Non-perturbative Renormalization

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Strategy

finite volume coupling $\alpha_{\rm SF}(\mu), \mu = 1/L$
defined at zero quark mass

$$L_{\rm max} = {\rm const.}/m_{\rm prot} = {\rm O}(\tfrac{1}{2}{\rm fm}) : \quad \longrightarrow \quad \alpha_{\rm SF}(\mu = 1/L_{\rm max})$$

$$\downarrow$$

$$\alpha_{\rm SF}(\mu = 2/L_{\rm max})$$

$$\downarrow$$

$$\bullet \quad \bullet \quad \bullet$$

always $a/L \ll 1$

$$\downarrow$$

$$\alpha_{\rm SF}(\mu = 2^n/L_{\rm max} = 1/L_{\rm min})$$

PT: $\downarrow$

$$\Lambda_{\rm SF} L_{\rm max} = \#$$

Result is a value for $\boxed{\Lambda_{\rm SF}/m_{\rm prot} = \#}$

The step scaling function

We leave the discussion of a finite volume coupling for later.
Discuss first the

Step scaling function

- It is a discrete β function:

$$\sigma(s, \bar{g}^2(L)) = \bar{g}^2(sL) \qquad \text{mostly } s = 2$$

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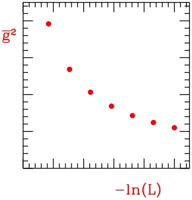
- It is a discrete β function:

$$\sigma(s, \bar{g}^2(L)) = \bar{g}^2(sL) \qquad \text{mostly } s = 2$$

- determines the non-perturbative running:

$$\begin{aligned} u_0 &= \bar{g}^2(L_{\max}) \\ &\downarrow \\ \sigma(2, u_{k+1}) &= u_k \\ &\downarrow \\ u_k &= \bar{g}^2(2^{-k} L_{\max}) \end{aligned}$$



The step scaling function

$\sigma(s,u) = \bar{g}^2(sL)$ with $u = \bar{g}^2(L)$

On the lattice:
additional dependence on the resolution
a/L

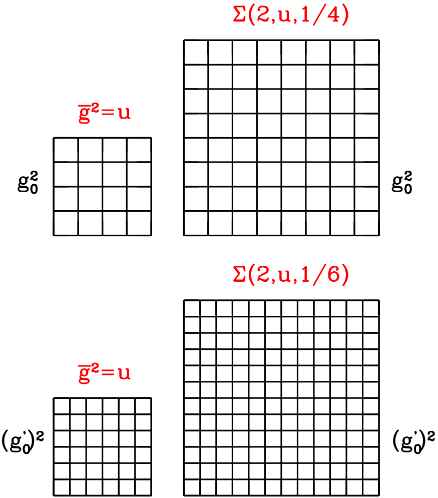
g_0 fixed, L/a fixed:

$$\bar{g}^2(L) = u, \qquad \bar{g}^2(sL) = u' \,,$$
$$\Sigma(s,u,a/L) \quad = \quad u'$$

continuum limit:

$$\Sigma(s,u,a/L) = \sigma(s,u) + \mathrm{O}(a/L)$$

in the following always $s = 2$



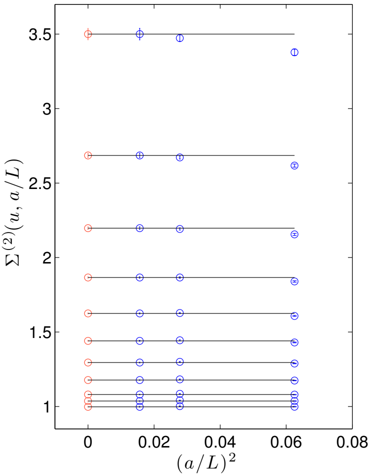
everywhere: $m = 0$ (PCAC mass defined in $(L/a)^4$ lattice)

The step scaling function

(Table from $N_f = 2$, ALPHA Collaboration)

