Non-perturbative Renormalization

Rainer Sommer

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References

to be completed

M. Lüscher, *Advanced lattice QCD*

P. Weisz: Les Houches lectures (especially for my first lecture)

RS: Nara lectures

M. Testa hep-th/9803147

A. Vladikas: Les Houches lectures (RI-MOM)

Reviews at various lattice conferences

Recent papers on Gradient Flow
...

NP renormalization of an effective theory:
RS: Les Houches lectures

QCD without CP-violating term, quark masses are real

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{2g_0^2}\,\mathrm{tr}\,\{F_{\mu\nu}F_{\mu\nu}\} + \sum_f \overline{\psi}_f\{D + m_f\}\psi_f$$

$$\mathcal{L}_{\mathrm{QCD}}(g_0, m_f) \leftrightarrow \overbrace{\begin{bmatrix} m_{\mathrm{proton}} \\ m_\pi \\ m_{\mathrm{K}} \\ m_{\mathrm{D}} \\ m_{\mathrm{B}} \end{bmatrix}}^{\text{Experiment}} \qquad (m_{\mathrm{u}} = m_{\mathrm{d}}, \quad \text{ignore top})$$

bare parameters $\rightarrow$ masses, observables

theory parametrized in terms of observables

Introduction:

What are we here interested in?

QCD without CP-violating term, quark masses are real

$$\mathcal{L}_{\rm QCD} = -\frac{1}{2g_0^2}\,\mathrm{tr}\,\{F_{\mu\nu}F_{\mu\nu}\} + \sum_f \overline{\psi}_f\{D + m_f\}\psi_f$$

$$\mathcal{L}_{\rm QCD}(g_0, m_f) \leftrightarrow \overbrace{\begin{bmatrix} m_{\rm proton} \\ m_\pi \\ m_{\rm K} \\ m_{\rm D} \\ m_{\rm B} \end{bmatrix}}^{\text{Experiment}} \qquad (m_{\rm u} = m_{\rm d}, \quad \text{ignore top})$$

bare parameters $\rightarrow$ masses, observables

theory parametrized in terms of observables

NP renormalization

LHC (and other collider physics):

$$p\bar{p} \to H \to \dots$$

SM (or MSSM) predictions depend on

renormalized perturbation theory (PT) in $\alpha_{\rm s}(\mu) \equiv \alpha_{\rm R}(\mu)$

$\mu = \mathrm{O}(10\mathrm{GeV}) \dots \mathrm{O}(300\mathrm{GeV})$

What is $\alpha_{\rm R}(\mu)$ in a given renormalization scheme?

What is $\Lambda_{\rm QCD}$:

$$\begin{aligned} m_{\rm proton} &= \# \times \Lambda_{\rm QCD} \\ \alpha_{\rm R}(\mu) &\overset{\mu/\Lambda \gg 1}{\sim} \frac{1}{b_0 \ln(\mu/\Lambda)} \left\{ 1 - \frac{b_1}{b_0^2 \ln(\mu/\Lambda)} \ln(\ln(\mu/\Lambda)) + \mathrm{O}(\ln(\mu/\Lambda)^{-2}) \right\} \end{aligned}$$

Weak decays (search for BSM physics) of quarks:

$$\begin{matrix}\text{effective theory} \\ \text{2-quark op's, 4-quark op's}\end{matrix} \quad \leftarrow \quad \begin{cases}\mathrm{SM} \\ \mathrm{BSM}\end{cases}$$

necessitates the **renormalization of composite fields**

What will we do?

- Renormalization in PT (repetition)
- RGE's, RGI
- NP renormalization (principle)
- Large scale ratios, step scaling functions (SSF)
- Finite volume schemes
- Gradient flow (new development)

- very incomplete covery of techniques
  concentrate on concepts

Renormalization in PT

Repetition

Consider continuum PT, $D = 4 - 2\epsilon$ dimensions as a regularisation

gauge-invariant, physical observable

G

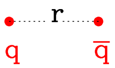
bare, regularised

$G_0(\epsilon, q, g_0, m_{0i})$

Example

force between static quarks

$F(r)$



$q \equiv 1/r$

G_0 is singular as $\epsilon \to 0$ *at fixed* q, g_0, m_{0i}

Renormalizability:

all observables G become **finite** after the

Renormalization:

dimensionful coupling in D dimensions

$$g_{\mathrm{R}}^2 \equiv g^2 \quad = Z_g(\epsilon, g^2)\mu^{-2\epsilon} g_0^2$$

$$m_{\mathrm{R},i} \equiv m_i \quad = Z_m(\epsilon, g^2) m_{0i}$$

$$G_{\mathrm{R}}(\mu, q, g, m_i) = \lim_{\epsilon \to 0} G_0(\epsilon, q, \underbrace{Z_g^{-1/2} g \mu^{\epsilon}}_{g_0}, \underbrace{Z_m^{-1} m_i}_{m_{0i}})$$

The limit exists with

$$Z_x = 1 + g^2 z_{x,1}\epsilon^{-1} + g^4[z_{x,2}\epsilon^{-2} + z_{x,3}\epsilon^{-1}] + \ldots$$

“minimal subtraction” (of ϵ poles; only those)

Renormalization in PT

$\overline{\mathrm{MS}}$ scheme

Renormalizability:

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dimensionful coupling in D dimensions

$$
\begin{aligned}
g_{\mathrm{R}}^2 \equiv \bar{g}^2 \quad &= Z_g(\epsilon, \bar{g}^2)\mu^{-2\epsilon} g_0^2 \\
m_{\mathrm{R},i} \equiv \bar{m}_i \quad &= Z_m(\epsilon, \bar{g}^2) m_{0i}
\end{aligned}
$$

mass–independent
renormalization scheme

$$
G_{\mathrm{R}}(\mu, q, \bar{g}, \bar{m}_i) = \lim_{\epsilon \to 0} G_0(\epsilon, q, \underbrace{Z_g^{-1/2} \bar{g} \mu^{\epsilon}}_{g_0}, \underbrace{Z_m^{-1} \bar{m}_i}_{m_{0i}})
$$

The limit exists with

$$
Z_x = 1 + \bar{g}^2 z_{x,1} \epsilon^{-1} + \bar{g}^4 [z_{x,2} \epsilon^{-2} + z_{x,3} \epsilon^{-1}] + \ldots
$$

"minimal subtraction" (of ϵ poles; only those) on the lattice: $G_0(a,q,g_0,m_{0i})$

$$\begin{aligned} g_{\rm lat}^2 \equiv g^2 &= Z_g(\ln(a\mu),g^2)\,g_0^2 \\ m_{{\rm lat},i} \equiv m_i &= Z_m(\ln(a\mu),g^2)\,\hat m_{{\rm q},i} \qquad (*) \\ G_{\rm R}^{\rm lat}(\mu,q,g_{\rm lat},m_i) &= \lim_{a\to 0} G_0(a,q,\underbrace{Z_g^{-1/2}g}_{g_0},\ \underbrace{m_{0i}}_{Z_m^{-1}m_{{\rm lat},i}}\ ) \end{aligned}$$

when $r_m = 1$

The limit exists (continuum limit) with different Z_x !

$$Z_x = 1 + g^2 z_{x,1}\ln(a\mu) + g^4[z_{x,2}(\ln(a\mu))^2 + z_{x,3}\ln(a\mu)] + \ldots$$

“**lattice minimal subtraction**” (of logs $\ln(a\mu)$; only those) $g = g_{\rm lat}, m = m_{\rm lat}$

Proven to all orders of PT for Wilson reg'n [T. Reisz].

