakov loops => competitive to the leading order only at κ >≈ 0.18 on Nt=4 lattices

    shifts κ_{cp} only ≈ 3%.

  - κ_{cp}(μ) --> 0 towards large μ: safe to use the κ expansion at all μ to study κ_{cp}.

Light quark QCD



Light quark QCD

With dynamical light quarks

- Ω no more plays an decisive role in the reweighting.

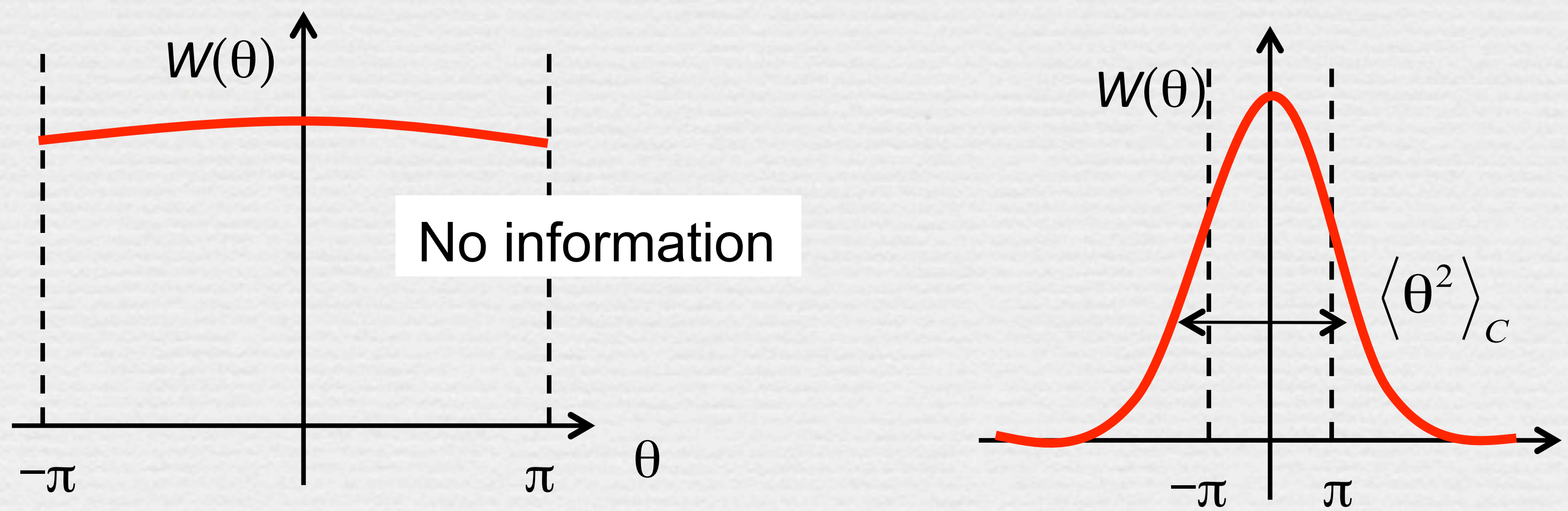
  We have to handle $\det M$ itself instead.

  Because $\det M$ corresponds to the quark energy, this should be sensitive to the phase too.

- $\theta = \arg[\det M]^{N_f}$ has an ambiguity mod. 2π

  Note that $\hat{\theta} = 12 \cdot 2^{N_t} N_F N_s^3 \kappa^{N_t} \sinh(\mu/T)\, \hat{\Omega}_\mathrm{I}$ in heavy quark QCD is not restricted in $(-\pi,\pi)$.

  We show that a good choice of θ (as an integral of local operator) leads to Gaussian distributions.



Light quark QCD

$$Z(\beta,\mu) = \int \mathscr{D}U\, e^{i\theta(\mu)}\, |\det M(\mu)|^{N_\mathrm{f}}\, e^{6\beta N_\mathrm{site} P}$$

- **Our strategy:** Y. Nakagawa et al., PoS Lattice 2012, 092
- Phase-quenched simulation with $|\det M|^{N_\mathrm{f}}\, e^{-S_g}$
- Reweight to incorporate the effects of the phase $e^{i\theta}$
- To cover a wide range of V_eff by another reweighting, we choose

$P = -S_g/6\beta N\mathrm{site}$: generalized plaquette
≈ glue energy

$F = N_\mathrm{f} \ln |\det M(\mu)/\det M(0)|$: abs. value of det$M$
≈ quark energy

as $\mathcal{O}$'s for V_eff

Light quark QCD

$$\frac{Z(\beta,\mu)}{Z(\beta,0)} = \int dPdF\, w_0(P,F;\beta,\mu)\, \left\langle e^{i\theta} \right\rangle = \int dPdF\, e^{-V(P,F;\beta,\mu)},$$

where

$$w_0(P,F;\beta,\mu) = \frac{1}{Z(\beta,0)} \int \mathcal{D}U \delta(P-\hat{P})\delta(F-\hat{F}(\mu)) \left|\det M(\mu)\right|^{N_\mathrm{f}} e^{6\beta N_\mathrm{site} P}$$

: phase-quenched histogram of (P,F)

normalized at μ=0

$$\begin{aligned}\left\langle e^{i\theta} \right\rangle (P,F;\mu) &= \frac{\int \mathscr{D}U e^{i\theta(\mu)} \delta(P-\hat{P})\delta(F-\hat{F}(\mu)) \left|\det M(\mu)\right|^{N_\mathrm{f}} e^{6\beta N_\mathrm{site} P}}{\int \mathscr{D}U \delta(P-\hat{P})\delta(F-\hat{F}(\mu)) \left|\det M(\mu)\right|^{N_\mathrm{f}} e^{6\beta N_\mathrm{site} P}} \\ &= \frac{\left\langle\!\left\langle e^{i\theta(\mu)} \delta(P-\hat{P})\delta(F-\hat{F}(\mu)) \right\rangle\!\right\rangle_{(\beta,\mu)}}{\left\langle\!\left\langle \delta(P-\hat{P})\delta(F-\hat{F}(\mu)) \right\rangle\!\right\rangle_{(\beta,\mu)}}\end{aligned}$$

: phase factor at fixed P and F

Light quark QCD

- **Reweighting factors**

$$w_0(P,F;\beta,\mu) = R(P,F;\beta,\beta_0,\mu,\mu_0)\, w_0(P,F;\beta_0,\mu_0)$$

$$R(P,F,\beta,\beta_0,\mu,\mu_0) = e^{6(\beta-\beta_0)N_{\mathrm{site}}P}\, \frac{\left\langle\!\left\langle \delta(P-\hat{P})\delta(F-\hat{F}) \left|\frac{\det M(\mu)}{\det M(\mu_0)}\right|^{N_{\mathrm{f}}} \right\rangle\!\right\rangle_{(\beta_0,\mu_0)}}{\left\langle\!\left\langle \delta(P-\hat{P})\delta(F-\hat{F}) \right\rangle\!\right\rangle_{(\beta_0,\mu_0)}}$$

$$\left\langle e^{i\theta}\right\rangle(P,F;\mu) = \frac{\left\langle\!\left\langle e^{i\theta} \left|\frac{\det M(\mu)}{\det M(\mu_0)}\right|^{N_{\mathrm{f}}} \delta(P-\hat{P})\delta(F-\hat{F}) \right\rangle\!\right\rangle_{(\beta_0,\mu_0)}}{\left\langle\!\left\langle \left|\frac{\det M(\mu)}{\det M(\mu_0)}\right|^{N_{\mathrm{f}}} \delta(P-\hat{P})\delta(F-\hat{F}) \right\rangle\!\right\rangle_{(\beta_0,\mu_0)}}$$

With *P* fixed, the ratios in r.h.s. are actually indep of ß.

Light quark QCD

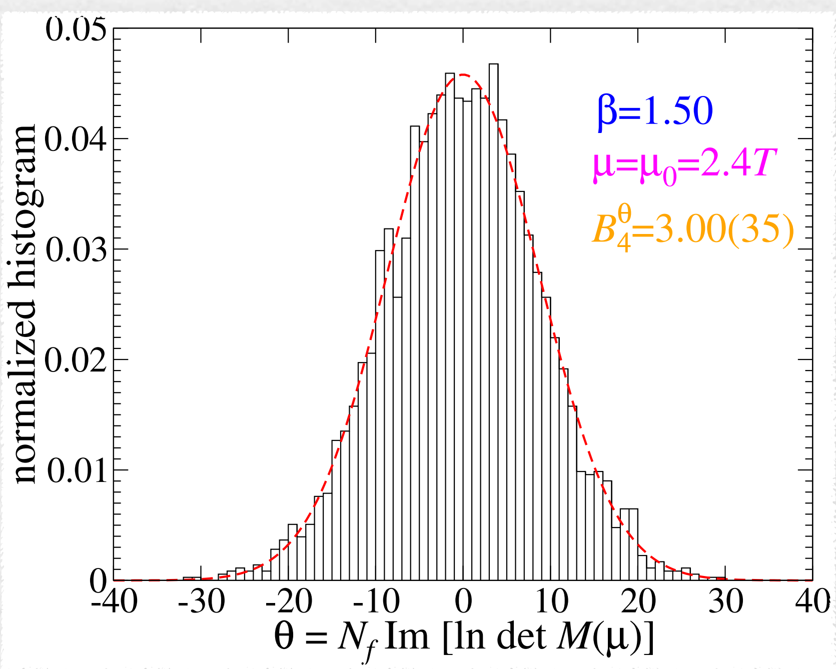
- **Our definition of θ**

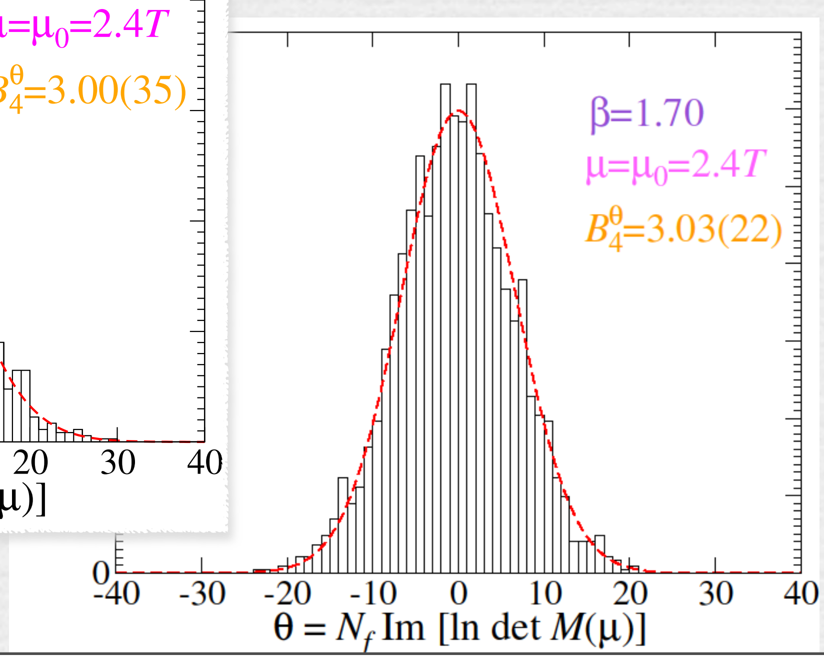
$$\theta(\mu) = N_\mathrm{f} \int_0^{\mu/T} \mathfrak{Im}\left[\frac{\partial(\ln\det M(\mu))}{\partial(\mu/T)}\right]_{\bar{\mu}} d\left(\frac{\bar{\mu}}{T}\right)$$

$$\partial(\ln\det M)/\partial\mu' = \mathrm{tr}[M^{-1}(\partial M/\partial\mu')]$$ *i.e.*, $\int d^4x$ of a local op.

=> uniquely defined in $(-\infty, +\infty)$.
Conventional θ in $(-\pi, +\pi)$ recovered by taking the modulus.

=> Gaussian distribution when the volume >> correlation range,
as in the case of QCD at $m_q > 0$.