L/a	β	κ	$\bar{g}^2$	$d\bar{g}^2$	m	dm
$u = 1.1814$						
4	8.2373	0.1327957	1.1814	0.0005	0.00100	0.00011
5	8.3900	0.1325800	1.1807	0.0012	-0.00018	0.00009
6	8.5000	0.1325094	1.1814	0.0015	-0.00036	0.00003
8	8.7223	0.1322907	1.1818	0.0029	-0.00115	0.00004
8	8.2373	0.1327957	1.3154	0.0055	0.00020	0.00005
10	8.3900	0.1325800	1.3287	0.0059	0.00097	0.00007
12	8.5000	0.1325094	1.3253	0.0067	-0.00102	0.00002
16	8.7223	0.1322907	1.3347	0.0061	-0.00194	0.00002
L/a			$\Sigma(1.1814, a/L)$	$\delta\Sigma$		
4			1.3154	0.0055		
5			1.3296	0.0061		
6			1.3253	0.0070		
8			1.3342	0.0071		

- tune κ, g_0 to have desired $m \approx 0$, fixed $\bar{g}^2(L)$
- propagate errors from $\bar{g}^2(L)$, shift means if necessary
  $\longrightarrow \Sigma, \delta\Sigma$



- *Constant fit:*

$$\Sigma^{(2)}(u, a/L) = \sigma(u)$$

for $L/a = 6, 8$

- *Global fit:*

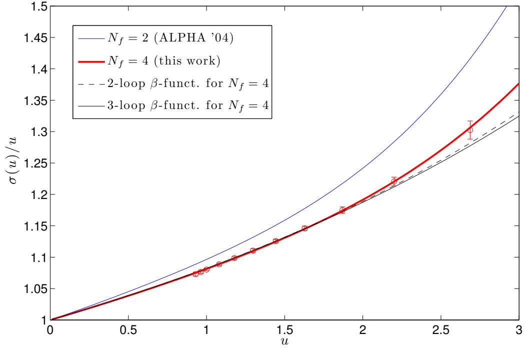
$$\Sigma^{(2)}(u, a/L) = \sigma(u) + \rho\, u^4\, (a/L)^2$$

for $L/a = 6, 8$
$\rightarrow \rho = 0.007(85)$

- $L/a = 8$ *data:*

$$\sigma(u) = \Sigma^{(2)}(u, 1/8)$$

[ALPHA Collaboration (S., Tekin & Wolff, 2010); update: M. Marinkovic, 2013] compare it to perturbation theory and other flavour numbers



[ALPHA Collaboration, 2010; update: M. Marinkovic, 2013]

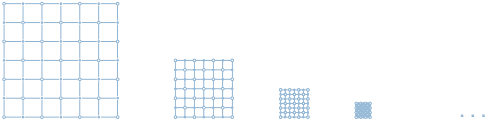
Excellent agreement with PT over a large range of couplings in this particular scheme.

$$
\begin{aligned}
u_i &\equiv \bar{g}^2\left(L_{\max}/2^i\right) \\
u_i &= \sigma(u_{i+1}), \quad i = 0, \ldots, n, \quad u_0 = u_{\max} = \bar{g}^2\left(L_{\max}\right),
\end{aligned}
$$

solve for u_{i+1}, $i = 0 \ldots n = 10$

The strategy

$$\frac{\Lambda_{\overline{\rm MS}}}{m_{\rm proton}} = \frac{1}{m_{\rm proton} L_{\rm max}} \times \frac{L_{\rm max}}{L_k} \times L_k \Lambda_{\rm SF} \times \frac{\Lambda_{\overline{\rm MS}}}{\Lambda_{\rm SF}}$$



$am_{\rm prot}$ $\quad L_{\rm max}/a$

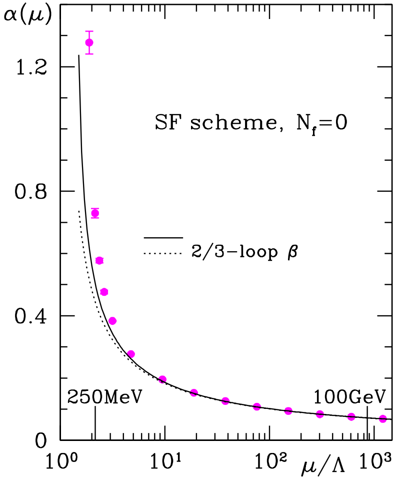
$\Lambda_{\overline{\rm MS}}$

1-lp (exact)