Expected also non-perturbatively and for other regularisations (universality).

(*) $\hat m_{{\rm q},i}$ are bare subtracted masses,

$$\begin{aligned} \hat m_{{\rm q},i} &= m_{{\rm q},i} + (r_m(g_0)-1)\,\frac{1}{N_{\rm f}}\,{\rm tr}\,M_{\rm q} \\ m_{{\rm q},i} &= m_{0i} - m_c(g_0)\,,\ \ M_{\rm q} = {\rm diag}(m_{{\rm q},1},m_{{\rm q},2},\ldots) \end{aligned}$$

with sufficient chiral symmetry: $r_m = 1, m_c = 0$

The limit is universal (does not depend on the regularisation) after changing the renormalization scheme: finite renormalization

$$
\begin{aligned}
g_{\mathrm{lat}}^2 &= \chi_g(g_{\mathrm{MS}})\, g_{\mathrm{MS}}^2\,, \quad \chi_g(g) = 1 + \chi_g^{(1)} g^2 + \ldots \\
m_{\mathrm{lat},i} &= \chi_m(g_{\mathrm{MS}})\, m_{\mathrm{MS},i}\,, \quad \chi_m(g) = 1 + \chi_m^{(1)} g^2 + \ldots \\
G_{\mathrm{R}}(\mu, q, g_{\mathrm{MS}}, m_{\mathrm{MS},i}) &= G_{\mathrm{R}}^{\mathrm{lat}}(\mu, q, \underbrace{\chi_g(g_{\mathrm{MS}})\, g_{\mathrm{MS}}^2}_{g_{\mathrm{lat}}}, \underbrace{\chi_m(g_{\mathrm{MS}})\, m_{\mathrm{MS},i}}_{m_{\mathrm{lat},i}})
\end{aligned}
$$

Renormalization in PT

μ-dependence

Renormalized masses and coupling depend on μ:

$$\lim_{a\to 0} \mu\partial_\mu g_{\rm lat}\big|_{g_0,m_{\mathrm{q},i}} \equiv \beta_{\rm lat}(g_{\rm lat}) = -g_{\rm lat}^3\,(b_0 + b_1 g_{\rm lat}^2 + \ldots)$$

$$\lim_{a\to 0} \mu\partial_\mu m_{{\rm lat},i}\big|_{g_0,m_{\mathrm{q},i}} \equiv \tau_{\rm lat}(g_{\rm lat})\, m_{{\rm lat},i}$$

$$\tau_{\rm lat}(g_{\rm lat}) = -g_{\rm lat}^2\,(d_0 + d_1^{\rm lat} g_{\rm lat}^2 + \ldots)$$

$$b_0 = \frac{1}{(4\pi)^2}\left(11 - \frac{2}{3}N_{\rm f}\right), \quad d_0 = \frac{8}{(4\pi)^2}$$

$$b_1 = \frac{1}{(4\pi)^4}\left(102 - \frac{38}{3}N_{\rm f}\right)$$

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or in the MS-scheme

$$\lim_{\epsilon\to 0} \mu\partial_\mu g_{\rm MS}|_{g_0, m_{0,i}} \equiv \beta_{\rm MS}(g_{\rm MS}) = -g_{\rm MS}^3\,(b_0 + b_1 g_{\rm MS}^2 + \ldots)$$

$$\lim_{\epsilon\to 0} \mu\partial_\mu m_{{\rm MS},i}|_{g_0, m_{0,i}} \equiv \tau_{\rm MS}(g_{\rm MS})\, m_{{\rm MS},i}$$

$$\tau_{\rm MS}(g_{\rm MS}) = -g_{\rm MS}^2\,(d_0 + d_1^{\rm MS} g_{\rm MS}^2 + \ldots)$$

A physical quantity $G_{\rm R}$ does not depend on μ, since G_0 does not depend on μ:

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}G_0 = 0 \quad \rightarrow \quad \mu\frac{\mathrm{d}}{\mathrm{d}\mu}G_{\rm R}(\mu,q,g,m_i) \;=\; 0$$
$$(\mu\partial_\mu + \beta(g)\partial_g + \tau(g)\, m_i\partial_{m_i})\, G_{\rm R} \;=\; 0$$

The general solution of the RGE can be expressed in terms of special solutions:

1. m_i, q-independent function $\Lambda(\mu, g)$: $m_i\partial_{m_i}\Lambda = 0$

$$(\mu\partial_\mu + \beta(g)\partial_g)\,\Lambda = 0$$
$$\Lambda = \mu\,\varphi_g(g)\,, \qquad \varphi_g \text{ dimensionless}$$
$$(1+\beta(g)\partial_g)\varphi_g = 0$$
$$\varphi_g = \exp\left\{-\int^g \mathrm{d}x\,\frac{1}{\beta(x)}\right\} \times \text{constant}$$

A physical quantity G_R does not depend on μ, since G_0 does not depend on μ:

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}G_0 = 0 \quad \rightarrow \quad \mu\frac{\mathrm{d}}{\mathrm{d}\mu}G_\mathrm{R}(\mu,q,g,m_i) = 0$$
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$$\varphi_g = \exp\left\{-\int^g \mathrm{d}x\,\tfrac{1}{\beta(x)}\right\} \times \text{constant}$$
$$= (b_0 g^2)^{-b_1/(2b_0^2)}\,\mathrm{e}^{-1/(2b_0 g^2)}\exp\left\{-\int_0^g \mathrm{d}x\left[\tfrac{1}{\beta(x)} + \tfrac{1}{b_0 x^3} - \tfrac{b_1}{b_0^2 x}\right]\right\}$$

A physical quantity G_{R} does not depend on an arbitrarily introduced μ:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\mu} G_{\mathrm{R}} &= 0 \\
\left(\mu\partial_\mu + \beta(g)\partial_g + \tau(g)\, m_i \partial_{m_i}\right) G_{\mathrm{R}} &= 0
\end{aligned}
$$

The general solution can be expressed in terms of special solutions:

