Non-perturbative Renormalization

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NIC @ DESY, Zeuthen

Natal, March 2013

Strategy

finite volume coupling $\alpha_{\rm SF}(\mu), \mu = 1/L$
defined at zero quark mass

$$L_{\rm max} = {\rm const.}/m_{\rm prot} = {\rm O}(\tfrac{1}{2}{\rm fm}) : \quad \longrightarrow \quad \alpha_{\rm SF}(\mu = 1/L_{\rm max})$$

$$\downarrow$$

$$\alpha_{\rm SF}(\mu = 2/L_{\rm max})$$

$$\downarrow$$

always $a/L \ll 1$ $\quad$ • • •

$$\downarrow$$

$$\alpha_{\rm SF}(\mu = 2^n/L_{\rm max} = 1/L_{\rm min})$$

PT: $\downarrow$

$$\Lambda_{\rm SF} L_{\rm max} = \#$$

Result is a value for $\boxed{\Lambda_{\rm SF}/m_{\rm prot} = \#}$

We leave the discussion of a finite volume coupling for later.
Discuss first the

Step scaling function

- It is a discrete β function:

$$\sigma(s, \bar{g}^2(L)) = \bar{g}^2(sL) \qquad \text{mostly } s = 2$$

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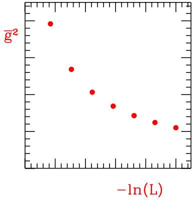
Step scaling function

- It is a discrete β function:

$$\sigma(s, \bar{g}^2(L)) = \bar{g}^2(sL) \qquad \text{mostly } s = 2$$

- determines the non-perturbative running:

$$\begin{aligned} u_0 &= \bar{g}^2(L_{\max}) \\ &\downarrow \\ \sigma(2, u_{k+1}) &= u_k \\ &\downarrow \\ u_k &= \bar{g}^2(2^{-k} L_{\max}) \end{aligned}$$

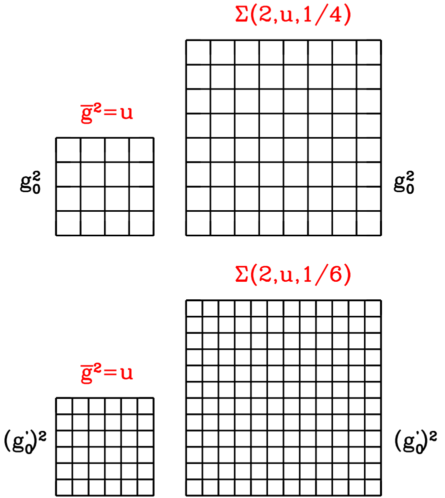


$\sigma(s,u) = \bar{g}^2(sL)$ with $u = \bar{g}^2(L)$

On the lattice:
additional dependence on the resolution a/L

g_0 fixed, L/a fixed:

$$\bar{g}^2(L) = u, \qquad \bar{g}^2(sL) = u'\,,$$
$$\Sigma(s,u,a/L) \quad = \quad u'$$



continuum limit:

$$\Sigma(s,u,a/L) = \sigma(s,u) + \mathrm{O}(a/L)$$

in the following always $s = 2$

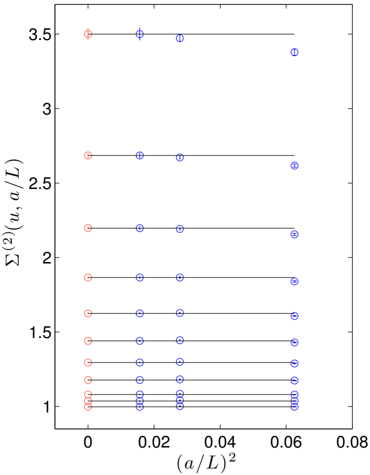
everywhere: $m = 0$ (PCAC mass defined in $(L/a)^4$ lattice)