$\bar{g}^2(L_{\rm max})$ —— non-perturbative SSF's, massless theory ——→ $\bar{g}^2(L_k)$ —— 3-lp ——→ $\Lambda_{\rm SF}$

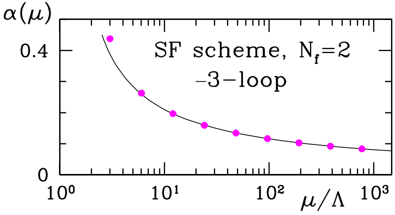
μ

$\mu = 0$ $\quad m_b \quad M_Z \quad \infty$

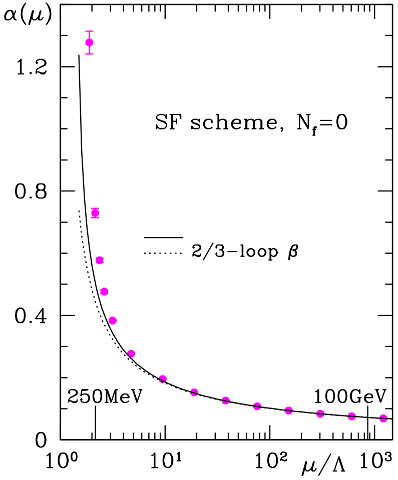


[ALPHA Collaboration, 2001]

Non-perturbative running of α_{SF}

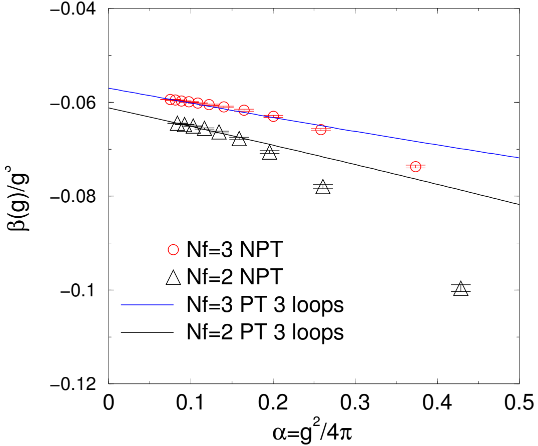


[ALPHA Collaboration, 2005]

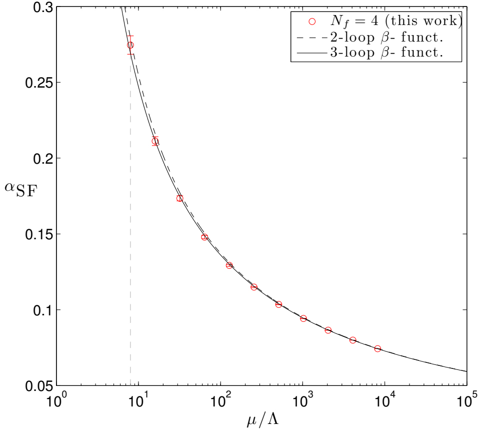


[ALPHA Collaboration, 2001]

Non-perturbative running of α_{SF}: β-function



$N_\mathrm{f} = 3$ [PACS-CS , 2009]
$N_\mathrm{f} = 2$ [ALPHA Collaboration, 2004]