2. m_i dependent function, independent of μ and $m_j,\ j \neq i$, $M_i(m_i, g)$:

$$
\begin{aligned}
&(m_i \partial_{m_i} + \beta(g)\partial_g)\, M_i = 0 \\
&M_i = m_i\, \varphi_m(g)\,, \qquad \varphi_m \text{ dimensionless} \\
&(\tau(g) + \beta(g)\partial_g)\varphi_m = 0 \\
&\varphi_m = \exp\left\{ -\int^g \mathrm{d}x \frac{\tau(g)}{\beta(x)} \right\} \times \text{ constant}
\end{aligned}
$$

A physical quantity G_R does not depend on an arbitrarily introduced μ:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\mu} G_\mathrm{R} &= 0 \\ \left(\mu\partial_\mu + \beta(g)\partial_g + \tau(g)\, m_i \partial_{m_i}\right) G_\mathrm{R} &= 0 \end{aligned}$$

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$$(m_i\partial_{m_i} + \beta(g)\partial_g)\, M_i = 0$$

$$M_i = m_i\, \varphi_m(g)\,, \qquad \varphi_m \text{ dimensionless}$$

$$(\tau(g) + \beta(g)\partial_g)\varphi_m = 0$$

$$\varphi_m = \exp\left\{ - \int^g \mathrm{d}x \frac{\tau(g)}{\beta(x)} \right\} \times \text{ constant}$$

$$= \left(2b_0 g^2\right)^{-d_0/(2b_0)} \exp\left\{ - \int_0^g \mathrm{d}x \left[\frac{\tau(x)}{\beta(x)} - \frac{d_0}{b_0 x} \right] \right\}$$

Now take G_R independent of q; example: $G_\mathrm{R} = m_\mathrm{hadron}$

$$\begin{aligned} G_\mathrm{R} &= G_\mathrm{R}(\mu, g(\mu), m_i(\mu)) \\ \mu \text{ independent :} \quad &= G_\mathrm{R}(k\Lambda, \underbrace{g(k\Lambda)}_{\varphi_g^{-1}(1/k)}, M_i/\underbrace{\varphi_m(g(k\Lambda))}_{\varphi_m(\varphi_g^{-1}(1/k))}) \\ \to \quad G_\mathrm{R} &= G^\mathrm{RGI}(\Lambda, M_i) \qquad \text{any } k\text{, e.g. } k=1 \end{aligned}$$

with mass dimension 1: $[G_\mathrm{R}] = 1$, e.g. m_hadron

$$m_\mathrm{hadron} = \Lambda \bar{f}_h(M_i/\Lambda)$$

Λ, M_i: fundamental parameters of QCD ($N_\mathrm{f} + 1$ parameters)

Renormalization Group Invariants (RGI)

Renormalization Group Invariants (RGI)

- non–perturbatively defined

  with the standard (undoubted) assumtions:
  NP “corrections” to RG functions vanish as $\mu^{-\eta}$, $\eta > 0$
  e.g. renormalons, instantons
- our job is to determine them

in the chiral limit $M_i = 0$

$$\begin{aligned} m_{\rm hadron} &= \Lambda \bar{f}_h(0) = \bar{f}_h(0)\, \mu \, \mathrm{e}^{-1/(2b_0 g(\mu)^2)} \times \ldots \\ & \partial_g^n m_{\rm hadron}\big|_{g=0} = 0 \\ \rightarrow & \quad m_{\rm hadron} = 0 \text{ to all orders of PT} \end{aligned}$$

$m_{\rm hadron}, \Lambda, M_i$ are non-perturbative quantities

Exercises

- Show that
$$M_i^s = M_i^{s'}$$
where s, s' are different schemes.
- Show that
$$\Lambda^s = k\,\Lambda^{s'}$$
Determine k in terms of $\chi_g^{(1)}\,,\ b_0$.
- What is needed to determine $\chi_g^{(1)}$?

$q = 1/r$ large: short (Euclidean) distances

$$\begin{aligned} G_\mathrm{R} &= G_\mathrm{R}(\mu, q, g(\mu), m_i(\mu)) \qquad \text{dimensionless (e.g. } r^2F(r)) \\ &= P(q/\mu, g(\mu), m_i(\mu)/q) \\ &= P(1, g(q), m_i(q)/q)\,, \quad \varphi_g(g(q)) = \Lambda/q \\ & \qquad\qquad m_i(q) = M_i/\varphi_m(g) \end{aligned}$$

- yields the RG improved prediction for P
- becomes more and more accurate for $q \to \infty$

$$\begin{aligned} g^2(q) &= \frac{1}{b_0 t}\left\{1 - \frac{b_1}{b_0^2 t}\ln(t) + \mathrm{O}(t^{-2})\right\} \\ &\to \; 0 \text{ as } t \to \infty \qquad t = 2\ln(q/\Lambda) \\ m_i(q) &= M_i\,\left(\tfrac{2}{t}\right)^{d_0/2b_0}\{1 + \ldots\} \end{aligned}$$

- unphysical μ-dependence of the coupling turned into physical q dependence

$q = 1/r$ large: short (Euclidean) distances
we also see that

$$G_{\rm R} \quad = \quad P(1, g(q), m_i(q)/q) \stackrel{q \gg \Lambda, M_i}{\sim} P(1, g(q), 0)$$

mass effects disappear at short distances

For weak interactions, chiral symmetry breaking order parameter, ...

Local composite fields ("operators")

$$
\begin{aligned}
S^{rs}(x) &= \overline{\psi}_r(x)\psi_s(x)\,, \quad P^{rs}(x) = \overline{\psi}_r(x)\gamma_5\psi_s(x) \quad r \neq s \text{ flavor indices} \\
S(x) &= S^{rr}(x) \equiv \sum_{r=1}^{N_\mathrm{f}} S^{rr}(x)\,, \quad P(x) = P^{rr}(x) \\
A_\mu^{rs}(x) &= \overline{\psi}_r(x)\gamma_\mu\gamma_5\psi_s(x)\ldots \\
O_{\mathrm{LL}}^{rs}(x) &= \overline{\psi}_r(x)\gamma_\mu(1-\gamma_5)\psi_s(x)\overline{\psi}_r(x)\gamma_\mu(1-\gamma_5)\psi_s(x)
\end{aligned}
$$