Thus the cumulant expansion converges, and the Gaussian approximation works well.

$$\left\langle e^{i\theta}\right\rangle_{P,F} \approx \exp\left[-\frac{1}{2}\left\langle\theta^2\right\rangle\right]$$

Light quark QCD

F and the reweighting factor can be evaluated similarly

using the same measurement

$$F(\mu) = N_\mathrm{f} \ln \left| \frac{\det M(\mu)}{\det M(0)} \right| = N_\mathrm{f} \int_0^{\mu/T} \mathfrak{Re} \left[\frac{\partial(\ln \det M(\mu))}{\partial(\mu/T)} \right]_{\bar{\mu}} d\left(\frac{\bar{\mu}}{T}\right),$$

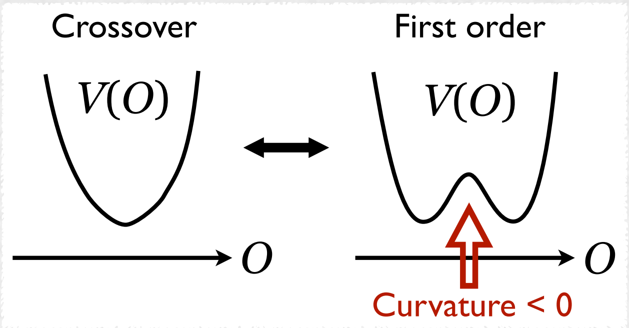
$$C(\mu) = N_\mathrm{f} \ln \left| \frac{\det M(\mu)}{\det M(\mu_0)} \right| = N_\mathrm{f} \int_{\mu_0/T}^{\mu/T} \mathfrak{Re} \left[\frac{\partial(\ln \det M(\mu))}{\partial(\mu/T)} \right]_{\bar{\mu}} d\left(\frac{\bar{\mu}}{T}\right)$$

- **Curvatures of V**

  to the lowest order of the cumulant expansion (Gaussian approx.)

$$\frac{\partial^2 V}{\partial P^2}(P,F;\beta,\mu) = \frac{\partial^2(-\ln w_0)}{\partial P^2}(P,F;\beta_0,\mu_0) - \frac{\partial^2 \ln R}{\partial P^2}(P,F;\beta,\beta_0,\mu,\mu_0) + \frac{1}{2}\frac{\partial^2 \langle \theta^2 \rangle_c}{\partial P^2}(P,F;\mu)$$

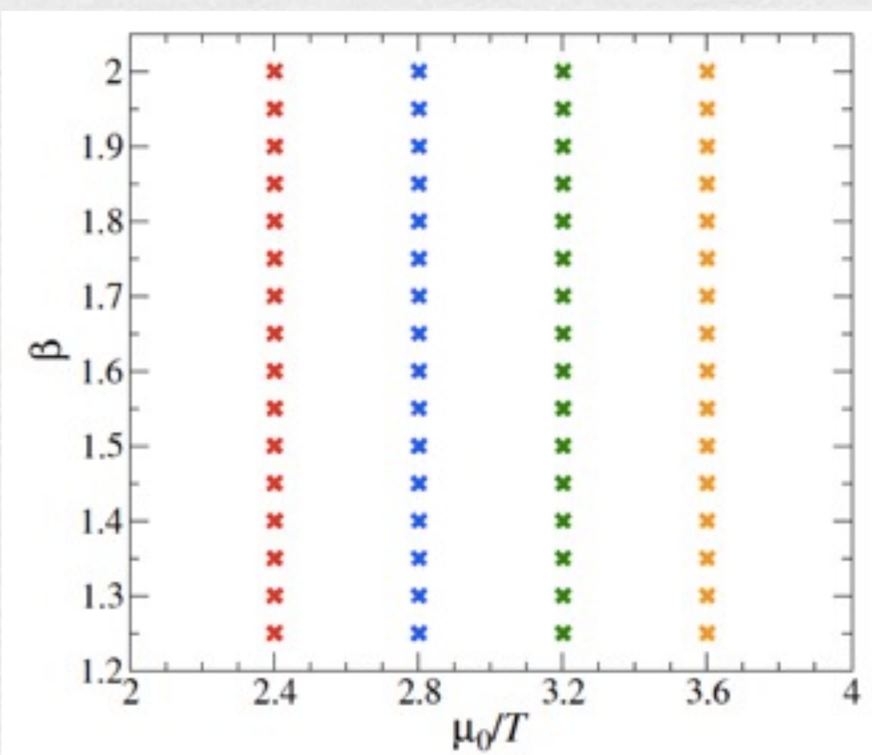
$$\frac{\partial^2 V}{\partial F^2}(P,F;\beta,\mu) = \frac{\partial^2(-\ln w_0)}{\partial F^2}(P,F;\beta_0,\mu_0) - \frac{\partial^2 \ln R}{\partial F^2}(P,F;\beta,\beta_0,\mu,\mu_0) + \frac{1}{2}\frac{\partial^2 \langle \theta^2 \rangle_c}{\partial F^2}(P,F;\mu)$$



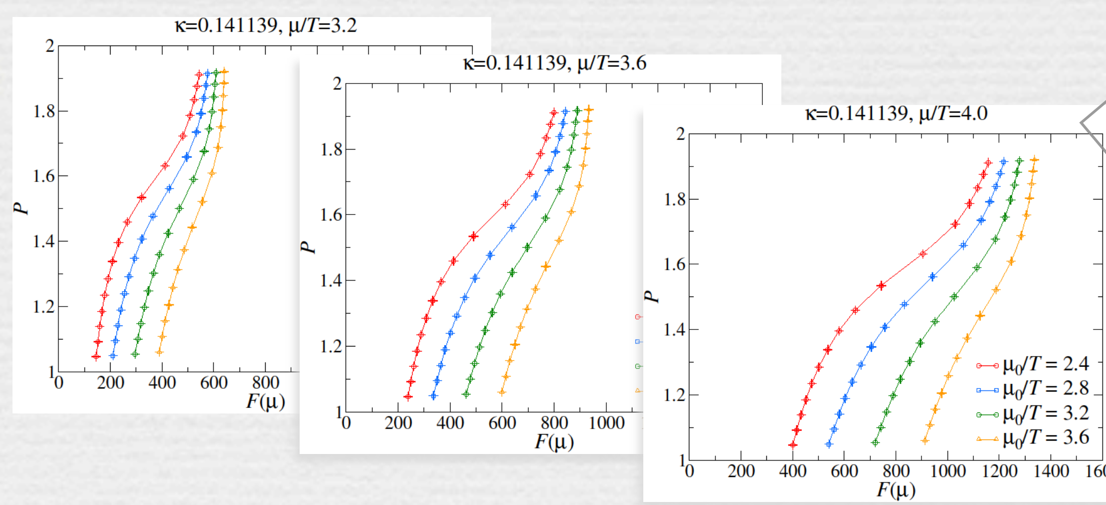
Negative curvature signals 1st order transitions.

Light quark QCD

- **Simulation of 2-flavor QCD on clusters**
- RG-improved Iwasaki gauge
- 2-flavor non-perturbatively O(a)-improved clover quarks
- 8^3x4 lattice at $m_\pi/m_\rho \approx 0.8$ (κ=0.141139)
- phase-quenched simulations at β_0=1.2–2.0, μ_0/T=2.0–4.0



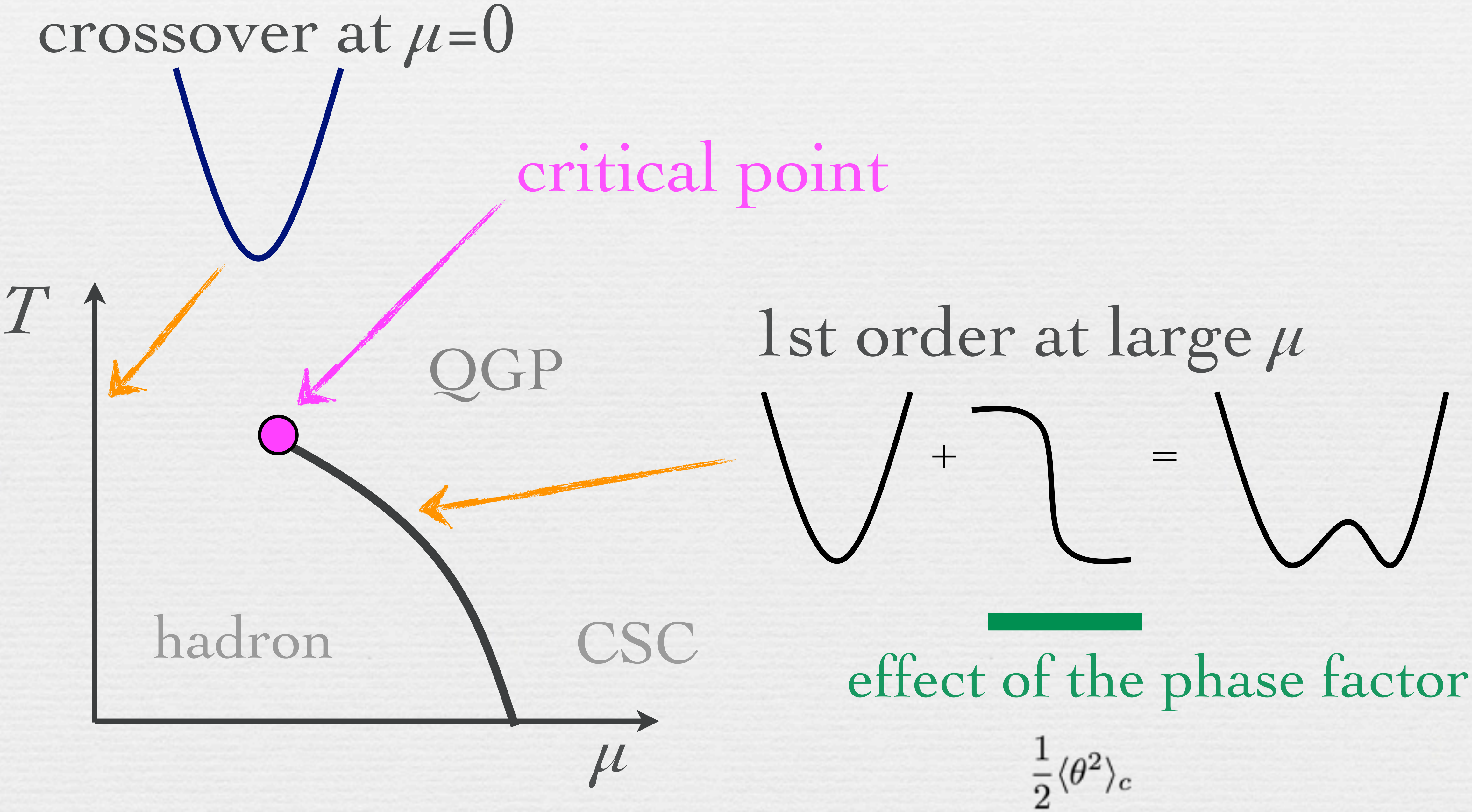
- random-noise method to compute derivatives of ln det$M$ using 50 noises every 10 traj's.
- ≥ 29,000 trajectories each, after thermalization
- peak of the phase-quenched distribution



ON-GOING

Light quark QCD

What we expect for 2-flavor QCD with $m_q > 0$:



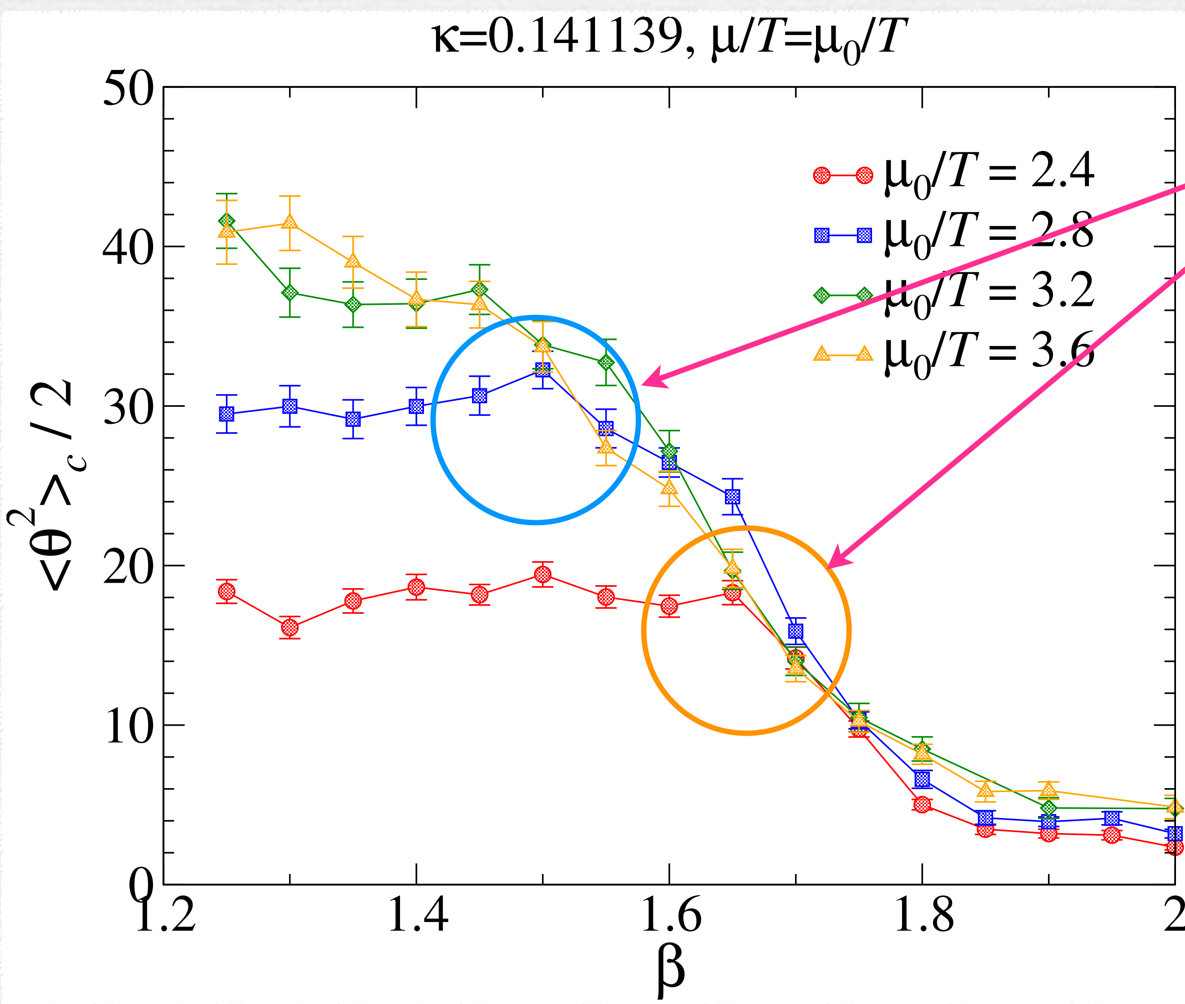
We try to find the crit. pt. through negative curvature in V_{eff}.