[ALPHA Collaboration, 2010; update: M. Marinkovic, 2013]

energies μ

$0.9\,\mathrm{GeV}$

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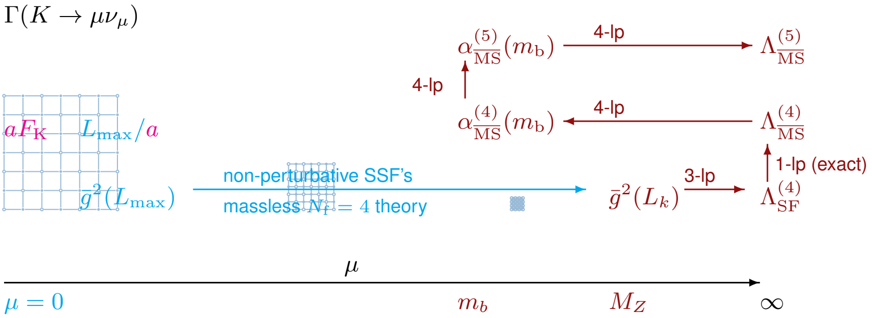
$900\,\mathrm{GeV}$

The complete strategy

$$\frac{\Lambda^{(5)}_{\overline{\rm MS}}}{F_{\rm K}} = \frac{1}{F_{\rm K} L_{\rm max}} \times \frac{L_{\rm max}}{L_k} \times L_k \Lambda^{(4)}_{\overline{\rm MS}} \times \frac{\Lambda^{(5)}_{\overline{\rm MS}}}{\Lambda^{(4)}_{\overline{\rm MS}}}$$



$F_{\rm K}L_{\rm max}$

- for $N_{\rm f}=4$ missing completely
- for $N_{\rm f}=3$, CP-PACS $m_\rho L_{\rm max}$ (with 1-loop $c_{\rm t}$)
- for $N_{\rm f}=2$

- for $N_{\mathrm{f}} = 4$ missing completely
- for $N_{\mathrm{f}} = 3$, CP-PACS $m_{\rho} L_{\mathrm{max}}$ (with 1-loop c_{t})
- for $N_{\mathrm{f}} = 2$

  issues:
  – autocorrelations
  – chiral extrapolation in u/d
  – continuum extrapolation

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- for $N_{\mathrm{f}} = 3$, CP-PACS $m_\rho L_{\mathrm{max}}$ (with 1-loop c_{t})
- for $N_{\mathrm{f}} = 2$

  issues:
  – autocorrelations
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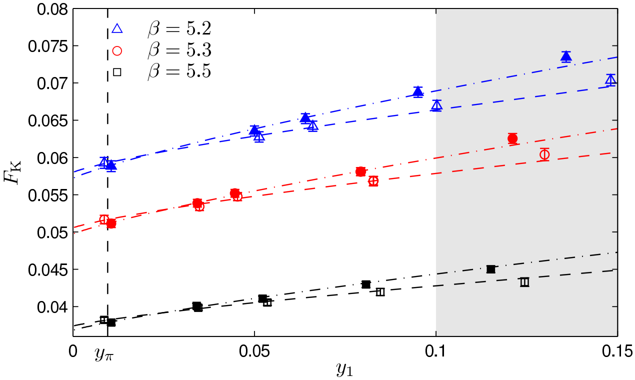
-

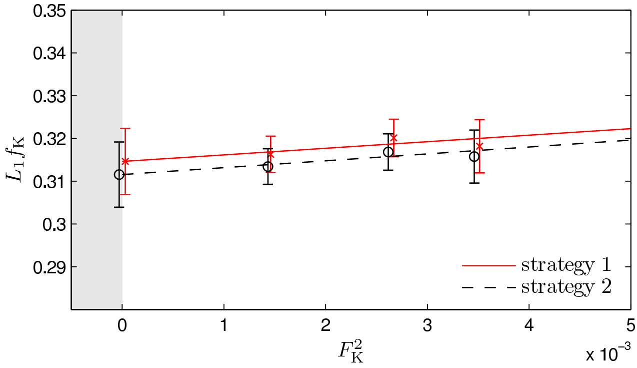
$$F_{\mathrm{K}} L_{\mathrm{max}} = f(\beta) \cdot l_{\mathrm{max}}(\beta) + \mathrm{O}(a^2 F_{\mathrm{K}}^2)$$
$$f(\beta) = F_{\mathrm{K}} a \,, \quad l_{\mathrm{max}}(\beta) = L_{\mathrm{max}}/a$$