In contrast to non-local composite fields

- Wilson loop
- smeared fields

$$S_t^{rs}(x) \qquad t \text{ a proper smearing parameter}$$

$\rightarrow$ see the final lecture

$$\langle \phi_{\mathrm{R}\,1}(x_1)\phi_{\mathrm{R}\,2}(x_2)\phi_{\mathrm{R}\,3}(x_3)\phi_{\mathrm{R}\,4}(x_4)\ldots\rangle_{\text{path integral average}}$$

is finite for $x_i \neq x_j$ for $i \neq j$ with

dimensional regularisation, MS

$$\phi_{\mathrm{R},i}^{(D)} = \sum_j Z_{ij}(\epsilon, g^2)\, \Phi_j^{(D)}\,, \qquad [\Phi_j^{(D)}] = [\Phi_i^{(D)}] = D$$

e.g. $[S] = [P^{rs}] = 3$

lattice MS

$$\phi_{\mathrm{R},i}^{(D)} = \sum_j Z_{ij}(\ln(a\mu), g^2)\, \Phi_{\mathrm{sub},j}^{(D)}\,, \quad [\Phi_{\mathrm{sub},j}^{(D)}] = [\Phi_i^{(D)}] = D$$

$$\Phi_{\mathrm{sub},j}^{(D)} = \Phi_j^{(D)} + \sum_{n\geq 1} \mathbf{a}^{-\mathbf{n}} \sum_k d_{jk}(g_0)\Phi_k^{(D-n)}$$

Subtraction coefficients d_{jk} can be chosen purely as functions of g_0, not $\ln(a\mu)$ [M. Testa, hep-th/9803147, Sect. 2]

Exercise: Go through the argument in hep-th/9803147. Does it hold beyond PT?

$$\phi_{\mathrm{R},i}^{(D)} = \sum_j Z_{ij}(\ln(a\mu), g^2)\, \Phi_{\mathrm{sub},j}^{(D)}\,, \quad [\Phi_{\mathrm{sub},j}^{(D)}] = [\Phi_i^{(D)}] = D$$

$$\Phi_{\mathrm{sub},j}^{(D)} = \Phi_j^{(D)} + \sum_{n\geq 1} \mathbf{a}^{-\mathbf{n}} \sum_k d_{jk}(g_0)\Phi_k^{(D-n)}$$

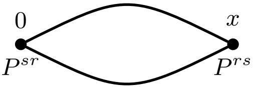
An example

$$S_{\mathrm{R}}(x) = Z_{\mathrm{S}}^{\mathrm{sing}}(\ln(a\mu), g^2)\left[\overline{\psi}(x)\psi(x) + \mathbf{a}^{-\mathbf{3}} d_1(g_0)\right]$$

- “mixing with the unit-operator”
- in theories without exact chiral symmetry
- in the chiral limit
  otherwise: $\frac{m^n}{a^{3-n}}$ terms

Just work with a simple example:

$$
\begin{aligned}
G_0(a,x,g_0) &= \langle P^{rs}(x)P^{sr}(0)\rangle \\
G_{\mathrm{R}}^{\mathrm{cont}}(\mu,x,g) &= \lim_{a\to 0} G_{\mathrm{R}}^{\mathrm{lat}}(\mu,x,g,a\mu) \\
G_{\mathrm{R}}^{\mathrm{lat}}(\mu,x,g,a\mu) &= \langle P_{\mathrm{R}}^{rs}(x)P_{\mathrm{R}}^{sr}(0)\rangle \\
&= Z_{\mathrm{P}}^2(a\mu,g_0)\,G(a,x,g_0)
\end{aligned}
$$



RGE:

$$
\mu\frac{\mathrm{d}}{\mathrm{d}\mu}G_0(a,x,g_0) = 0 = \mu\frac{\mathrm{d}}{\mathrm{d}\mu}Z_{\mathrm{P}}^{-2}G_{\mathrm{R}}
$$

$$
\begin{aligned}
\to \qquad & Z_{\mathrm{P}}^2\mu\frac{\mathrm{d}}{\mathrm{d}\mu}[Z_{\mathrm{P}}^{-2}G_{\mathrm{R}}] = 0 \\
& (\mu\partial_\mu + \beta(g)\partial_g + \tau(g)\,m_i\partial_{m_i} - 2\gamma)\,G_{\mathrm{R}} = 0 \\
& \gamma = Z_{\mathrm{P}}^{-1}\,\mu\partial_\mu Z_{\mathrm{P}}(\mu a, g_0)
\end{aligned}
$$

Now turn to a renormalization group invariant form:

$$P_{\mathrm{RGI}}^{rs} = \varphi_{\mathrm{P}}(\mu, g) P_{\mathrm{R}}^{rs}$$

$$\text{with} \qquad (\mu\partial_\mu + \beta\partial_g)\varphi_{\mathrm{P}} = -\gamma\,\varphi_{\mathrm{P}}$$

then

$$G_{\mathrm{RGI}} = \langle P_{\mathrm{RGI}}^{rs}(x) P_{\mathrm{RGI}}^{sr}(0)\rangle = \varphi_{\mathrm{P}}^2\, G_{\mathrm{R}}$$

$$\varphi_{\mathrm{P}} = \exp\left\{-\int^g \mathrm{d}x\,\frac{\gamma(g)}{\beta(x)}\right\} \times \text{constant} \ \ldots$$

Then we get the RGE for a renormalization group invariant (without an anomalous dimension term).

$$(\mu\partial_\mu + \beta(g)\partial_g + \tau(g)\, m_i\partial_{m_i})\, G_{\mathrm{RGI}} = 0$$

- We have the prediction for the short distance behavior as before.
- $G_{\mathrm{RGI}}(x, \Lambda, M_i)$: scheme-independent functions, uniquely given by QCD.

Now turn to a renormalization group invariant form:

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then

$$G_{\mathrm{RGI}} = \langle P^{rs}_{\mathrm{RGI}}(x) P^{sr}_{\mathrm{RGI}}(0)\rangle = \varphi_{\mathrm{P}}^2\, G_{\mathrm{R}}$$

$$\varphi_{\mathrm{P}} = \exp\left\{-\int^g \mathrm{d}x\, \tfrac{\gamma(g)}{\beta(x)}\right\} \times \text{ constant } \ldots$$

$$= \left(2b_0 g^2\right)^{-\gamma_0/(2b_0)} \exp\left\{-\int_0^g \mathrm{d}x \left[\tfrac{\gamma(x)}{\beta(x)} - \tfrac{\gamma_0}{b_0 x}\right]\right\}$$

Then we get the RGE for a renormalization group invariant (without an anomalous dimension term).

$$(\mu\partial_\mu + \beta(g)\partial_g + \tau(g)\, m_i\partial_{m_i})\, G_{\mathrm{RGI}} = 0$$

- We have the prediction for the short distance behavior as before.
- $G_{\mathrm{RGI}}(x, \Lambda, M_i)$: scheme-independent functions, uniquely given by QCD.
- **It is the job of lattice QCD to determine them.**

The general principle (lot's of evidence)

- Mixing with all local operators of same and lower dimensions, allowed by the symmetries
  in renormalizable theories (normal propagators, no couplings with negative mass dimension)

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- Mixing with all local operators of same and lower dimensions, allowed by the symmetries

$$S = a^4 \sum_{\text{space}-\text{time}} \sum_{n=1}^{4} \sum_i g_{in} \Phi_i^{(n)}(x) + a^3 \sum_{\text{boundary}} \sum_{n=1}^{3} \sum_i c_{in} \Phi_i^{(n)}(x)$$