The step scaling function

(Table from $N_\mathrm{f} = 2$, ALPHA Collaboration)

L/a	β	κ	$\bar{g}^2$	$d\bar{g}^2$	m	dm
$u = 1.1814$						
4	8.2373	0.1327957	1.1814	0.0005	0.00100	0.00011
5	8.3900	0.1325800	1.1807	0.0012	-0.00018	0.00009
6	8.5000	0.1325094	1.1814	0.0015	-0.00036	0.00003
8	8.7223	0.1322907	1.1818	0.0029	-0.00115	0.00004
8	8.2373	0.1327957	1.3154	0.0055	0.00020	0.00005
10	8.3900	0.1325800	1.3287	0.0059	0.00097	0.00007
12	8.5000	0.1325094	1.3253	0.0067	-0.00102	0.00002
16	8.7223	0.1322907	1.3347	0.0061	-0.00194	0.00002
L/a			$\Sigma(1.1814, a/L)$	$\delta\Sigma$		
4			1.3154	0.0055		
5			1.3296	0.0061		
6			1.3253	0.0070		
8			1.3342	0.0071		

- tune κ, g_0 to have desired $m \approx 0$, fixed $\bar{g}^2(L)$
- propagate errors from $\bar{g}^2(L)$, shift means if necessary $\longrightarrow \Sigma, \delta\Sigma$



- *Constant fit:*

$$\Sigma^{(2)}(u, a/L) = \sigma(u)$$

for $L/a = 6, 8$

- *Global fit:*

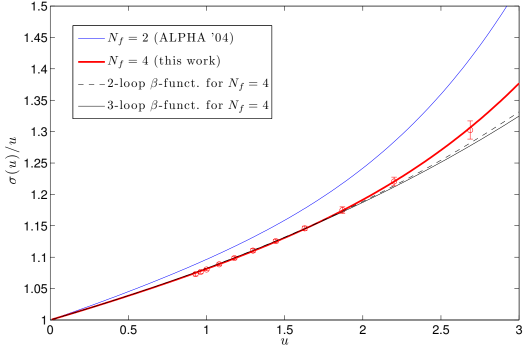
$$\Sigma^{(2)}(u, a/L) = \sigma(u) + \rho\, u^4\, (a/L)^2$$

for $L/a = 6, 8$
$\to \rho = 0.007(85)$

- $L/a = 8$ *data:*

$$\sigma(u) = \Sigma^{(2)}(u, 1/8)$$

[ALPHA Collaboration (S., Tekin & Wolff, 2010); update: M. Marinkovic, 2013] compare it to perturbation theory and other flavour numbers



[ALPHA Collaboration, 2010; update: M. Marinkovic, 2013]

Excellent agreement with PT over a large range of couplings in this particular scheme.

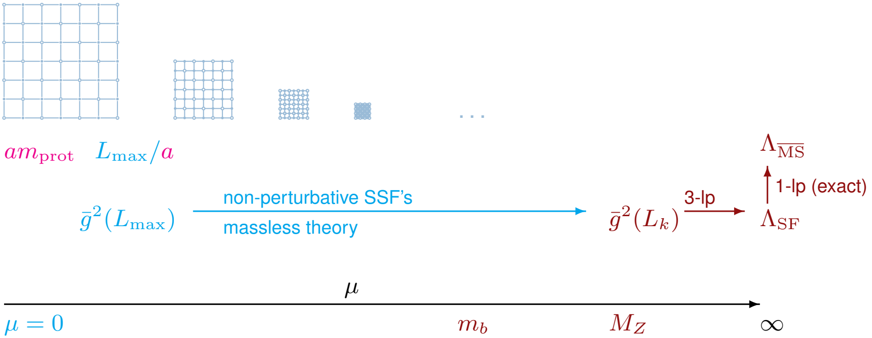
Recursive reconstruction of $\bar{g}_{\rm SF}(L)$