Light quark QCD

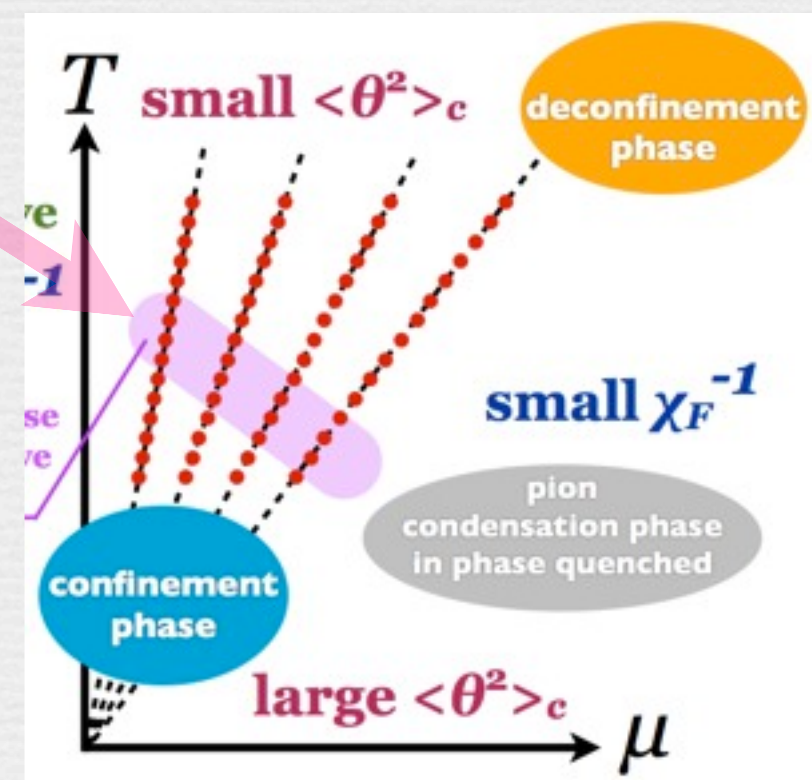
$$\frac{\partial^2 V}{\partial P^2}(P,F;\beta,\mu) = \frac{\partial^2(-\ln w_0)}{\partial P^2}(P,F;\beta_0,\mu_0) - \frac{\partial^2 \ln R}{\partial P^2}(P,F;\beta,\beta_0,\mu,\mu_0) + \frac{1}{2}\frac{\partial^2 \langle \theta^2 \rangle_c}{\partial P^2}(P,F;\mu)$$

$$\frac{\partial^2 V}{\partial F^2}(P,F;\beta,\mu) = \frac{\partial^2(-\ln w_0)}{\partial F^2}(P,F;\beta_0,\mu_0) - \frac{\partial^2 \ln R}{\partial F^2}(P,F;\beta,\beta_0,\mu,\mu_0) + \frac{1}{2}\frac{\partial^2 \langle \theta^2 \rangle_c}{\partial F^2}(P,F;\mu)$$

See Y. Nakagawa et al., PoS Lattice 2012, 092 for details of the calculation.

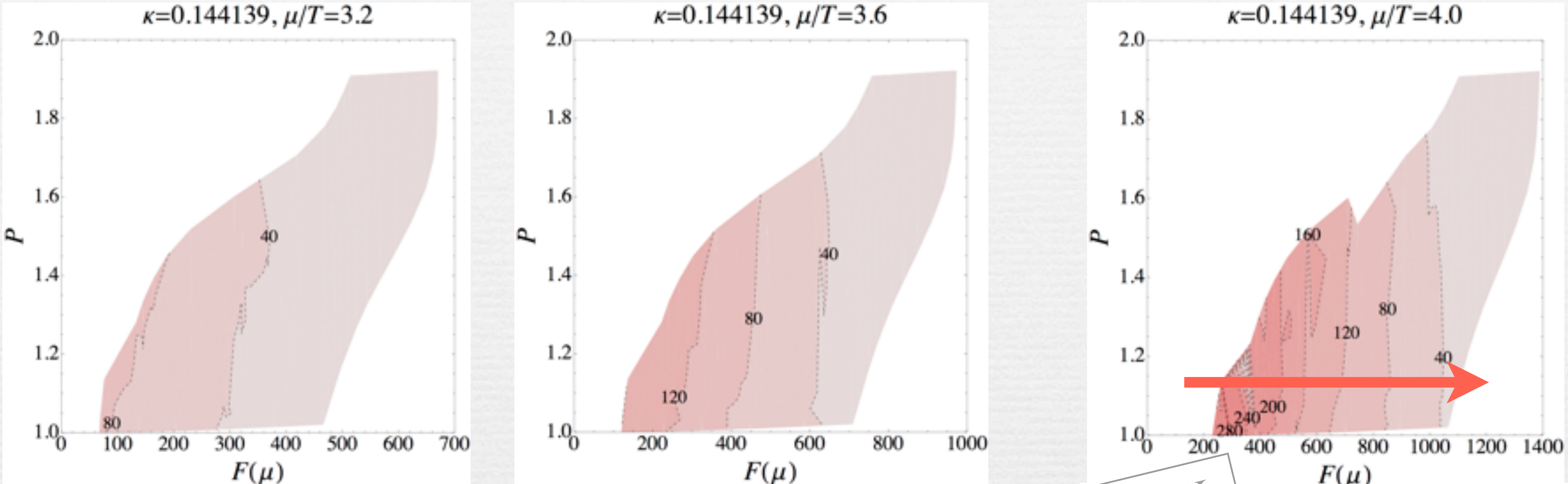


negative contribution to the curvature



Light quark QCD

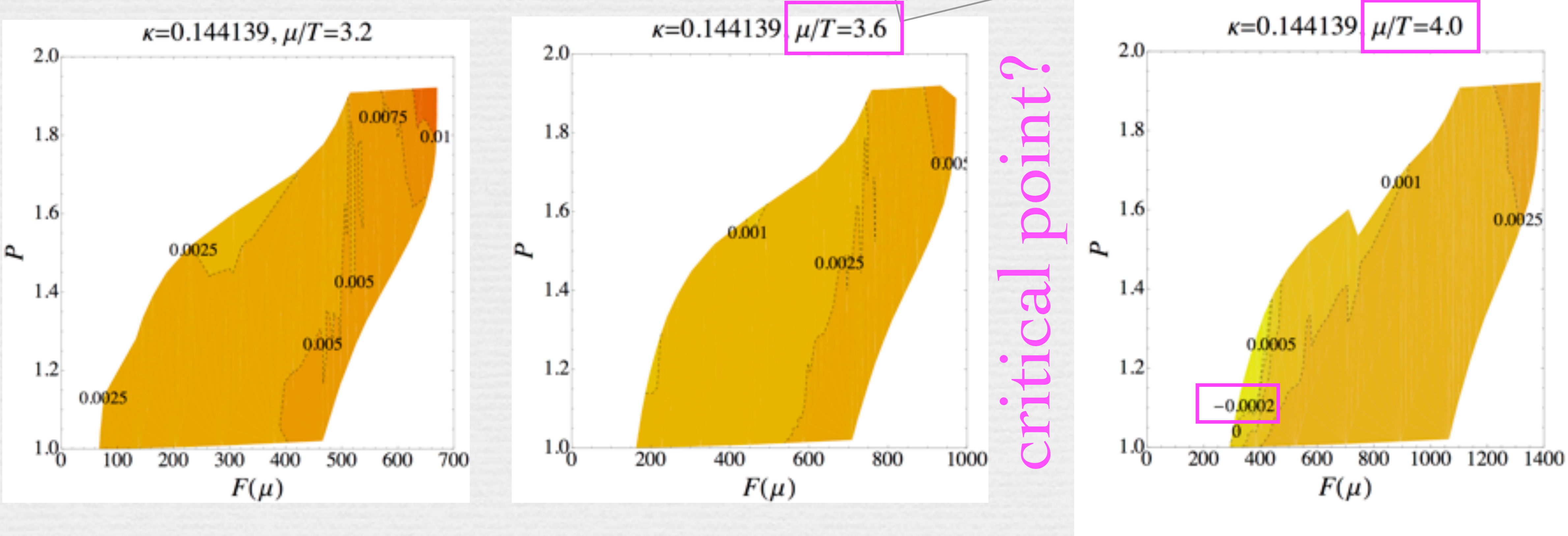
$$\frac{1}{2}\langle\theta^2\rangle_c$$



- Curvature in the F-direction:

PRELIMINARY

$$\frac{\partial^2 V}{\partial F^2}$$



critical point?

Summary

- Histogram method to detect 1st order transitions
- Sign problem avoided by the cumulant expansion up to $\mu \sim \mu_c$
- Validity of the method confirmed in heavy quark QCD
- Light quark QCD studied by phase-quenched simulations + reweighting
- Critical point terminating the 1st order deconfining transition line suggested numerically
- Simulations/analyses on-going

thank you

- Backup slides from recent talks
- New results are not included.

PROSPECTED PHASE STRUCTURE AT $\mu = 0$

GL effective models + lattice results

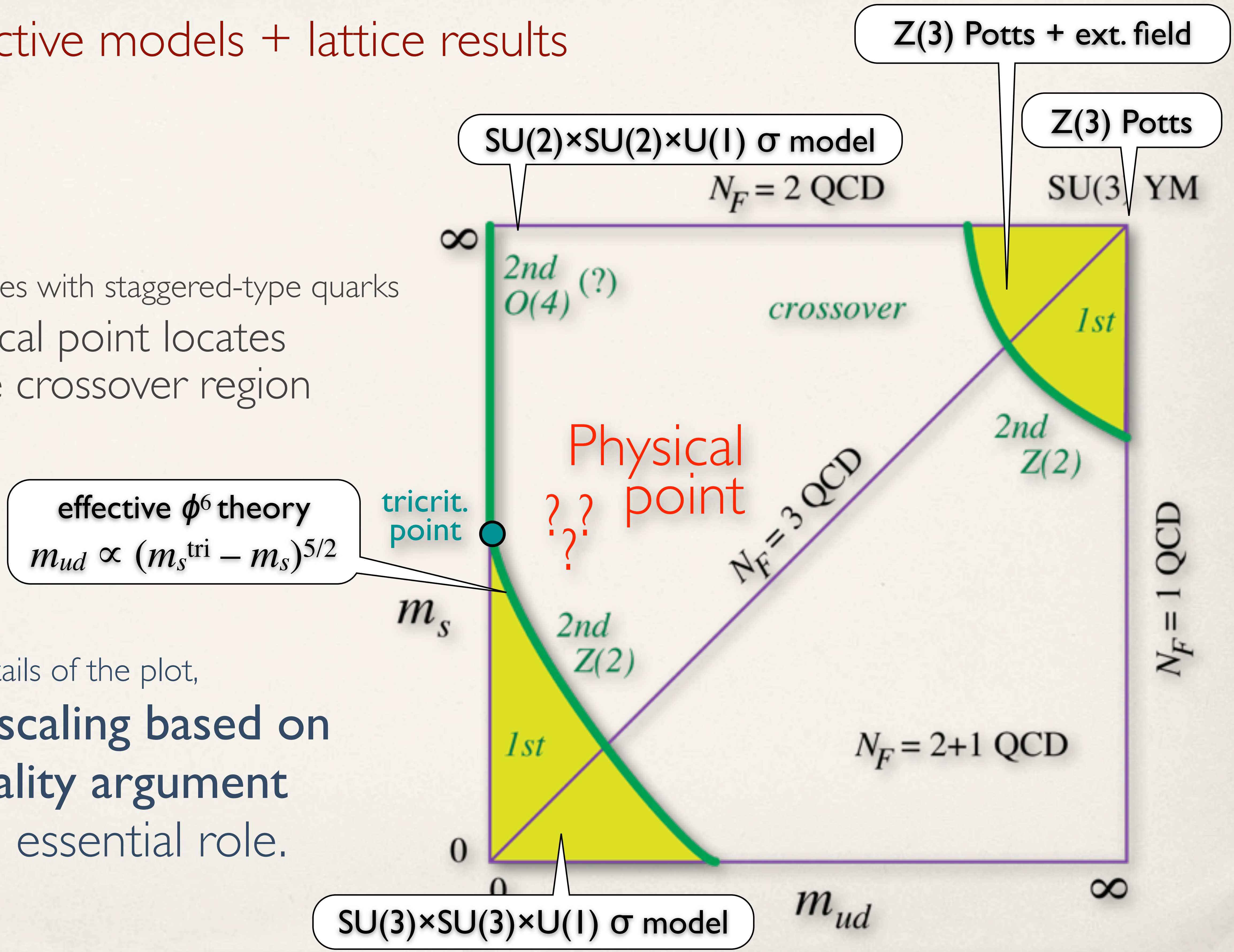
Lattice studies with staggered-type quarks

=> Physical point locates in the crossover region

In fixing details of the plot,

critical scaling based on universality argument

plays an essential role.