Interpolate $l_{\mathrm{max}}(\beta)$ to CLS (used in large volume) β's



$$y_1 = m_\pi^2/(4\pi f_{\mathrm{K}})^2$$

Continuum extrapolation of $F_{\rm K} L_{\rm max}$



$$F_{\rm K} = a f_{\rm K}$$

N_f dependence of $\Lambda_{\overline{MS}}$ and comparison to phenomenology

$\Lambda_{\overline{MS}}$[MeV] status 2011

Experiment	Theory N_f:	0	2	3	4	5
M_K, $K \to l2, l3$	SF [ALPHA Collaboration]	238(19)	310(20)			
M_K, M_ρ	SF [PACS-CS]			362(23)(25)		239(10)(6)(-22)
DIS, HERA	PT, PDF-fits [ABM11]				234(14)	160(11)
DIS, HERA	PT, PDF-fits [MSTW09]				285(23)	198(16)
“world av. ”[2011]	PT					212(12)
$e^+e^- \to$ had (LEP)	4-loop PT					275(57)

- Non-trivial, non-perturbative N_f-dependence.
- Small errors are cited, but overall consistency is not that great.
- More precision and rigor (PT only at high energy) will be very useful.

Boundary conditions matter in finite volume. Which ones?

A most relevant criterion is zero modes

- Zero modes of gauge fields
  $\rightarrow$ perturbative expansion (+ MC)
- Zero modes of Dirac operator
  $\rightarrow$ HMC stability

Path integral w.o. fermions

$$\langle O(U)\rangle = \frac{1}{\mathcal{Z}}\int \mathrm{D}[U]\mathrm{e}^{-\beta\bar{S}(U)}O(U)$$
$$\bar{S}(U) = \sum_p \mathrm{tr}\,(1-U(p))\,,\quad \beta=\frac{6}{g_0^2}$$

PT, sketchy

$\beta\to\infty$ $\quad U\approx U_{\rm min}\equiv V$ dominates (classical solution)

$$U(x,\mu)=V(x,\mu)\mathrm{e}^{\bar{q}^b_\mu(x)T^b}\,,\quad \bar{q}^b_\mu(x)\ll 1\,,\quad \int\mathrm{D}[U]\to\int\mathrm{D}[\bar{q}]$$
$$\bar{S}(U) = \bar{S}(V)+\sum_{n,m}q_mK_{mn}q_n+\mathrm{O}(q^3)\,,\quad q_n=\bar{q}^b_\mu(x)\,,\ n=(\tfrac{x}{a},\mu,b)$$
$$O(U) = O(V)+\ldots$$

Gauss intergrals $\to$ Wick contractions ... IFF K has no zero modes $(Kv=\lambda v\,,\ \lambda>0)$

Generically there are zero modes

- gauge modes $\rightarrow$ gauge fixing
- finite volume modes (gauge invariant)
  "Ground state metamorhosis"[Gonzales Arroyo, Jurkiewicz, Korthals-Altes] with periodic BC's

Toy example: SU(2), L^4, $L = a$ lattice, PBC, $d = 2$, single point

- $\bar{S} = 2 - \mathrm{tr}\,(U_2 U_1 U_2^\dagger U_1^\dagger)$
- $\mathrm{tr}\, U_i$ is gauge invariant, U_i can't be gauged away
- minima: $U_1 = U_2 = V$ ... pick $U_1 = U_2 = 1$.
- fluctuations $U_i = \mathrm{e}^{i\sigma^b q_i^b}$

$$\bar{S} \;=\; 2 - \mathrm{tr}\,\mathrm{e}^{i\sigma^b q_2^b}\mathrm{e}^{i\sigma^b q_1^b}\mathrm{e}^{-i\sigma^b q_2^b}\mathrm{e}^{-i\sigma^b q_1^b} = \mathrm{O}(q^4) \quad \to K = 0$$