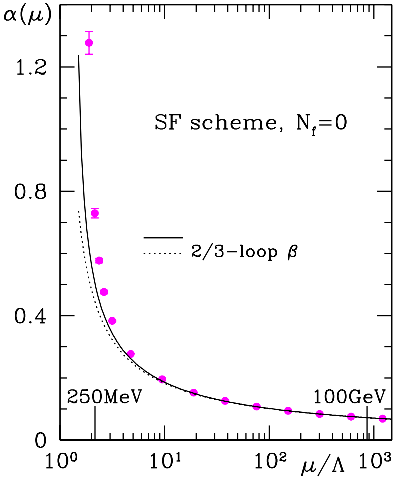
$$u_i \equiv \bar{g}^2\left(L_{\max}/2^i\right)$$
$$u_i = \sigma(u_{i+1}), \quad i=0,\ldots,n, \quad u_0 = u_{\max} = \bar{g}^2\left(L_{\max}\right),$$

solve for u_{i+1}, $i = 0\ldots n = 10$

The strategy

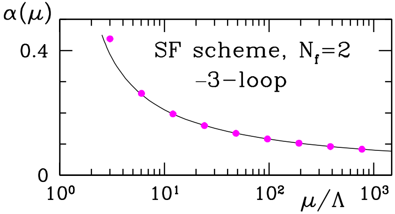
$$\frac{\Lambda_{\overline{\rm MS}}}{m_{\rm proton}} = \frac{1}{m_{\rm proton} L_{\rm max}} \times \frac{L_{\rm max}}{L_k} \times L_k \Lambda_{\rm SF} \times \frac{\Lambda_{\overline{\rm MS}}}{\Lambda_{\rm SF}}$$



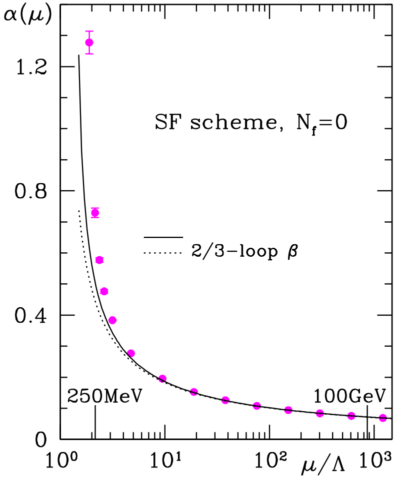


[ALPHA Collaboration, 2001]

Non-perturbative running of α_{SF}



[ALPHA Collaboration, 2005]



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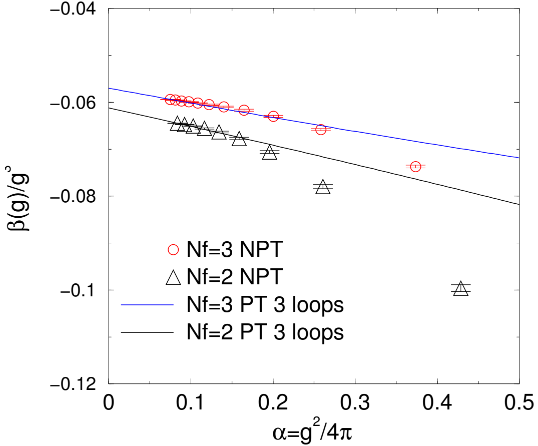
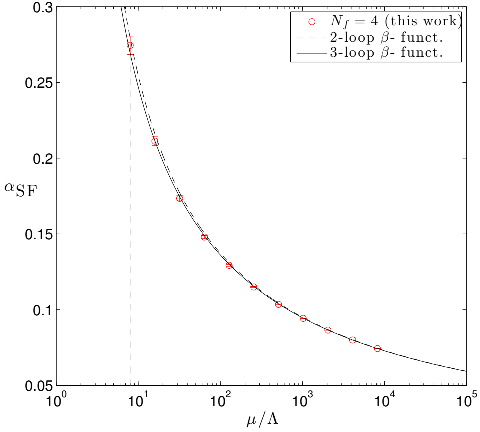


$N_{\rm f} = 3$ [PACS-CS , 2009]
$N_{\rm f} = 2$ [ALPHA Collaboration, 2004]

Non-perturbative running of α_{SF}



[ALPHA Collaboration, 2010; update: M. Marinkovic, 2013]

energies μ

$0.9\,\mathrm{GeV}$

.