QCD transition around the chiral limit

Effective 3d σ model with the same flavor-chiral symmetry of massless QCD (continuum)

Pisarski-Wilczek, PRD('84); Wilczek, IJMP('92); Rajagopal-Wilczek NPB('93)

$$\mathcal{L} = \mathrm{Tr}\partial_i M^\dagger \partial_i M + \mu^2 \mathrm{Tr} M^\dagger M + \lambda_1 \mathrm{Tr}\left(M^\dagger M\right)^2 + \lambda_2 \left(\mathrm{Tr} M^\dagger M\right)^2$$
$$+ c_{\mathrm{U}(1)_A}\left(\det M + \det M^\dagger\right) \quad \Longleftarrow \quad \mathrm{U}(1)_A \text{ anomaly} \qquad M_{ab} \sim \left\langle \bar{q}_a \frac{1+\gamma_5}{2} q_b \right\rangle$$

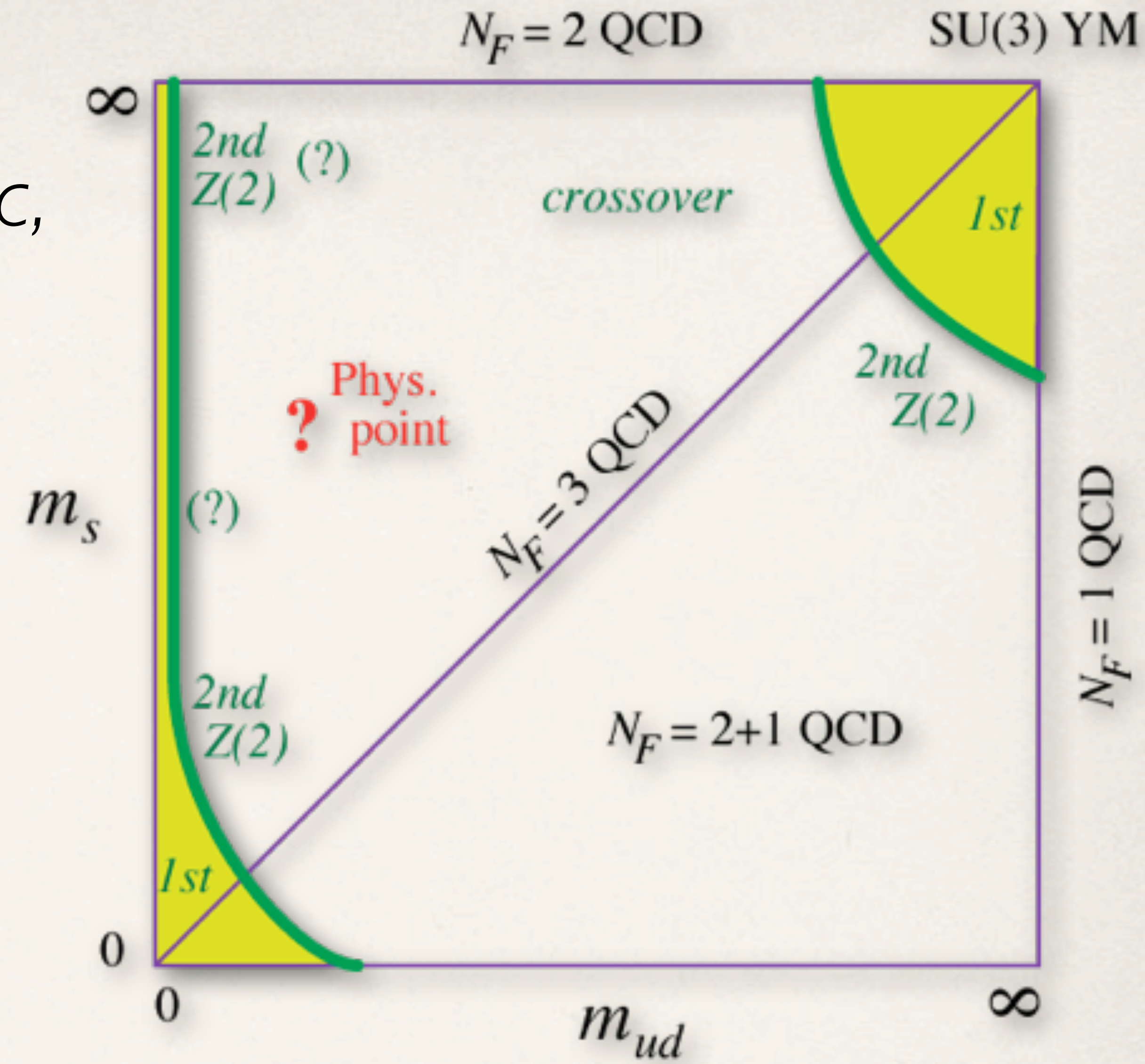
- $N_F \geq 3$: **1st order**
- $N_F = 2$: depends on the magnitude of the anomaly
  - when anomaly strong: the **σ** model ≈ O(4) Heisenberg model
    => **2nd order** with established critical properties
    $$M/h^{1/\delta} = f(t/h^{1/\beta\delta}) \qquad \begin{array}{l} 1/\beta\delta = 0.537(7) \\ 1/\delta = 0.2061(9) \end{array}$$
    $h \sim m_q$ <= chiral violating coupling (*external mag. field*)
    $t \sim (T - T_c)/T_c$ <= chiral symmetric coupling (*reduced temperature*)
  - when anomaly negligible around T_c
    => fluctuation-induced (weakly) **1st order**

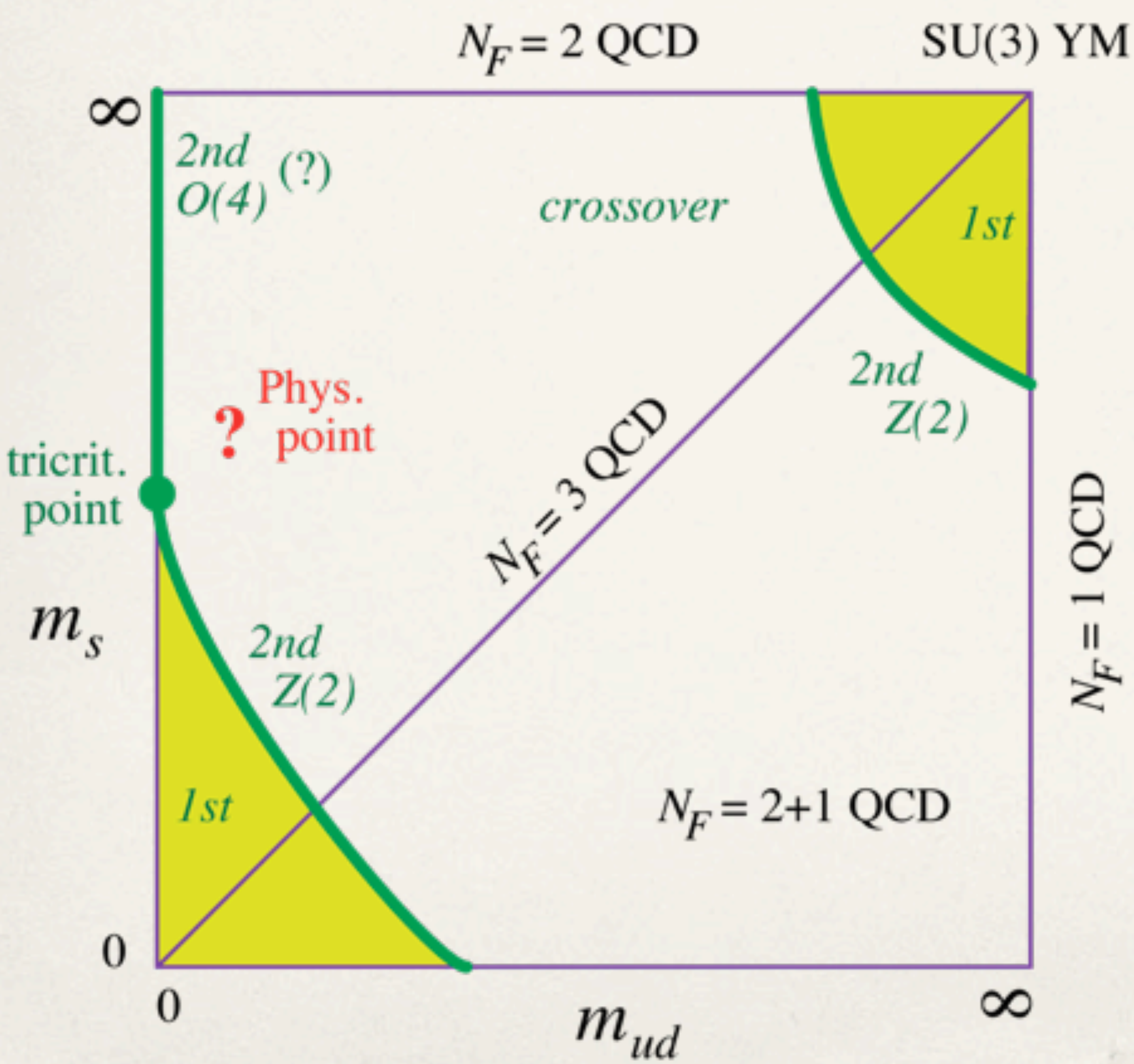
In case anomaly negligible around *Tc*,

$N_F = 2$: 1st order chiral trans. with Ising crit. end point

$N_F = 3$: smaller 1st order region

<= anomaly was a source of the M^3 term





To discriminate the pictures, non-perturbative test on the lattice needed.

O(4) scaling is a powerful guide here.

Flavor-chiral sym. on the lattice

No-go theorem (Nielesen-Ninomiya): one flavor lattice fermion cannot be local and chiral simultaneously.

Chiral symmetry cannot be simply realized on the lattice.
=> several options for the quark action with different flavor-chiral properties:

Wilson-type / staggered-type / domain-wall / overlap / ...

Wilson-type quarks: violate the chiral symmetry at $a > 0$.

Many studies are being made.

Pros:
- ✓ Describes a single flavor. => Flavor symmetry exact.
- ✓ Continuum limit exists. <= The chiral sym. is restored in the cont. limit.

Cons:
- ❖ Explicit violation of the chiral sym. at $a > 0$.
- ❖ Light quarks expensive.

Lattice chiral quarks: domain-wall / overlap

Still quite expensive to simulate.
Real applications to $T > 0$ have just started! => HotQCD, JLQCD

So far, most large-scale simulations at finite T and μ have been made with

Staggered-type quarks

Pros: ✓ Relatively cheap to simulate.

✓ A modified chiral sym. preserved: U(1) [=O(2)] taste-chiral sym.

Cons: ❖ 4 copies of identical fermions ("**tastes**") in the cont. lim. for each flavor.

=> "**4th root trick**" to remove unwanted 3 : $\det M$ => $[\det M]^{1/4}$

⇓

❖ **Non-local => Universality arguments fragile.**

? continuum limit?

<= Empirically OK *if* the continuum limit is taken first.

? chiral scaling on finite lattices? ???

❖ **Taste violation problem** at a>0 => errors in flavor identifications.

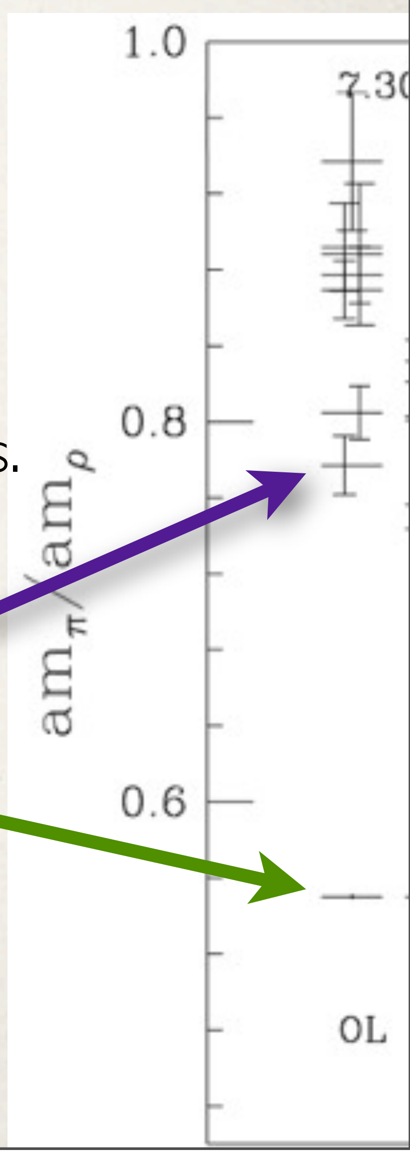
(e.g.) many π's in the taste space, one is light due to the taste-chiral sym.

Lightest π (pNG π) usually treated as "physical".

Other π's do contribute in dynamical effects.

=> lattice artifacts.

m_q is effectively much heavier.