- $q = \mathrm{O}(\beta^{-1/4}) = \mathrm{O}(g_0^{1/2})$
  PT in powers of g_0, not g_0^2 NOT regular
- In general: mixture of gaussian and non-gaussian modes
  integrate over non-gaussian ones exactly ...
  complicated, non-universal β-function
  it can be worse, divergent behavior, $1/\log(g)$ terms , see [Nogradi et al., 2012]
- think of these U_i as Polyakov loops $\to$ relevant for 4-d gauge theory.
  "Ground state metamorhosis"[Gonzales Arroyo, Jurkiewicz, Korthals-Altes]

$$V(x,\mu) = 1\,, \quad \text{PBC: } \psi(x+L\hat{\mu}) = \psi(x)$$

massless Dirac operator has a zero mode (constant mode, $p = 0$)

easily fixed by

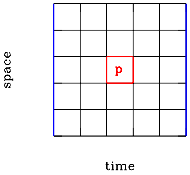
$$\psi(x+L\hat{\mu}) = \mathrm{e}^{i\alpha}\psi(x)$$

e.g. $\alpha = \pi/2$ in SU(2), $\alpha = \pi/3$ in SU(3)

Exercise: why these values of α?

Boundary conditions

- Space: PBC
- Time: Dirichlet, breaks translation invariance!



Yang Mills theory [Lüscher, Narayanan, Weisz & Wolff]:

$$\mathcal{Z}(V,V') = \int \mathrm{D}(U)_{\text{inside}}\, \mathrm{e}^{-S_{\mathrm{SF}}(U)}$$

$$S_{\mathrm{SF}}(U) = \sum_{p \text{ inside}} \beta\, \mathrm{tr}\,(1 - U(p))\,, \quad U(x,k) = \begin{cases} V(\mathbf{x},k) & x_0 = 0 \\ V'(\mathbf{x},k) & x_0 = T \end{cases}$$

Standard introduction of Hilbert space, transfer matrix:

$$\mathcal{Z}(V,V') = \langle V'|\underbrace{\mathrm{e}^{-\hat{H}T}}_{\mathbb{T}^{T/a}}\underbrace{\mathbb{P}_0}_{\uparrow}|V\rangle\,, \quad \hat{U}(\mathbf{x},k)|U\rangle = U(\mathbf{x},k)|U\rangle$$

projector onto gauge invariant states

$\mathcal{Z}(V,V') =$ Euclidean time propagation kernel by time T = Schrödinger functional

Wilson Dirac operator (also others are possible)

$$D_{\rm W} \quad = \quad \tfrac{1}{2}\{\gamma_\mu(\nabla_\mu + \nabla^*_\mu) - a\nabla^*_\mu\nabla_\mu\}$$

$$\nabla_\mu\psi(x) = \frac{1}{a}\left[U(x,\mu)\psi(x+a\hat\mu) - \psi(x)\right]$$

$$\nabla^*_\mu\psi(x) = \frac{1}{a}\left[\psi(x) - U(x-a\hat\mu,\mu)^{-1}\psi(x-a\hat\mu)\right]$$

Schrödinger functional action

$$S_{\rm F} \quad = \quad a^4\sum_x \overline{\psi}(x)[m_0 + D_{\rm W}]\psi(x)\,,$$

with $\psi(x) = 0\,,\ \overline{\psi}(x) = 0$ for $x_0 \leq 0\,,$ and $x_0 \geq T$

In the continuum theory this corresponds to BC's [Sint, 1994]

$$P_+\psi(x)|_{x_0=0} = 0 \qquad \overline{\psi}(x)P_-\Big|_{x_0=0} = 0 \qquad P_\pm = \tfrac{1}{2}(1\pm\gamma_0)$$

$$P_-\psi(x)|_{x_0=T} = 0 \qquad \overline{\psi}(x)P_+\Big|_{x_0=T} = 0$$

These BC's are stable: emerge in the cont. limit without fine-tuning. Universality! [Lüscher, 2006]
The Universality class is characterised by Parity invariance, discrete rot. invariance (not chiral symm).