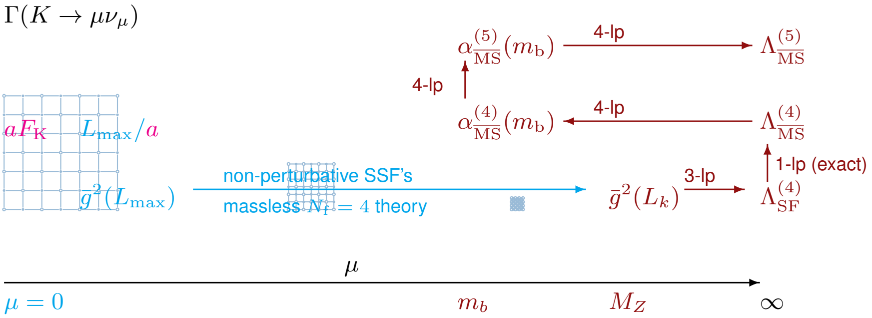
.

.

$900\,\mathrm{GeV}$

The complete strategy

$$\frac{\Lambda_{\overline{\rm MS}}^{(5)}}{F_{\rm K}} = \frac{1}{F_{\rm K} L_{\rm max}} \times \frac{L_{\rm max}}{L_k} \times L_k \Lambda_{\overline{\rm MS}}^{(4)} \times \frac{\Lambda_{\overline{\rm MS}}^{(5)}}{\Lambda_{\overline{\rm MS}}^{(4)}}$$



- for $N_\mathrm{f} = 4$ missing completely
- for $N_\mathrm{f} = 3$, CP-PACS $m_\rho L_\mathrm{max}$ (with 1-loop c_t)
- for $N_\mathrm{f} = 2$

- for $N_f = 4$ missing completely
- for $N_f = 3$, CP-PACS $m_\rho L_{max}$ (with 1-loop c_t)
- for $N_f = 2$

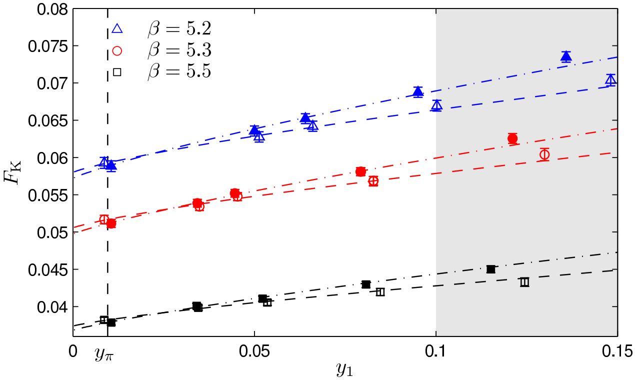
  issues:
  – autocorrelations
  – chiral extrapolation in u/d
  – continuum extrapolation

- for $N_{\mathrm{f}} = 4$ missing completely
- for $N_{\mathrm{f}} = 3$, CP-PACS $m_\rho L_{\mathrm{max}}$ (with 1-loop c_{t})
- for $N_{\mathrm{f}} = 2$

  issues:
  – autocorrelations
  – chiral extrapolation in u/d
  – continuum extrapolation
-