Recently, it was noted that **a good control of the taste-violation is essential to obtain physical results with staggered-type quarks.**

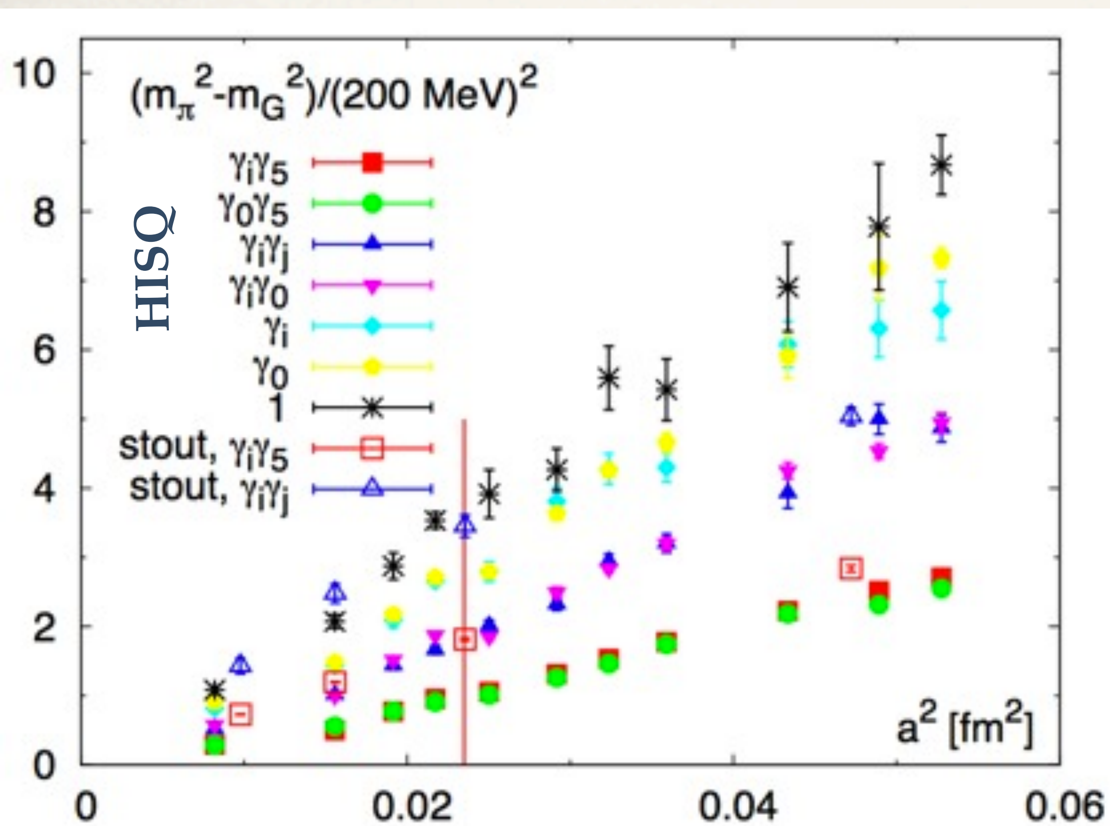
Improved staggered quarks

Various actions proposed to milden lattice artifacts including the taste violation:

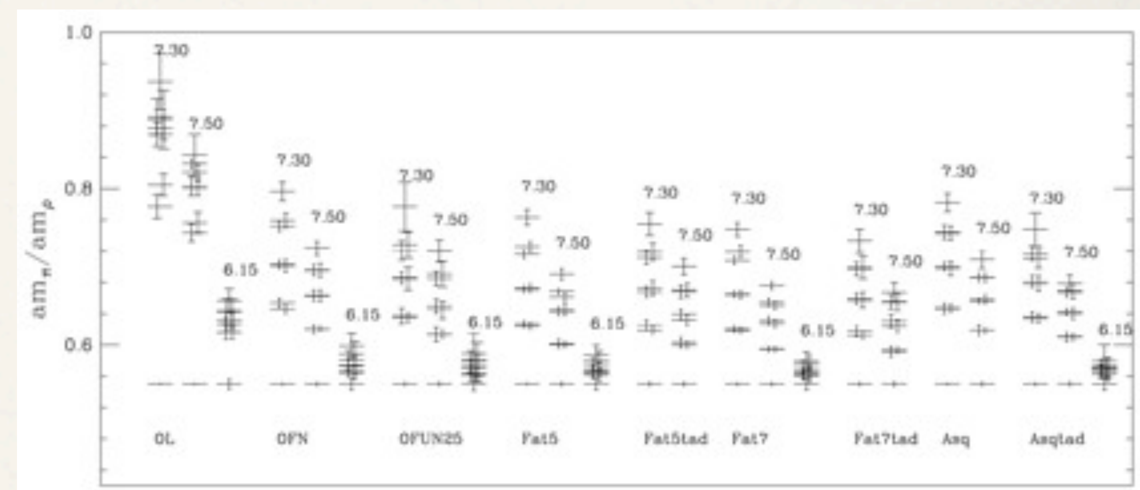
asqtad / p4 / HYP / stout / HISQ / ...

The extent of improvement differs depending on the action.

Orginos et al,
hep-lat/9909087
$m_\pi/m_\rho=0.55$



Bazavov-Petreczky
(HotQCD)
arXiv:1012.1257



unimproved **asqtad**

The magnitude of the taste violation

HISQ < stout < asqtad < p4

heavy "**π**" masses (at *T*~170MeV with m_π^{pNG}~135 MeV)

Nt~8	~ 400-600	asqtad < p4
	~ 300-500	stout
	~ 200-400	HISQ
Nt~12	~ 200-350	stout

O(4) scaling tests on the lattice

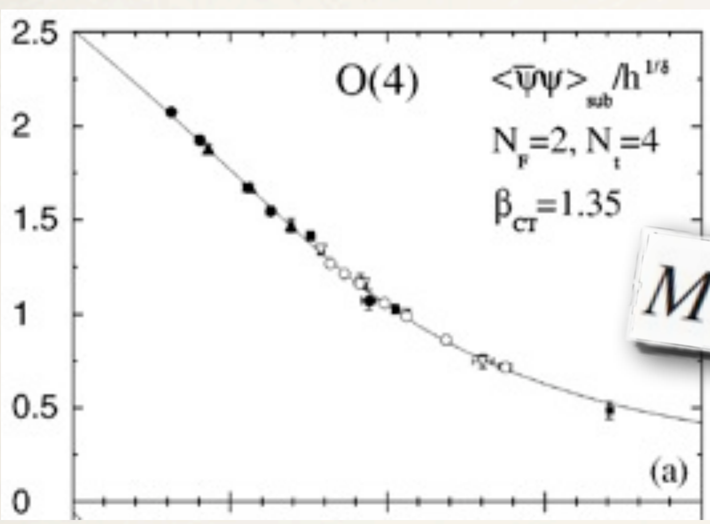
Wilson-type quarks (N_F=2)

Proper renormalization needed to recover the chiral symmetry in the continuum limit.

$M \sim \langle \bar{\Psi}\Psi \rangle_{\mathrm{sub}} = 2m_q aZ \sum_x \langle \pi(x)\pi(0) \rangle$ via axial W.I Bochicchio et al.('85)

QCD data vs. O(4) scaling function and exponents

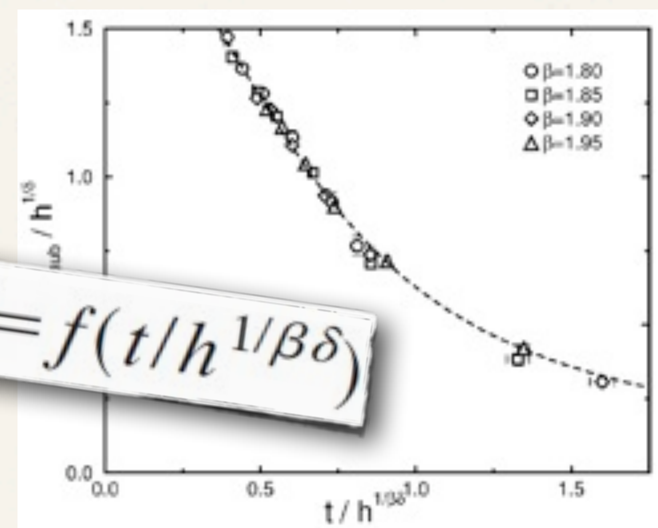
$$M/h^{1/\delta} = f(t/h^{1/\beta\delta})$$



Iwasaki et al. (QCDPAX)

PRL78('97)

- Iwasaki gauge + Wilson
- Nt=4, m_π ~ 600-900 MeV

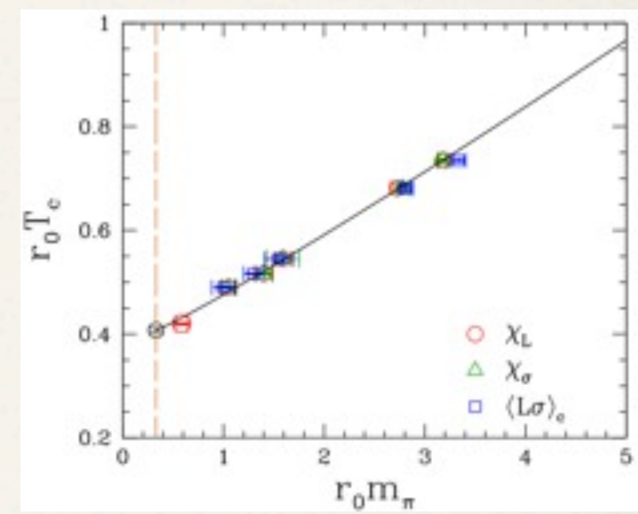


AliKhan et al. (CP-PACS)

PRD63('01)

- Iwasaki gauge + Clover
- Nt=4, m_π ~ 600-1000 MeV

O(4) scaling fit for *Tc*



Bornyakov et al. (QCDSF)

PRD82('10)

- plaquette gauge + Clover
- Nt = 8,10,12, $m_\pi \approx$ 420-1300 MeV

No indication of 1st order chiral transition.

QCD data well described by the O(4) scaling function with O(4) exponents.

➡ **Consistent with the O(4) scaling**, though quarks are heavy.

Nógrádi (Budapest-Wuppertal) @ Lat11

- Symanzik/tree + 6-level-**stout-clover** N_F=2+1
- fixed scale approach at 6 ß values

$$M \sim \Delta_{\bar{\Psi}\Psi} = \langle \bar{\Psi}\Psi \rangle_T - \langle \bar{\Psi}\Psi \rangle_{T=0}$$

Comparison with staggered (stout, N_t=8-12) at m_π≈545 MeV, m_K≈612 MeV

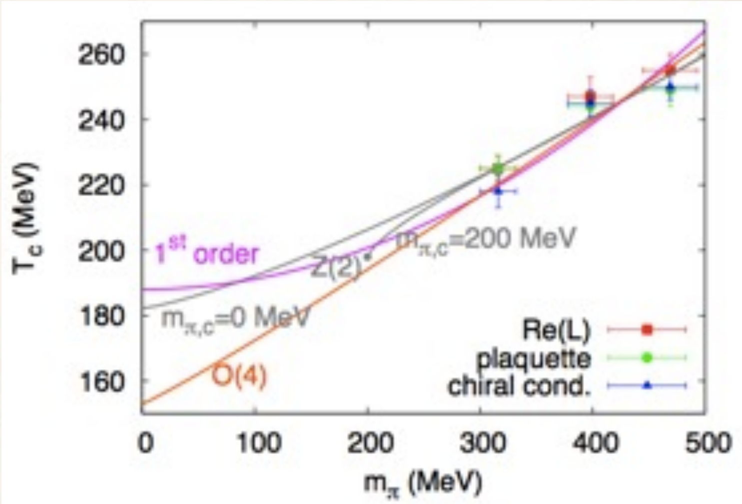
susceptibilities, renormalized Polyakov loop => well consistent with each other.

Continuum thermodynamics feasible with improved Wilson.

No O(4) scaling tests yet.

Burger (tmfT) @ Lat11, 1102.4530

- Symanzik/tree + **maximally twisted Wilson** N_F=2
- Nt=8–12 m_π≈320-480 MeV



O(4) fit for m_π≈320-480 MeV works well

=> T_c = 160–270 MeV.

Difficult to discriminate between O(4) and 1st order yet.

O(4) scaling tests on the lattice

Staggered-type quarks

★ O(4) vs. O(2)

The symmetry of 4-taste staggered quark action is the O(2) taste-chiral symmetry.

This is so also with the 4th root trick $\det M \Rightarrow [\det \boldsymbol{M}]^{1/4}$, therefore,

Sym. of the system in the chiral limit = **O(2)** for any N_F.

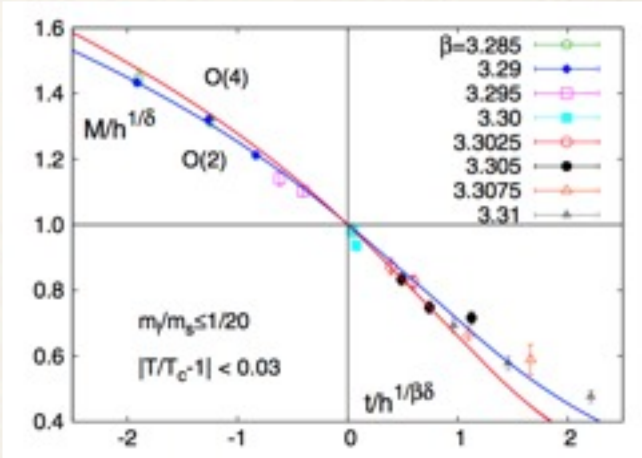
=> When the chiral transition is 2nd order on the lattice, we expect **O(2) scaling** not O(4).
O(4) may be realized when (1) continuum extrapolation,
and then (2) chiral extrapolation.

In practice, O(2) ≈ O(4) numerically.

N	β	γ	δ
2	0.349	1.319	4.780
4	0.380	1.453	4.824

Ejiri et al. (BNL-Bielefeld)
PRD80('09)

- p4, N_F=2+1
- Nt = 4, m_{ud}/m_s ≈ 1/80-1/20



Caveat: Universality may be inapplicable on finite lattices
due to the non-locality.