Correlation functions can be formed with the usual fields in the interior (bulk) **and the boundary quark fields**

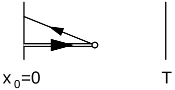
$$\zeta(\mathbf{x}) = P_- U(x,0)\psi(x+a\hat{0})\big|_{x_0=0} \qquad \bar{\zeta}(\mathbf{x}) = \bar{\psi}(x+a\hat{0})P_+ U(x,0)^{-1}\big|_{x_0=0}$$

$$\zeta'(\mathbf{x}) = P_+ U(x-a\hat{0},0)^{-1}\psi(x-a\hat{0})\big|_{x_0=T} \qquad \bar{\zeta}'(\mathbf{x}) = \bar{\psi}(x-a\hat{0})P_- U(x-a\hat{0},0)\big|_{x_0=T}$$

A very interesting feature of these is that one can form correlation functions where the **quark** fields are projected to $\mathbf{p}=0$. (Note that the gauge fields at the boundaries are fixed).

$$f_{\rm P}^{rs}(x_0) = a^6 \sum_{\mathbf{v},\mathbf{y}} \langle \bar{\zeta}_s(\mathbf{v})\gamma_5\zeta_r(\mathbf{y}) P^{rs}(x)\rangle$$

$$P^{rs}(x) = \bar{\psi}_r(x)\gamma_5\psi_s(x)$$



boundary quark fields

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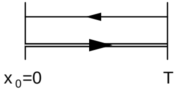
These boundary quark fields renormalize multiplicatively.

$$\zeta_{\rm R}(\mathbf{x}) = Z_\zeta \zeta(\mathbf{x})\,,\; \ldots\;,\; \bar\zeta'_{\rm R} = Z_\zeta \bar\zeta'(\mathbf{x})$$

Define also boundary-to-boundary correlation functions

$$f_1^{rs} = \frac{a^{12}}{L^6}\sum_{\mathbf{v},\mathbf{y},\mathbf{u},\mathbf{x}} \langle \bar\zeta_s(\mathbf{v})\gamma_5\zeta_r(\mathbf{y})\bar\zeta'_r(\mathbf{u})\gamma_5\zeta'_s(\mathbf{x})\rangle$$



Then

$$(f_1^{rs})_{\rm R} = Z_\zeta^4\,(f_1^{rs})\,, \quad (f_{\rm P}^{rs}(x_0))_{\rm R} = Z_\zeta^2\, Z_{\rm P}\, (f_{\rm P}^{rs}(x_0))$$

- Regular PT (no gauge field zero modes)
- Gap for Dirac operators
- Momentum zero boundary quark fields
  (spatially one takes pbc up to a phase, cf "flavor twisted bc")
- Schrödinger functional coupling defined with non-trivial V, V'
  β-function known to 3-loops [LNWW; LW; Bode, Weisz, Wolff]

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- We can define Z-factors (schemes) for composite fields, e.g.

$$Z_{\rm P} = \frac{1}{c(a/L)} \frac{\sqrt{f_1^{rs}}}{f_{\rm P}^{rs}(T/2)}\,, \quad c(a/L) = \left.\frac{\sqrt{f_1^{rs}}}{f_{\rm P}^{rs}(T/2)}\right|_{g_0=0}$$

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- There is also a new SF coupling ...

Exercise

Consider the free Schrödinger functional , i.e. $U(x, \mu) = 1$ with pbc in space for the fermions.

- Show that $f_{\rm P}(x_0) = \text{constant}$ for mass-less quarks.
  hints:
  - write down the Wick-contraction in terms of the Schrödinger functional propagator
  - note that it is apropriate to go to momentum space concerning the space components, but to remain in coordinate space concerning the time coordinates
  - what is the the equation for the spatial $\mathbf{p} = 0$ contribution to the propagator? note how it splits into $P_{\pm}$ pieces
  - solve the equation by “inspection”, iteration
  - obtain the result for arbitrary quark mass
- Could this result be guessed by dimensional reasoning?