$[g_{in}] = 4 - n\,, \quad [c_{in}] = 3 - n$ bare couplings and masses

- adjust (= tune = renormalize) all coefficienst g_{in}, c_{in} such that the continuum limit exists
- no couplings with negative mass dimensions!
- Including theories with boundaries (Schrödinger functional , Gradient Flow) all-order proof for GF but not for SF.
- $\mathrm{O}(a)$ effects: go higher in powers of a and include $[g_{in}] = 5 - n\,, \quad [c_{in}] = 4 - n$ Symanzik *effective* theory $\longrightarrow$ Steve Sharpe

$$S = a^4 \sum_{\text{space}-\text{time}} \sum_{n=1}^{5} \sum_i g_{in} \Phi_i^{(n)}(x) + a^3 \sum_{\text{boundary}} \sum_{n=1}^{4} \sum_i c_{in} \Phi_i^{(n)}(x)$$

$g_{i5} \sim a \qquad c_{i4} \sim a$

Wilson fermions

We had:

$$\begin{aligned} m_{\mathrm{lat},i} &= Z_m(\ln(a\mu), g^2)\, \hat{m}_{\mathrm{q},i} \\ \hat{m}_{\mathrm{q},i} &= m_{\mathrm{q},i} + (r_m(g_0) - 1)\tfrac{1}{N_\mathrm{f}}\, \mathrm{tr}\, M_\mathrm{q} \\ m_{\mathrm{q},i} &= m_{0i} - m_\mathrm{c}(g_0)\,,\;\; M_\mathrm{q} = \mathrm{diag}(m_{\mathrm{q},1}, m_{\mathrm{q},2}, \ldots) \end{aligned}$$

Why is that?

Write the mass-term as

$$\begin{aligned} \mathcal{L}_\mathrm{mass} &= \sum_i \overline{\psi}_i m_{0i} \psi_i \\ &= \sum_{a\in 3,8,\ldots} \mu_0^a \underbrace{\overline{\psi} T^a \psi}_{S^a\text{, nonsinglet}} + \underbrace{\overline{\psi}\psi}_{S_0\text{ singlet}} \tfrac{1}{N_\mathrm{f}}\, \mathrm{tr}\, M_0 \end{aligned}$$

Therefore there is (in general) $Z_m = (Z_\mathrm{S}^\mathrm{NS})^{-1}$ and $(r_m - 1)Z_m = (Z_\mathrm{S}^\mathrm{sing})^{-1}$

Quark mass renormalization on the lattice

In general, for $N_\mathrm{f} > 2$, any regularisation

There is a **non**–anomalous chiral Ward identity: PCAC–relation

$$\langle\, [\partial_\mu A_\mu^{rs}(x) - (m_r + m_s) P^{rs}(x)] \;\; [\text{fields not at } x] \,\rangle = 0$$

- Can be obtained formally, performing a chiral rotation $\longrightarrow$ Gregorio Herdoiza
- Can be obtained with lattice exact chiral symmetry (overlap)
- Is therefore (universality) a property of QCD in the continuum limit after renormalization

see also Gregorio Herdoiza's tutorial

Quark mass renormalization on the lattice

In general, for $N_\mathrm{f} > 2$, any regularisation

Renormalized relation

$$\langle\, [Z_\mathrm{A}\partial_\mu A_\mu^{rs}(x) - (m_r + m_s)_\mathrm{R} Z_\mathrm{P} P^{rs}(x)]\ \text{[fields not at } x]\,\rangle = 0$$

$$(m_r + m_s)_\mathrm{R} = \frac{Z_\mathrm{A}}{Z_\mathrm{P}}(m_r + m_s)$$

- defines $(m_r + m_s)_\mathrm{R} \longrightarrow$ with $N_\mathrm{f} > 2$ enough combinations to define/determine $m_r\,,\ r = 1 \ldots N_\mathrm{f}$ with $N_\mathrm{f} = 2$ use also PCVC
- $Z_\mathrm{A}, Z_\mathrm{P}$ standard problem which we will discuss
- RGI masses from μ-dependent masses as discussed. Unambiguous.
- NB: Z_A is actually more simple

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- This does not say that anything special happens at $M_\mathrm{u}=0$. There is no symmetry enhancement as explained by Mike Creutz.

First consider just the renormalization of the coupling, set

$$m_i = 0\,.$$

Properties of a renormalised coupling

a finite: $g = f(g_0, \mu a)$, such that $\lim_{a\to 0}\, G_0(qa, g_0)|_g$ exists

b gauge invariant (physical)

most natural

$$\begin{aligned} & G_0 = G_0(qa, g_0) \\ \to \qquad & G_\mathrm{R} = G_\mathrm{R}(q/m_\mathrm{proton}, qa)\,, \qquad am_\mathrm{proton} = f(g_0) \\ & G_\mathrm{R}^\mathrm{cont} = G_\mathrm{R}(xm_\mathrm{proton}, 0) \end{aligned}$$

"hadronic scheme"

- but we want a **coupling**, i.e. the relation to the Λ - parameter, the relation to perturbative QCD

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c

$$g^2_{\mathrm{NP}} \overset{g_{\mathrm{lat}}\to 0}{\sim} g^2_{\mathrm{lat}}\chi_g^{\mathrm{NP,lat}}(g_{\mathrm{lat}})$$

$$\chi_g^{x,y}(g) = \underset{\substack{\uparrow}}{1} + \chi_{g,1}^{x,y}g^2 + \ldots$$

convention, good idea

d It depends on a single scale μ $\quad\to$ RGE!
For $\mu \to \infty$ it is purely short distance

Take $G_0(\mu a, g_0)$ dimensionless (in the massless theory) satisfying a,b,d.