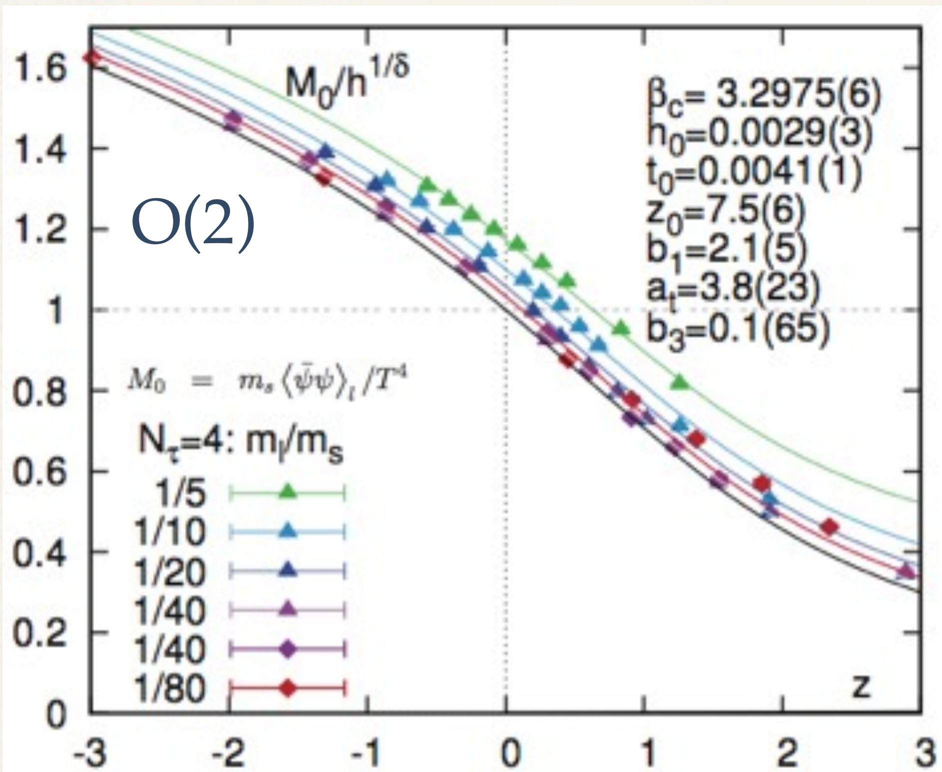
It turned out from intensive studies of T>0 QCD with staggered-type quarks around '09-'11,

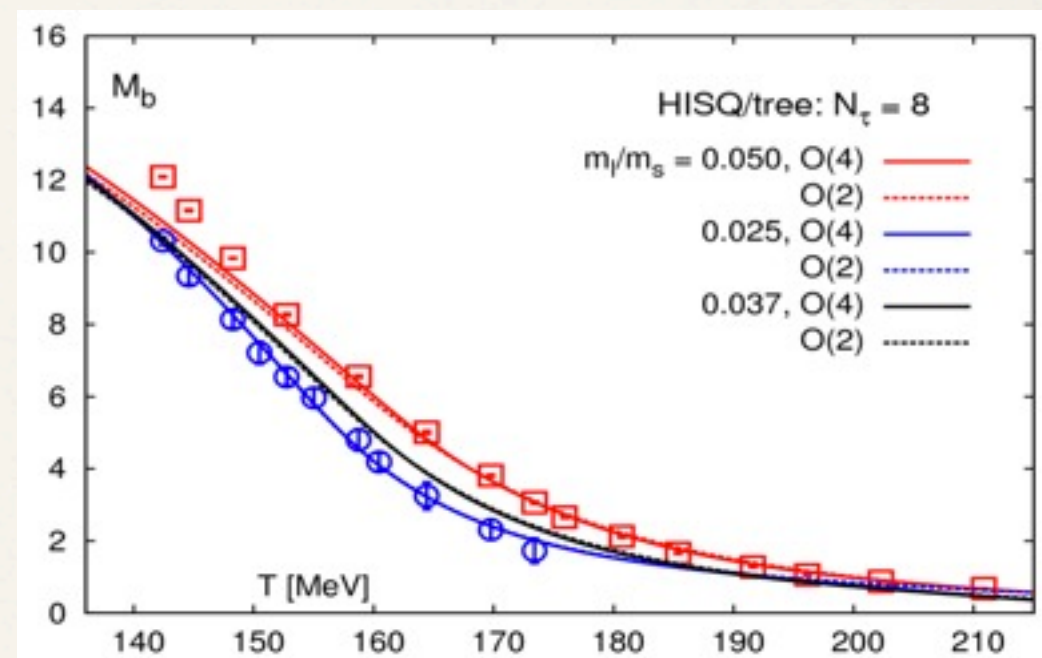
a good control of taste violation essential to extract physical conclusions with staggered-type quarks.

Improved staggered quarks (N_F=2+1)

Ejiri et al. (BNL-Bi) PRD80('09) (N_t = 4); Lat10 (N_t = 8)

- p4, Nt=4, m_s≈physical, m_l/m_s= 1/80 – 1/20 (m_π^{pNG} ≈ 75 – 150 MeV)





HotQCD @ QM11, Lat11

- HISQ
- m_s ≈ physical, m_l/m_s = 1/27 – 1/20

➡ Consistent with O(2)

Improved staggered quarks (N_F=2+1)

With staggered-type quarks, we should have taken the cont. limit prior to the chiral scaling studies.

If these properties remain also after taking the cont. limit, ...

➡ The phys. point dominated by the O(N) scaling

➡ 2nd order chiral transition for N_F=2

➡ Tricritical point locates lower than m_s^{phys}

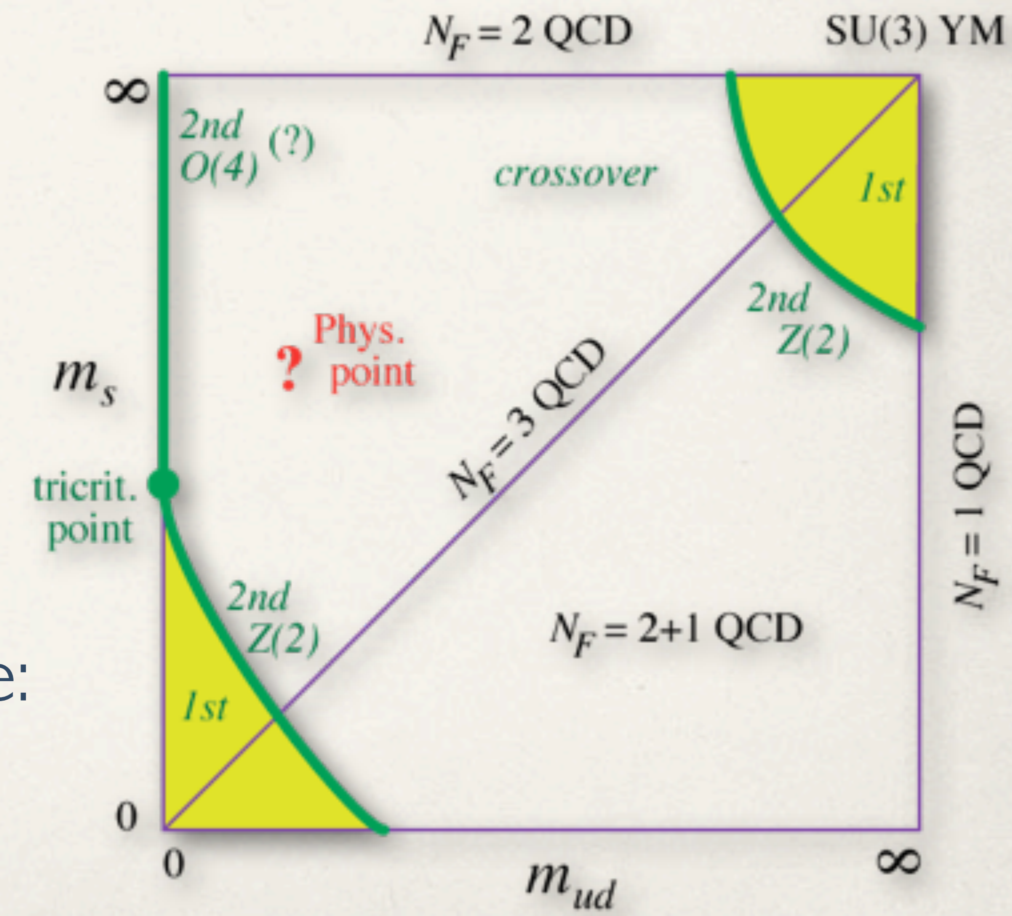
Consistent with small m_c for N_F=3:

$m_\pi^c \lesssim$ 45 MeV **HISQ** Nt=6 Ding et al. @ Lat11

$m_c/m_{ud}^{phys} \lesssim$ 0.12 **stout** Nt=6 Endrodi et al. @ Lat07

Consistent with broken U(1)$_A$ at $T \approx T_c$ and above:

HISQ, DW N_F=2+1 HotQCD @ Lat11
Overlap N_F=2 Cossu (JLQCD) @ Lat11



Fate of $U(1)_A$ at $T \approx Tc$

$U(1)_A$ explicitly broken at all T, but will restore at $T = \infty$.

Is $U(1)_A$ "effectively" restored at Tc ??

<= e.g. by formation of instanton-antiinstanton molecules

If so, the 1st order scenario becomes preferable,

though 2nd order transition not excluded.

N_F=2: $\vec{\pi} = \bar{q}\,\gamma_5 \dfrac{\vec{\tau}}{2}\, q$ $\qquad$ $\sigma = \bar{q}\, q$

SU(2)$_L$ xSU(2)$_R$

U(1)$_A$

$\vec{\delta} = \bar{q}\, \dfrac{\vec{\tau}}{2}\, q$ $\qquad$ $\eta = \bar{q}\,\gamma_5\, q$

^
disconnected diagrams required

If U(1)$_A$ "restored" => π-δ, σ-η degeneracy

Fate of $U(1)_A$ at T>Tc

If $U(1)_A$ "restored" => π-δ degeneracy

=> $\chi_\pi = \chi_\delta$ (note: =>'s are not <=>)

where $\chi_\pi = \frac{T}{V}\langle \mathrm{Tr} M^{-1}\gamma_5 M^{-1}\gamma_5\rangle$

$$\chi_\delta = \frac{T}{V}\langle \mathrm{Tr} M^{-1} M^{-1}\rangle$$

Banks-Casher: $-\langle\bar{q}q\rangle \xrightarrow{V\to\infty} \int_0^\infty d\lambda \, \frac{2m_q\,\rho(\lambda)}{\lambda^2+m_q^2} \xrightarrow{m_q\to 0} \pi\rho(0)$

$SU(N_F)_A$ restoration <=> ρ(0)=0 in the massless limit.

$$\chi_\pi - \chi_\delta \xrightarrow{V\to\infty} \int_0^\infty d\lambda \, \frac{4m_q^2\,\rho(\lambda)}{(\lambda^2+m_q^2)^2} \xrightarrow{m_q\to 0} ??$$

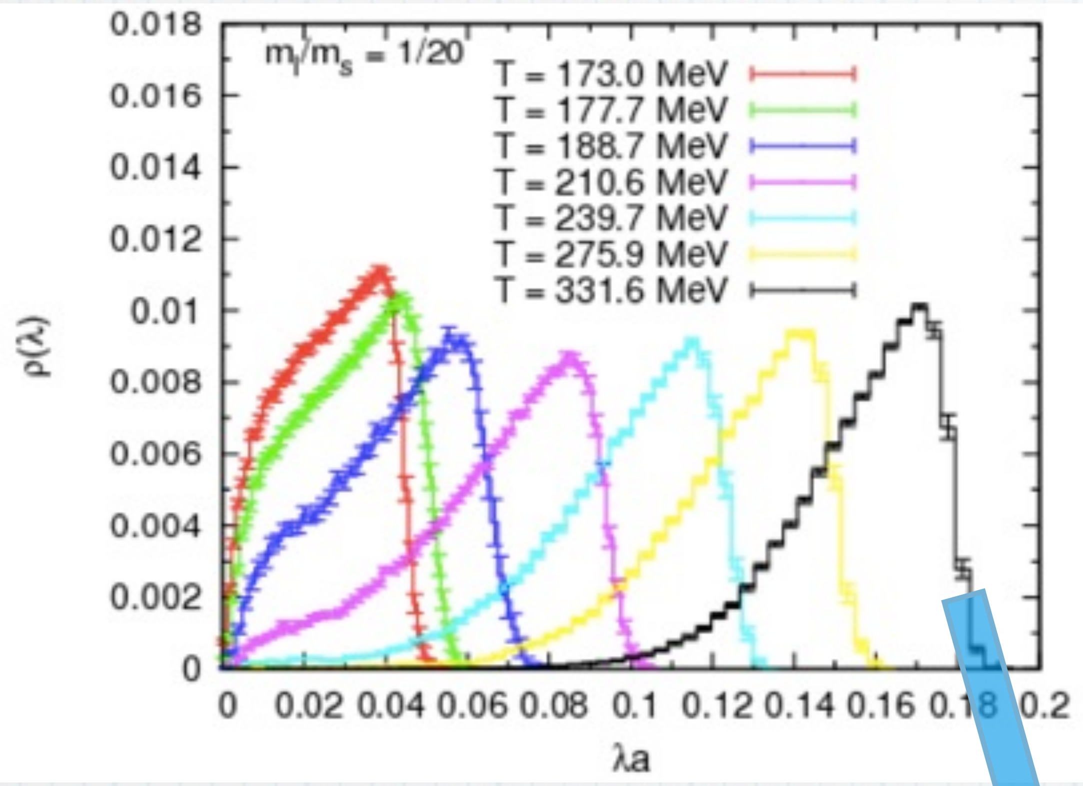
$$\rho(\lambda) \sim m^a\lambda^b \quad (a+b>0) \implies \chi_\pi - \chi_\delta \neq 0 \text{ if } a+b\leq 1$$