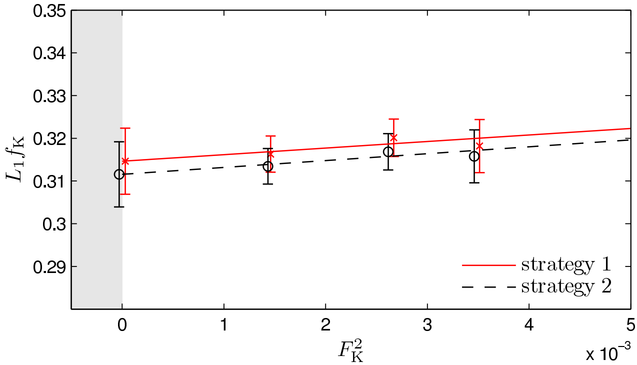
$$
\begin{aligned}
F_{\mathrm{K}} L_{\mathrm{max}} &= f(\beta) \cdot l_{\mathrm{max}}(\beta) + \mathrm{O}(a^2 F_{\mathrm{K}}^2) \\
& \qquad f(\beta) = F_{\mathrm{K}} a \,, \quad l_{\mathrm{max}}(\beta) = L_{\mathrm{max}}/a
\end{aligned}
$$

Interpolate $l_{\mathrm{max}}(\beta)$ to CLS (used in large volume) β's



$$y_1 = m_\pi^2/(4\pi f_K)^2$$

Continuum extrapolation of $F_K L_{max}$



$$F_K = a f_K$$

N_f dependence of $\Lambda_{\overline{MS}}$ and comparison to phenomenology

$\Lambda_{\overline{MS}}$[MeV] status 2011

Experiment	Theory	N_f: 0	2	3	4	5
M_K, $K \to l2, l3$	SF [ALPHA Collaboration]	238(19)	310(20)			
M_K, M_ρ	SF [PACS-CS]			362(23)(25)		239(10)(6)(-22)
DIS, HERA	PT, PDF-fits [ABM11]				234(14)	160(11)
DIS, HERA	PT, PDF-fits [MSTW09]				285(23)	198(16)
"world av. "[2011]	PT					212(12)
$e^+e^- \to$ had (LEP)	4-loop PT					275(57)

- Non-trivial, non-perturbative N_f-dependence.
- Small errors are cited, but overall consistency is not that great.
- More precision and rigor (PT only at high energy) will be very useful.

Boundary conditions matter in finite volume. Which ones?

A most relevant criterion is zero modes

- Zero modes of gauge fields
  $\rightarrow$ perturbative expansion (+ MC)
- Zero modes of Dirac operator
  $\rightarrow$ HMC stability

Path integral w.o. fermions

$$\langle O(U)\rangle = \frac{1}{\mathcal{Z}}\int \mathrm{D}[U]\mathrm{e}^{-\beta\bar{S}(U)}O(U)$$

$$\bar{S}(U) = \sum_p \mathrm{tr}\,(1-U(p))\,, \quad \beta = \frac{6}{g_0^2}$$

PT, sketchy

$\beta \to \infty$ $\quad U \approx U_{\mathrm{min}} \equiv V$ dominates (classical solution)

$$U(x,\mu) = V(x,\mu)\mathrm{e}^{\bar{q}^b_\mu(x)T^b}\,, \quad \bar{q}^b_\mu(x) \ll 1\,, \quad \int \mathrm{D}[U] \to \int \mathrm{D}[\bar{q}]$$

$$\bar{S}(U) = \bar{S}(V) + \sum_{n,m} q_m K_{mn} q_n + \mathrm{O}(q^3)\,, \quad q_n = \bar{q}^b_\mu(x)\,,\ n = (\tfrac{x}{a},\mu,b)$$

$$O(U) = O(V) + \ldots$$

Gauss intergrals $\to$ Wick contractions ... IFF K has no zero modes ($Kv = \lambda v\,,\ \lambda > 0$)

Generically there are zero modes

- gauge modes $\to$ gauge fixing
- finite volume modes (gauge invariant)
  "Ground state metamorhosis"[Gonzales Arroyo, Jurkiewicz, Korthals-Altes] with periodic BC's