G_0 has a regular PT

$$
\begin{aligned}
G_0(qa, g_0) &= G_0^{(0)}(qa) + G_0^{(1)}(qa) g_0^2 + G_0^{(2)}(qa) g_0^4 + \ldots \\
\text{lat scheme:} \quad & g_0^2 = g_{\rm lat}^2 + 2b_0 \ln(a\mu) g_{\rm lat}^4 + \ldots \\
\rightarrow \quad G_0 = G_{\rm R} &= G_{\rm R}^{(0)}(qa) + G_{\rm R}^{(1)}(qa) g_{\rm lat}^2 + G_{\rm R}^{(2)}(q/\mu, qa) g_{\rm lat}^4 + \ldots \\
i = 0, 1: \quad & G_{\rm R}^{(i)}(qa) = G_0^{(i)}(qa) = C^{(i)} + \mathrm{O}(a^2 q^2) \qquad (*) \\
& G_{\rm R}^{(2)}(q/\mu, qa) = G_0^{(2)}(qa) + 2b_0 \ln(a\mu) G_0^{(1)}(qa) \\
& \qquad = H(qa) + 2b_0 \ln(\mu/q) G_0^{(1)}(qa) \qquad (*) \\
& \qquad H(qa) = C^{(2)} + \mathrm{O}(a^2 q^2) \qquad (*)
\end{aligned}
$$

(*): the continuum limit exists

Set $\mu = q$:

$$\begin{aligned} G_0(\mu a, g_0^2) &= G_\mathrm{R}^{(0)}(\mu a) + G_\mathrm{R}^{(1)}(\mu a) g_\mathrm{lat}^2(\mu) + G_\mathrm{R}^{(2)}(1,\mu a) g_\mathrm{lat}^4(\mu) + \ldots \\ &= G_0^{(0)}(\mu a) + G_0^{(1)}(\mu a) g_\mathrm{lat}^2(\mu) + \underbrace{G_\mathrm{R}^{(2)}(1,\mu a)}_{C^{(2)}+\mathrm{O}(a^2 q^2)} g_\mathrm{lat}^4(\mu) + \ldots \end{aligned}$$

then

$$\bar{g}_G^2(\mu) \equiv \frac{G_0(\mu a, g_0^2) - G_0^{(0)}(\mu a)}{G_0^{(1)}(\mu a)}$$

satisfies a,b,c,d

Generic definition of a renormalized coupling

Set $\mu = q$:

$$\begin{aligned} G_0(\mu a, g_0^2) &= G_{\mathrm{R}}^{(0)}(\mu a) + G_{\mathrm{R}}^{(1)}(\mu a) g_{\mathrm{lat}}^2(\mu) + G_{\mathrm{R}}^{(2)}(1, \mu a) g_{\mathrm{lat}}^4(\mu) + \ldots \\ &= G_0^{(0)}(\mu a) + G_0^{(1)}(\mu a) g_{\mathrm{lat}}^2(\mu) + \underbrace{G_{\mathrm{R}}^{(2)}(1, \mu a)}_{C^{(2)} + \mathrm{O}(a^2 q^2)} g_{\mathrm{lat}}^4(\mu) + \ldots \end{aligned}$$

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satisfies a,b,c,d

“physical” coupling

note

$$\bar{g}_G^2(\mu) = 1 \times g_{\mathrm{lat}}^2 + \mathrm{O}(g_{\mathrm{lat}}^4)$$

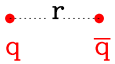
$\uparrow$

no a^2 effects here

Nonperturbative Renormalization

Example 1

$Q\bar{Q}$ potential, force:



$$G_0(\mu a, g_0^2) \quad = \quad r^2 F_{\rm impr}(r)\,, \qquad \mu = 1/r$$

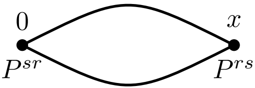
$$G_0^{(0)} = 0\,, \quad G_0^{(1)} = \frac{C_F}{4\pi} \qquad C_F = \frac{N^2-1}{6N}$$

def. of $F_{\rm impr}$ later

$$\rightarrow \qquad \bar{g}^2_{\rm qq}(\mu) = \frac{4\pi}{C_F} r^2 F_{\rm impr}(r)$$

[there is a little caveat with this ... not 100% short distance ... but ok]

Two-point function of multiplicatively renormalisable field



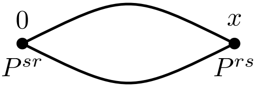
$$G_0(\mu a, g_0^2) \quad = \quad \frac{\langle P^{rs}(x) P^{sr}(0) \rangle_{x=(0,0,0,1/\mu)}}{\langle P^{rs}(x) P^{sr}(0) \rangle_{x=(0,0,0,2/\mu)}}$$

- Factors Z_{P} cancel
- Theoretically fine, but not really recommended in practise

$$\langle P^{rs}(x) P^{sr}(0) \rangle \overset{x \to 0}{\sim} x^{-6}$$

steep function, large $(a/x)^n$ effects

[Martinelli, Rossi, Sachrajda, Sharpe, talevi, Testa, 1997; . . . , Cichy, Jansen, Korcyl, 2012]



- case by case
- eg. (in principle)

$$Z_{\rm P}^2(a\mu, g_0)\,\langle P^{rs}(x)P^{sr}(0)\rangle_{x=(0,0,0,1/\mu)} = \langle P^{rs}(x)P^{sr}(0)\rangle_{x=(0,0,0,1/\mu),g_0=0}$$

this defines $Z_{\rm P}(a\mu, g_0)$

- many correlation functions of P^{rs} can be used, as long as they are sufficiently short distance dominated
- but be careful with integrals, e.g.

$$\int \mathrm{d}^4 x \langle P^{rs}(x)P^{sr}(0)\rangle$$

does not exist, since $\langle P^{rs}(x)P^{sr}(0)\rangle \overset{x\to 0}{\sim} x^{-6}$

drop gauge invariance requirement (b): fix a gauge (e.g. Landau gauge)
numerical evidence that this can be done non-perturbatively

then

$$\begin{aligned}
S(p) &= a^4 \sum_x \exp(-ipx)\ \langle \psi_r(x_1)\bar{\psi}_r(x_2)\rangle \\
G_P(p_1,p_2) &= a^8 \sum_{x_1,x_2} \exp(-ip_1x_1 + ip_2x_2)\ \langle\ \psi_r(x_1)\ P^{rs}(0)\ \bar{\psi}_s(x_2)\ \rangle \\
\Lambda_P(p_1,p_2) &= S^{-1}(p_1)\ G_P(p_1,p_2)\ S^{-1}(p_2) \\
\Gamma_P(p_1,p_2) &= \tfrac{1}{12}\mathrm{Tr}\ [\ \gamma_5\ \Lambda_P(p,p)\ ]\ , \qquad \Gamma_V(p_1,p_2) = \ldots
\end{aligned}$$

Define $\Gamma_V(p)$ similarly from the conserved vector current, then

$$Z_{\mathrm{P}}\Gamma_P(p_1,p_2)/\Gamma_V(p_1,p_2) = [\Gamma_P(p_1,p_2)/\Gamma_V(p_1,p_2)]_{g_0=0}$$

defines $Z_{\mathrm{P}}(\mu)$ [or use a similar condition for the quark propagator to define the quark field renormalization constant and divide it out in Λ_P]

symmetric point $p^2 = p_1^2 = p_2^2 = (p_1 - p_2)^2 = \mu^2$
$\rightarrow$ short distance dominated "RI-sMOM"

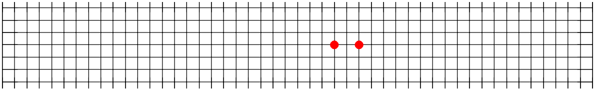
[C. Sturm, Y. Aoki, N. H. Christ, T. Izubuchi, C. T. C. Sachrajda & A. Soni, ARXIV:0901.2599]