at $T > T_c$

Bazavov et al., arXiv:1205.3535

Fate of $U(1)_A$ at T>Tc

* Ohno (HotQCD) N_F=2+1 HISQ Nt=8, $V=32^3$-48^3

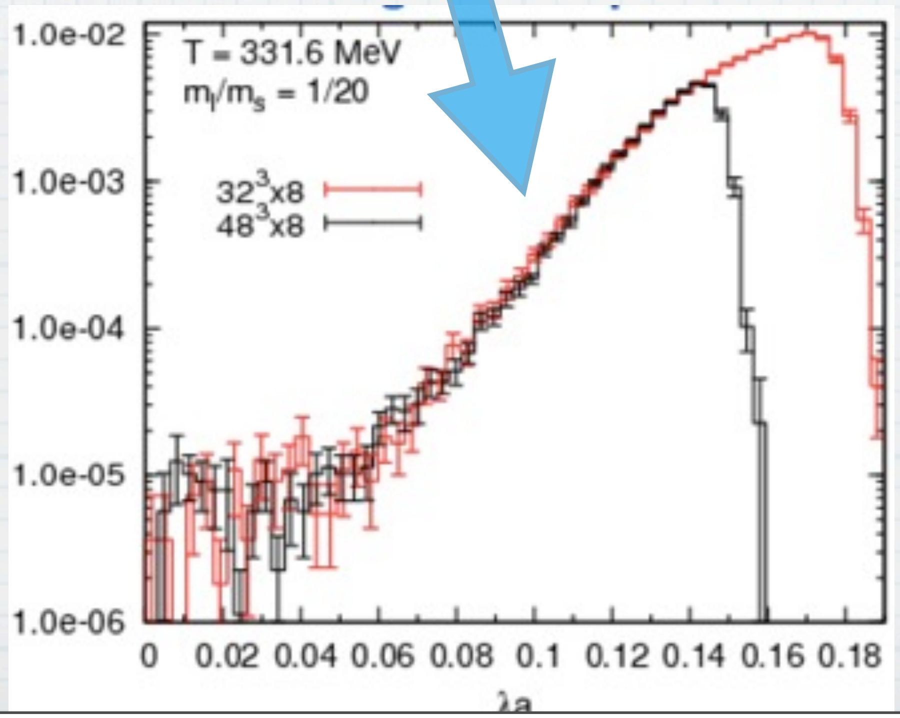


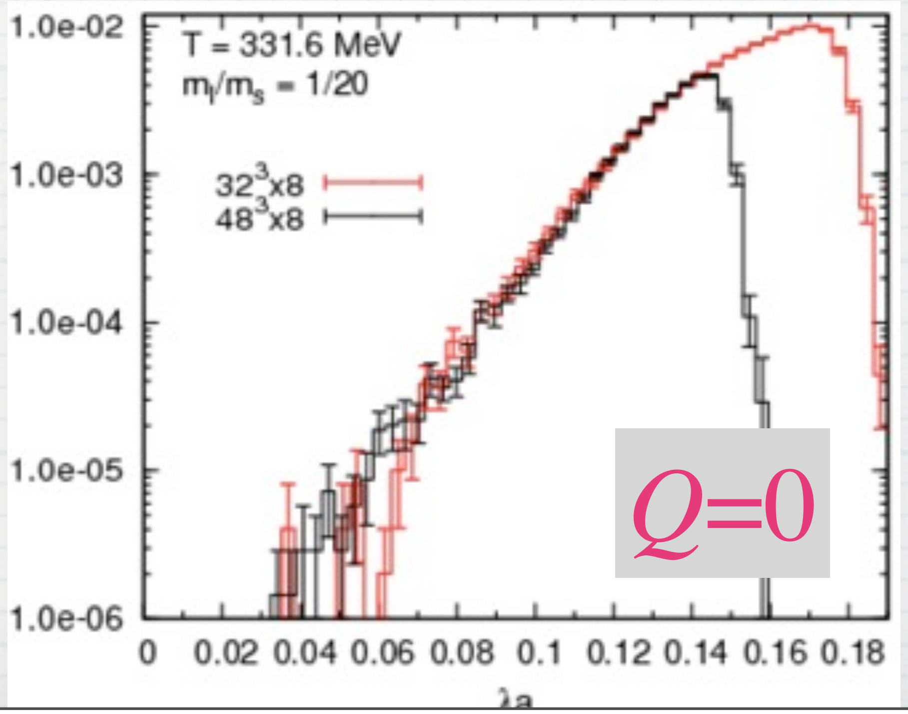
V-indep. tails remain => rho(0)≠ 0

But they are $Q\neq 0$ contributions.
=> large statistics needed to conclude at large *V*.

i.e. not clear

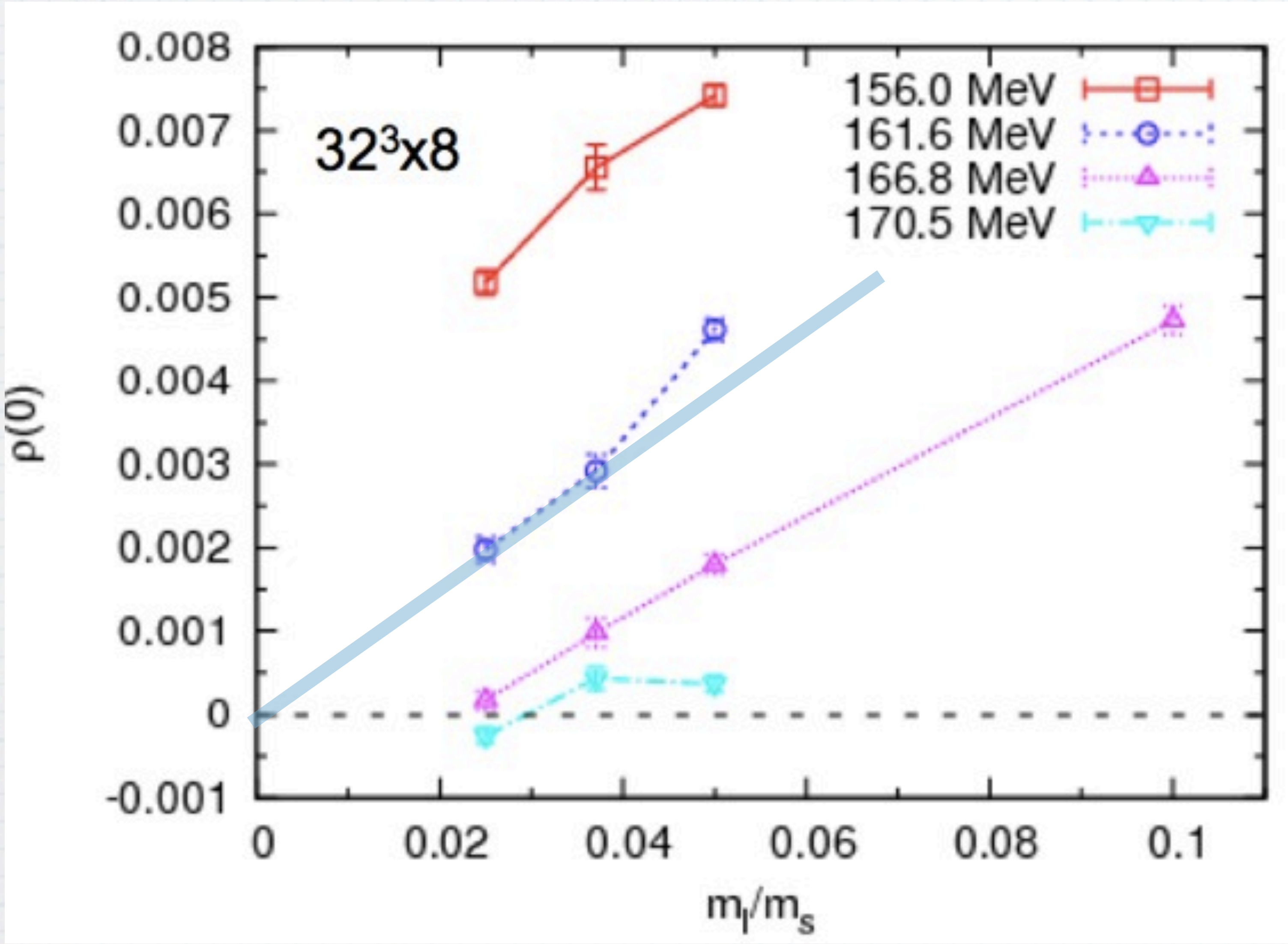
log-scale





Fate of $U(1)_A$ at T>Tc

* Ohno (HotQCD) N_F=2+1 HISQ Nt=8, V=32^3-48^3



c.f. T_{pc} estimated from the chiral susceptibility:
157(3) MeV (m_l/m_s=1/40)
162.9(1.8) MeV (m_l/m_s=1/20)

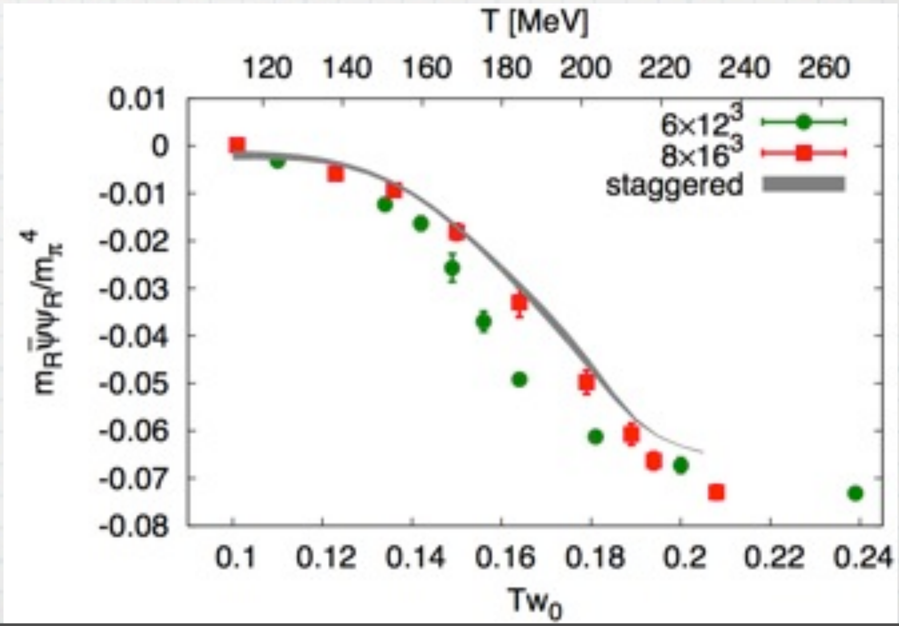
Assuming ρ(0) = 0 in the chiral limit, ρ(0) in the small quark mass region seems to linearly approach the origin up to *T* = 161.6 MeV.

This suggests that $U_A(1)$ symmetry remains broken in the chiral limit just above T_c.

Fate of $U(1)_A$ at T>Tc

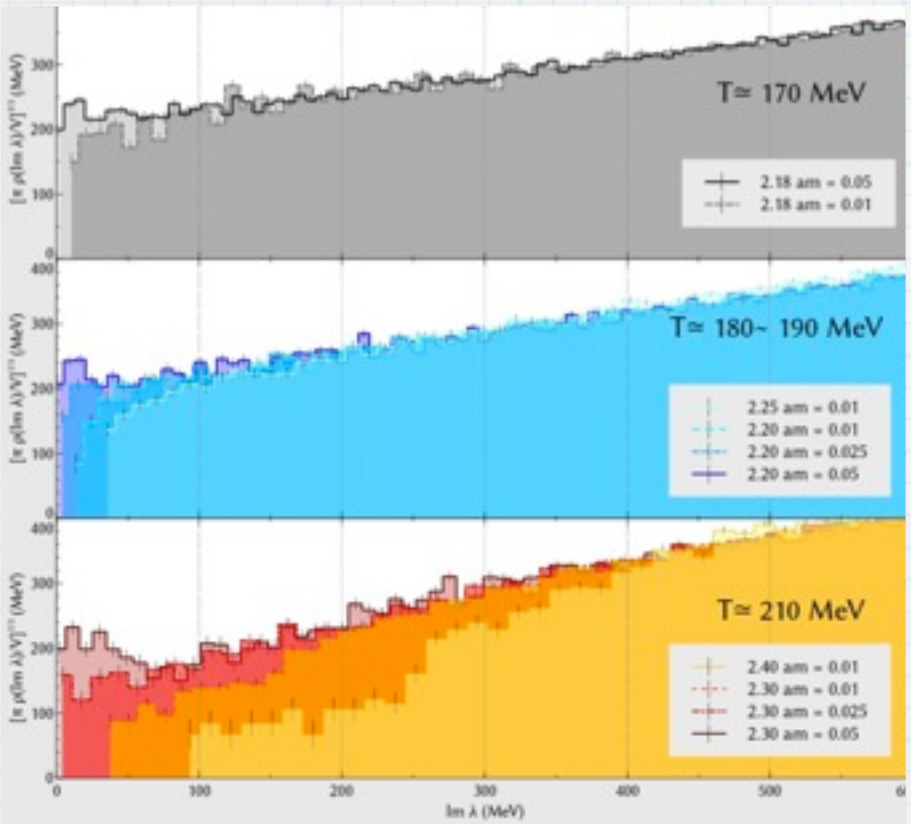
	SU(N)XSU(N)	UA(1)
Staggered	Remnant U(1)	Broken
Wilson	Broken	Broken
Domain Wall	Exact (for L → ∞)	Exact (for L → ∞)
Overlap	Exact	Exact

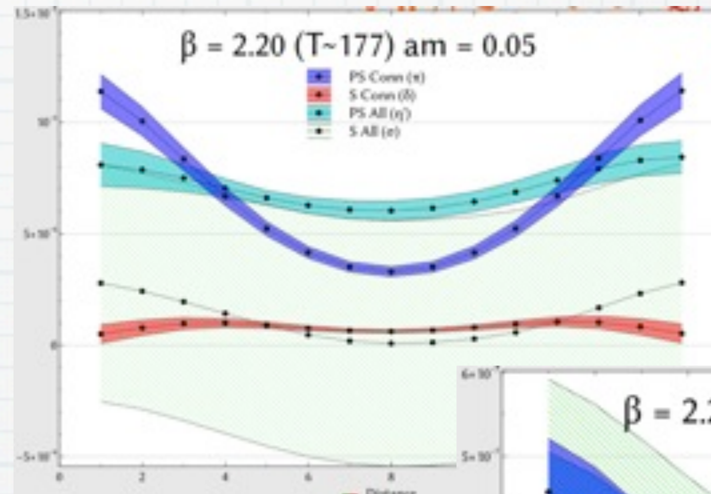
* **Cossu (JLQCD)** N_F=2 overlap + fixed-topology Iwasaki gauge
* **Lin (HotQCD)** N_F=2+1 DW + Iwasaki
* **Krieg (Budapest-Wuppertal)**
  N_F = 2+1 overlap + fixed-topology Symanzik
  m_π=350MeV, 12^3x6, 16^3x8
  => good agreement with stag.