Finite volume schemes

Ground state metamorhosis

Toy example: SU(2), $L^4, L = a$ lattice, PBC, $d = 2$, single point

- $\bar{S} = 2 -$ tr $(U_2 U_1 U_2^\dagger U_1^\dagger)$
- tr U_i is gauge invariant, U_i can't be gauged away
- minima: $U_1 = U_2 = V$ ...pick $U_1 = U_2 = 1$.
- fluctuations $U_i = \mathrm{e}^{i\sigma^b q_i^b}$

$$\bar{S} \quad = \quad 2 - \mathrm{tr}\, \mathrm{e}^{i\sigma^b q_2^b} \mathrm{e}^{i\sigma^b q_1^b} \mathrm{e}^{-i\sigma^b q_2^b} \mathrm{e}^{-i\sigma^b q_1^b} = \mathrm{O}(q^4) \quad \to K = 0$$

- $q = \mathrm{O}(\beta^{-1/4}) = \mathrm{O}(g_0^{1/2})$
  PT in powers of g_0, not g_0^2 NOT regular
- In general: mixture of gaussian and non-gaussian modes
  integrate over non-gaussian ones exactly ...
  complicated, non-universal β-function
  it can be worse, divergent behavior, $1/\log(g)$ terms , see [Nogradi et al., 2012]
- think of these U_i as Polyakov loops $\to$ relevant for 4-d gauge theory.
  "Ground state metamorhosis"[Gonzales Arroyo, Jurkiewicz, Korthals-Altes]

$$V(x,\mu) = 1\,, \quad \text{PBC: } \psi(x+L\hat{\mu}) = \psi(x)$$

massless Dirac operator has a zero mode (constant mode, $p = 0$)

easily fixed by

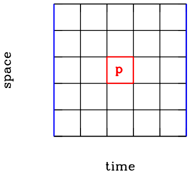
$$\psi(x+L\hat{\mu}) = \mathrm{e}^{i\alpha}\psi(x)$$

e.g. $\alpha = \pi/2$ in SU(2), $\alpha = \pi/3$ in SU(3)

Exercise: why these values of α?

Boundary conditions

- Space: PBC
- Time: Dirichlet, breaks translation invariance!



Yang Mills theory [Lüscher, Narayanan, Weisz & Wolff]:

$$\mathcal{Z}(V,V') = \int \mathrm{D}(U)_{\text{inside}}\, \mathrm{e}^{-S_{\mathrm{SF}}(U)}$$

$$S_{\mathrm{SF}}(U) = \sum_{p \text{ inside}} \beta\, \mathrm{tr}\,(1 - U(p))\,, \quad U(x,k) = \begin{cases} V(\mathbf{x},k) & x_0 = 0 \\ V'(\mathbf{x},k) & x_0 = T \end{cases}$$

Standard introduction of Hilbert space, transfer matrix:

$$\mathcal{Z}(V,V') = \langle V'|\underbrace{\mathrm{e}^{-\hat{H}T}}_{\mathbb{T}^{T/a}} \underbrace{\mathbb{P}_0}_{\uparrow} |V\rangle\,, \quad \hat{U}(\mathbf{x},k)|U\rangle = U(\mathbf{x},k)|U\rangle$$

projector onto gauge invariant states

$\mathcal{Z}(V,V') =$ Euclidean time propagation kernel by time T = Schrödinger functional

Wilson Dirac operator (also others are possible)

$$D_{\rm W} = \tfrac{1}{2}\{\gamma_\mu(\nabla_\mu+\nabla_\mu^*) - a\nabla_\mu^*\nabla_\mu\}$$

$$\nabla_\mu\psi(x) = \frac{1}{a}\left[U(x,\mu)\psi(x+a\hat\mu)-\psi(x)\right]$$

$$\nabla_\mu^*\psi(x) = \frac{1}{a}\left[\psi(x)-U(x-a\hat\mu,\mu)^{-1}\psi(x-a\hat\mu)\right]$$