Scale Problem

(α_{qq} as an example)

We need to reach large μ where perturbation theory is reliable to be able to use the perturbative relation (perturbative β-function) in

$$\frac{\Lambda}{\mu} = \varphi_g(g(\mu))$$



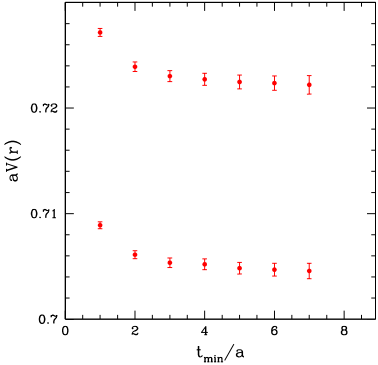
$$\underset{\substack{\uparrow \\ \text{box size}}}{L} \gg \underset{\substack{\uparrow \\ \text{QCD scale, } m_\pi}}{\frac{1}{0.2\mathrm{GeV}}} \gg \frac{1}{\mu} \sim \frac{1}{10\mathrm{GeV}} \gg \underset{\substack{\uparrow \\ \text{spacing}}}{a}$$

$$\Downarrow$$

$$L/a \gg 50$$

Let us see this in more detail

Potential $V(r)$ from single exponential fits with $t \geq t_{\rm min}$, SU(3) pure gauge theory.
$\beta = 6/g_0^2 = 6.4$, $L/a = T/a = 32$, Δ from variational method.
$r/a = 12, 13$ from [M. Guagnelli, R.Sommer, H. Wittig, 1998]



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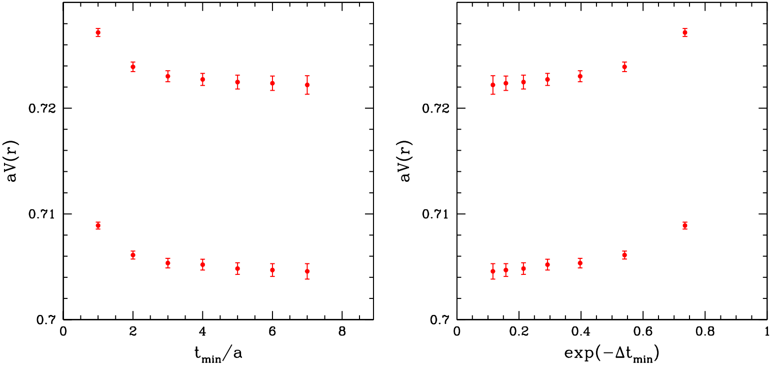
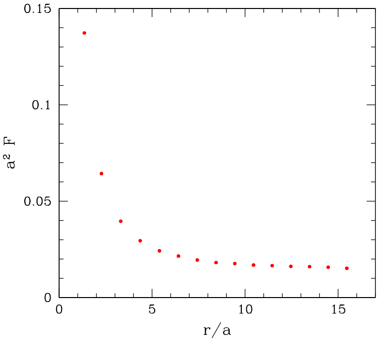


Define

$$F_{\mathrm{impr}}(\mathbf{r}_{\mathrm{I}}) = \frac{1}{a}[V(r) - V(r-a)] \qquad r \text{ along an axis}$$

with

$$\frac{C_{\mathrm{F}}}{4\pi\mathbf{r}_{\mathrm{I}}^{\mathbf{2}}} = \frac{1}{a}[D(r,0,0)) - D(r-a,0,0)] = [V(r) - V(r-a)]_{\mathrm{TL}}/g_0^2$$



Renormalize in a hadronic scheme

- replace g_0 $\longrightarrow$ physical quantity
- pure gauge theory: length scale from potential instead of a hadron mass

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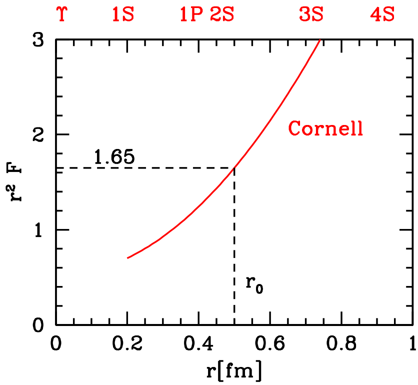
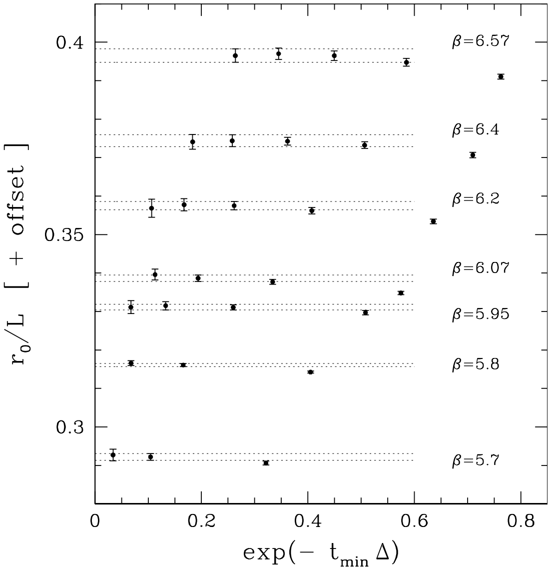
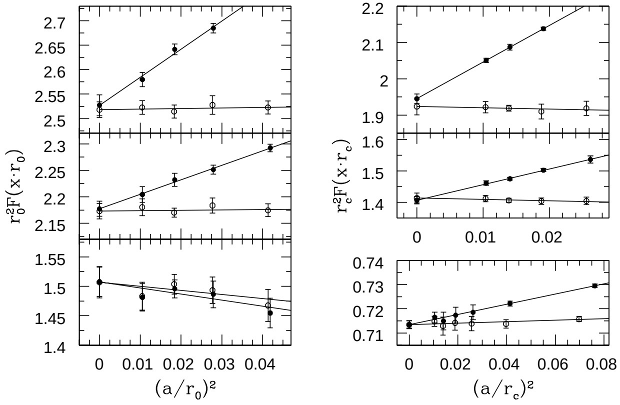


- $$r_0^2 F(r_0) = 1.65$$

Reference scale

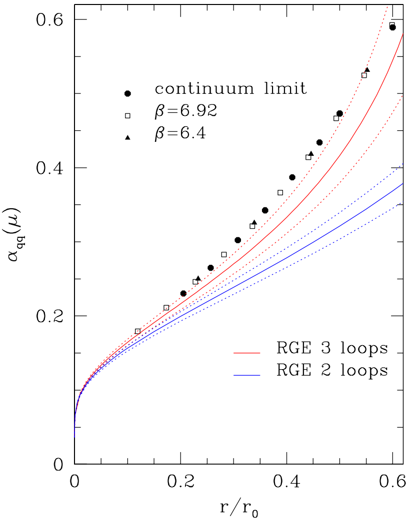


Continuum extrapolation [Necco, S, 2001]