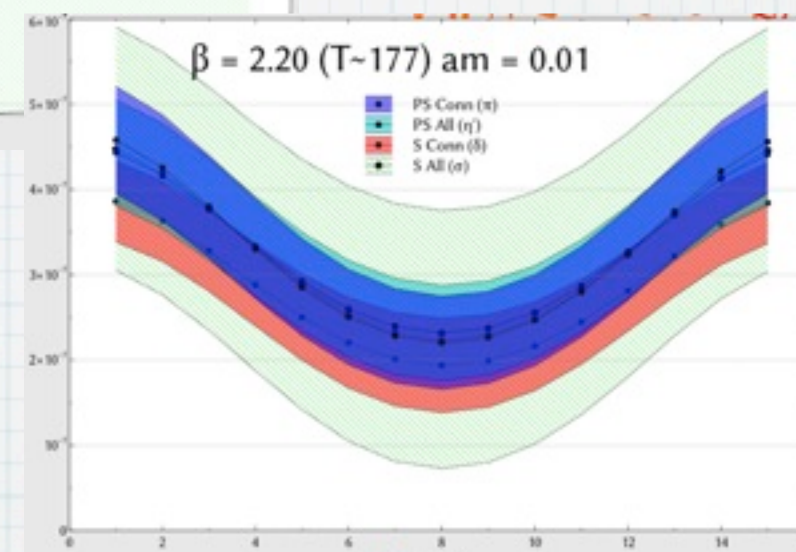


Fate of $U(1)_A$ at T>Tc

* **Cossu (JLQCD)** N_F=2 overlap + fixed-topology Iwasaki gauge
  Nt=8, V=16^3, m_π≈290MeV







- Full QCD spectrum shows <u>a gap at high temperature</u> even at pion masses ~250 MeV
- Correlators <u>show degeneracy of all channels</u> when mass is decreased
- Results support effective restoration of $U(1)_A$ symmetry

at these T's. / How about at Tc ?? / V-dep. should be checked.

Fate of $U(1)_A$ at T>Tc

* **Lin (HotQCD)** N_F=2+1 DW + Iwasaki gauge
also arXiv:1205.3535

V=64^3 in progress

Nt=8, V=16^3-32^3, m_π=200MeV DSDR (or Ls=96) to reduce m_{res}

DSDR allows topological tunnelings

Lowest 100 eigenvalues:

FIGURE: 159MeV

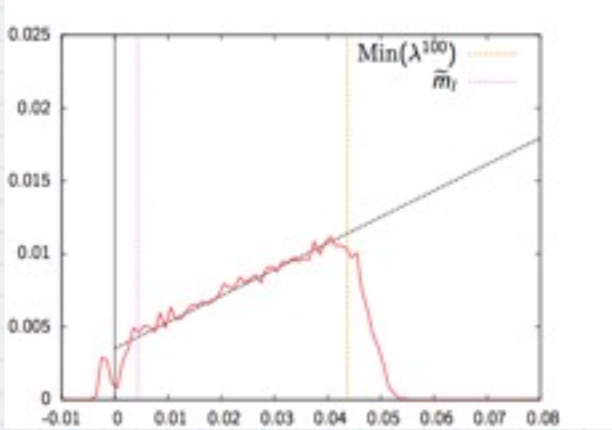
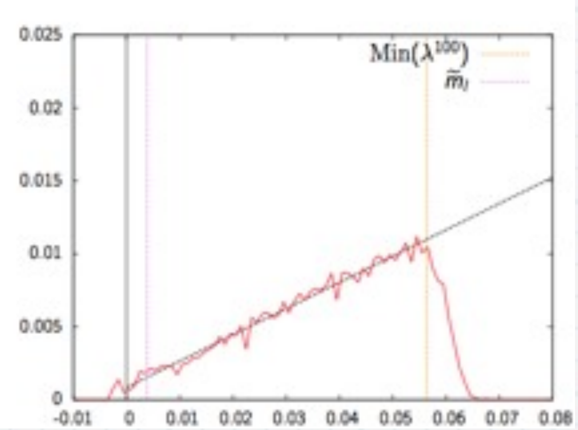


FIGURE: 168MeV



T =177 MeV

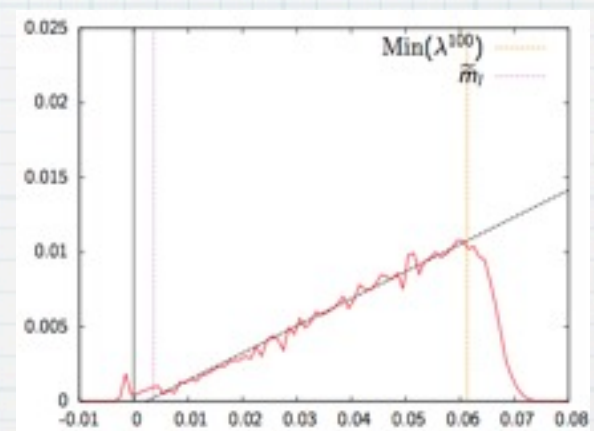


FIGURE: 186MeV

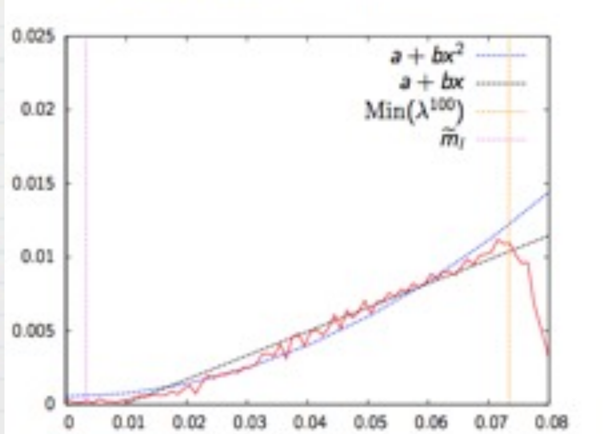
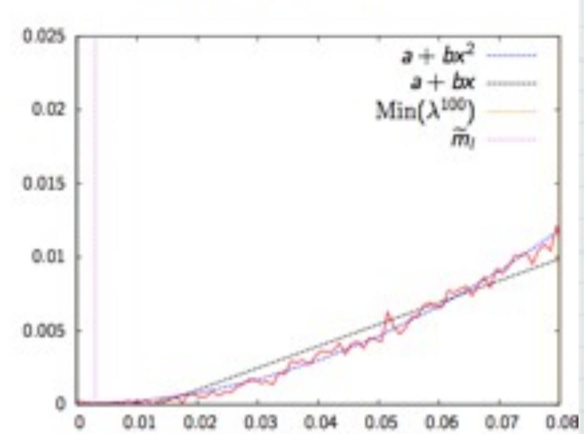


FIGURE: 195MeV



Intercept ~ 0
Linear slope visible.

$$\frac{\chi_\pi - \chi_\delta}{T^2} = 48.72+9.70$$

Consistent with their correlation functions.

Fate of $U(1)_A$ at T>Tc

* S. Aoki
  N_F=2 Chiral WT of Gisparg-Wilson fermions

$$\rho^A(\lambda) \equiv \lim_{V\to\infty} \frac{1}{V} \sum_n \delta\left(\lambda - \sqrt{\bar{\lambda}_n^A \lambda_n^A}\right) = \sum_{k=0}^{\infty} \rho_k^A \frac{|\lambda|^k}{k!}$$

$$D(A)\phi_n^A = \lambda_n^A \phi_n^A$$

$$\langle \rho_0^A \rangle_m = O(m^4) \qquad \langle \rho_1^A \rangle_m = O(m^2) \qquad \langle \rho_2^A \rangle_m = O(m^2)$$

$$\lim_{m\to 0} \chi^{\pi-\eta} = \lim_{m\to 0} \lim_{V\to\infty} \frac{N_f^2}{m^2 V^2} \langle Q(A)^2 \rangle_m = 0$$

at all *T*'s above *Tc*.

More generally, for $\mathcal{O} = \mathcal{O}_{n_1,n_2,n_3,n_4} = (P^a)^{n_1}(S^a)^{n_2}(P^0)^{n_3}(S^0)^{n_4}$

$$\lim_{m\to 0} \lim_{V\to\infty} \frac{1}{V^k} \langle \delta^0 \mathcal{O} \rangle_m = 0$$

δ^0 : singlet rotation

Breaking of U(1)_A symmetry is absent for these "bulk quantities".

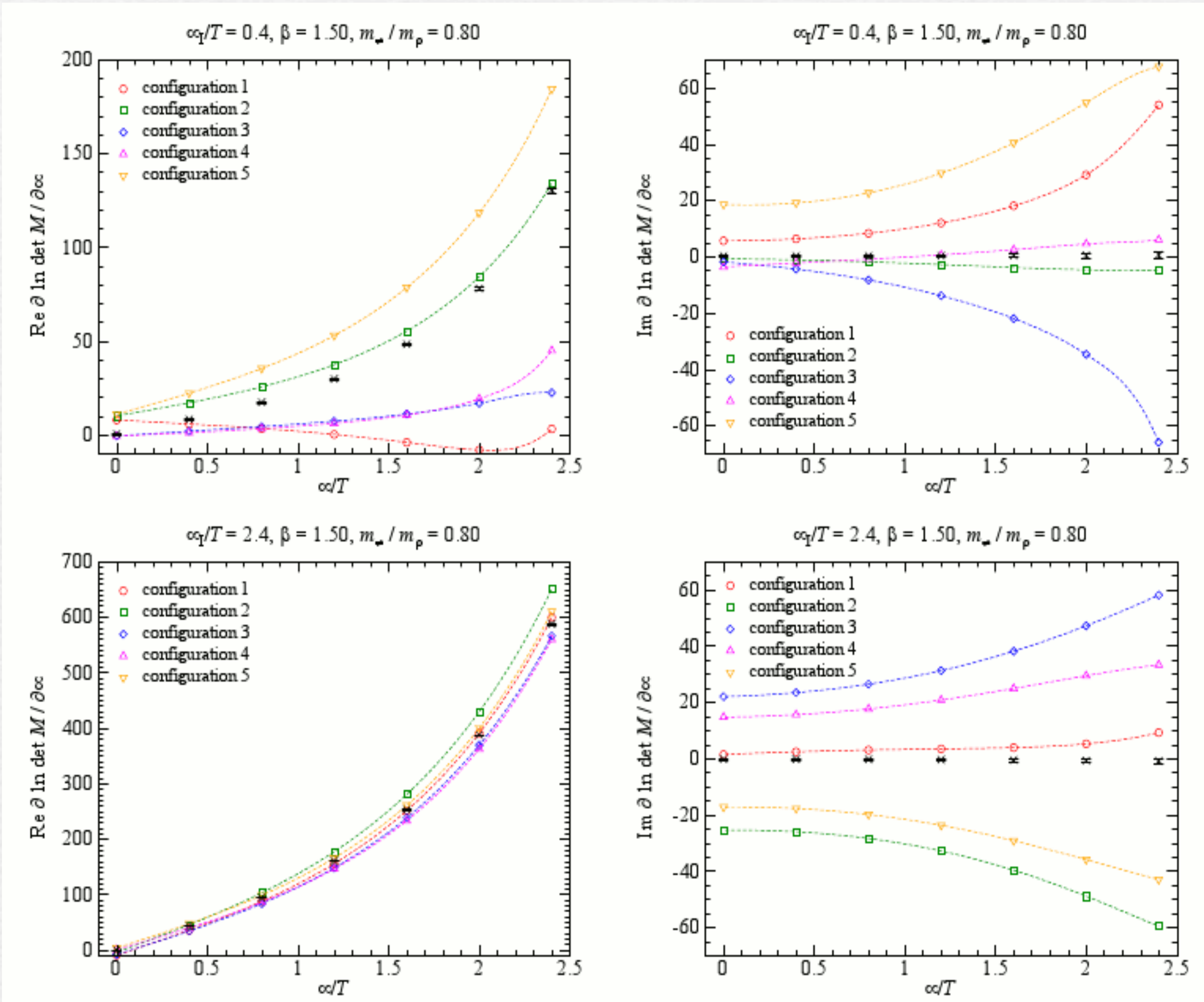
V-dep. important to check in the lattice results.

Light quark QCD

det$M$ via μ-integration

Random noise method to compute

$$\frac{\partial \ln \det M}{\partial(\mu/T)}$$



$\mu/T = 0.4$

$\mu/T = 2.4$

Real part Imaginary part

π-condensed phase

$$\boxed{W(P,F,\beta,m,\mu)=\left\langle e^{i\theta}\right\rangle_{P,F}\times W_0(P,F,\beta,m,\mu)}$$

$\left\langle e^{i\theta}\right\rangle = 0$ is suggested in the pion condensed phase by phenomenological studies.

[Han-Stephanov '08, Sakai et al. '10]

Near the phase boundary, large fluctuations in θ: expected.

$$\left\langle e^{i\theta}\right\rangle_{P,F}\to 0\quad\left(\ln\left\langle e^{i\theta}\right\rangle_{P,F}\to -\infty\right)$$

phase-quenched QCD

QCD