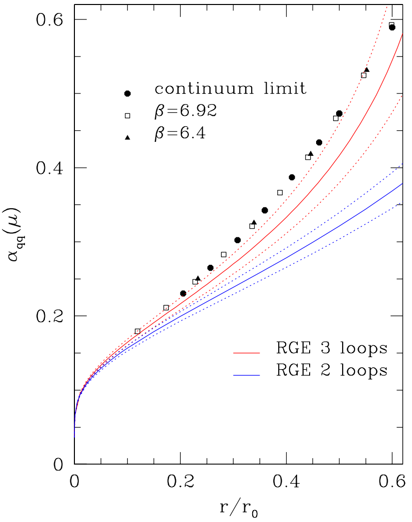


○: $r_{\rm I} \to F_{\rm impr}$ naive: ●: $r_{\rm n} = r - a/2$

- Effect of observable improvement is substantial! (Cutoff effects are substantial)
- Do not rely too much on this improvement!

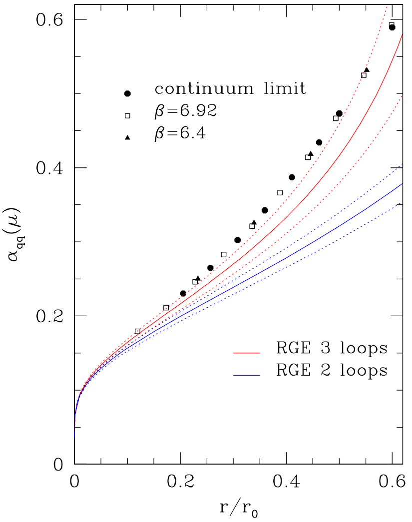


- RGE solutions with known Λ

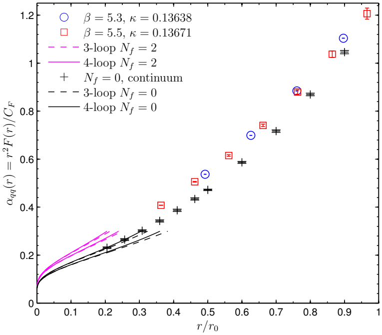


- RGE solutions with known Λ
- Qualitative contact to PT is made

Force



- RGE solutions with known Λ
- Qualitative contact to PT is made
- But is this safe to determine the Λ-parameter?



graph from [Leder & Knechtli, 2011]

$N_{\rm f} = 0$, continuum limit

[Necco & S., 2001]

$N_{\rm f} = 2$, small lattice spacing

[Leder & Knechtli, 2011]

$$\Lambda/\mu = \varphi_g(g) = \; = \left(b_0 g^2\right)^{-b_1/(2b_0^2)} \mathrm{e}^{-1/(2b_0 g^2)} \exp\left\{-\int_0^g \mathrm{d}x \left[\frac{1}{\beta(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0^2 x}\right]\right\}$$

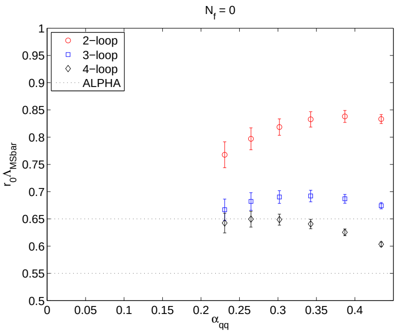
approximations:

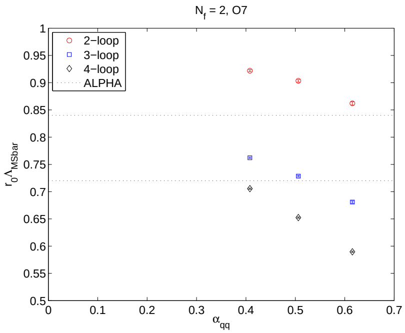
$$\left.\frac{\Lambda}{\mu}\right|^{\mathrm{eff}}_{n-\text{loop}} = \left(b_0 g^2\right)^{-b_1/(2b_0^2)} \mathrm{e}^{-1/(2b_0 g^2)} \exp\left\{-\int_0^g \mathrm{d}x \left[\frac{1}{\beta_{n-\text{loop}}(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0^2 x}\right]\right\}$$

$$\left.\frac{\Lambda}{\mu}\right|^{\mathrm{eff}}_{2-\text{loop}} = \frac{\Lambda}{\mu} + \mathrm{O}(\alpha_{\mathrm{qq}})$$

$$\left.\frac{\Lambda}{\mu}\right|^{\mathrm{eff}}_{n-\text{loop}} = \frac{\Lambda}{\mu} + \mathrm{O}(\alpha_{\mathrm{qq}}^{n-1})$$

$$r_0\,\Lambda|^{\rm eff}_{n-\text{loop}} = \frac{r_0}{r}\,[r\,\Lambda]^{\rm eff}_{n-\text{loop}} = r_0\,\Lambda + \mathrm{O}(\alpha_{\rm qq}^{n-1}) \qquad [r = 1/\mu]$$



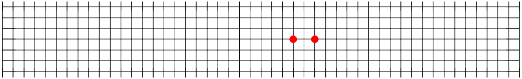


A realistic estimate of the uncertainty is impossible.

There are other opinions on this [Brambilla eta al.; Jansen, Karbstein, Nagy, Wagner, 2011]

Scale Problem

($\alpha_{\rm qq}$ as an example)



$$L \gg \frac{1}{0.2\mathrm{GeV}} \gg \frac{1}{\mu} \sim \frac{1}{10\mathrm{GeV}} \gg a$$

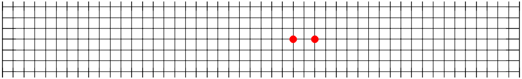
$\uparrow$ box size — $\uparrow$ QCD scale, m_π — $\uparrow$ spacing

$$\Downarrow$$

$$L/a \gg 50$$

Scale Problem

(α_{qq} as an example)



$$\underset{\substack{\uparrow\\ \text{box size}}}{L} \gg \underset{\substack{\uparrow\\ \text{QCD scale, } m_\pi}}{\frac{1}{0.2\mathrm{GeV}}} \gg \frac{1}{\mu} \sim \frac{1}{10\mathrm{GeV}} \gg \underset{\substack{\uparrow\\ \text{spacing}}}{a}$$

$$\Downarrow$$

$$L/a \gg 50$$

Solution: $\boxed{L = 1/\mu}$ $\longrightarrow$ left with $L/a \gg 1$ [Wilson, ... , Lüscher, Weisz, Wolff]

Finite size effect as a physical observable; finite size scaling!