Schrödinger functional action

$$S_{\rm F} = a^4\sum_x \overline{\psi}(x)[m_0+D_{\rm W}]\psi(x)\,,$$

with $\psi(x)=0\,,\ \overline{\psi}(x)=0$ for $x_0\leq 0\,,$ and $x_0\geq T$

In the continuum theory this corresponds to BC's [Sint, 1994]

$$P_+\psi(x)|_{x_0=0}=0 \qquad \overline{\psi}(x)P_-\Big|_{x_0=0}=0 \qquad P_\pm = \tfrac{1}{2}(1\pm\gamma_0)$$

$$P_-\psi(x)|_{x_0=T}=0 \qquad \overline{\psi}(x)P_+\Big|_{x_0=T}=0$$

These BC's are stable: emerge in the cont. limit without fine-tuning. Universality! [Lüscher, 2006]
The Universality class is characterised by Parity invariance, discrete rot. invariance (not chiral symm).

Correlation functions can be formed with the usual fields in the interior (bulk) **and the boundary quark fields**

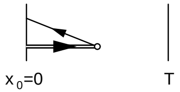
$$\zeta(\mathbf{x}) = P_{-}U(x,0)\psi(x+a\hat{0})\big|_{x_0=0} \qquad \bar{\zeta}(\mathbf{x}) = \bar{\psi}(x+a\hat{0})P_{+}U(x,0)^{-1}\big|_{x_0=0}$$

$$\zeta'(\mathbf{x}) = P_{+}U(x-a\hat{0},0)^{-1}\psi(x-a\hat{0})\big|_{x_0=T} \qquad \bar{\zeta}'(\mathbf{x}) = \bar{\psi}(x-a\hat{0})P_{-}U(x-a\hat{0},0)\big|_{x_0=T}$$

A very interesting feature of these is that one can form correlation functions where the **quark** fields are projected to $\mathbf{p}=0$. (Note that the gauge fields at the boundaries are fixed).

$$f_{\rm P}^{rs}(x_0) = a^6 \sum_{\mathbf{v},\mathbf{y}} \langle \bar{\zeta}_s(\mathbf{v})\gamma_5\zeta_r(\mathbf{y})P^{rs}(x)\rangle$$

$$P^{rs}(x) = \bar{\psi}_r(x)\gamma_5\psi_s(x)$$



boundary quark fields

$$\zeta(\mathbf{x}) = P_- U(x,0)\psi(x+a\hat{0})\big|_{x_0=0} \qquad \bar{\zeta}(\mathbf{x}) = \bar{\psi}(x+a\hat{0})P_+ U(x,0)^{-1}\big|_{x_0=0}$$

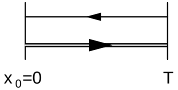
$$\zeta'(\mathbf{x}) = P_+ U(x-a\hat{0},0)^{-1}\psi(x-a\hat{0})\big|_{x_0=T} \qquad \bar{\zeta}'(\mathbf{x}) = \bar{\psi}(x-a\hat{0})P_- U(x-a\hat{0},0)\big|_{x_0=T}$$

These boundary quark fields renormalize multiplicatively.

$$\zeta_{\rm R}(\mathbf{x}) = Z_\zeta \zeta(\mathbf{x})\,,\;\ldots\;,\; \bar{\zeta}'_{\rm R} = Z_\zeta \bar{\zeta}'(\mathbf{x})$$

Define also boundary-to-boundary correlation functions

$$f_1^{rs} = \frac{a^{12}}{L^6} \sum_{\mathbf{v},\mathbf{y},\mathbf{u},\mathbf{x}} \langle \bar{\zeta}_s(\mathbf{v})\gamma_5\zeta_r(\mathbf{y})\bar{\zeta}'_r(\mathbf{u})\gamma_5\zeta'_s(\mathbf{x})\rangle$$



Then

$$(f_1^{rs})_{\rm R} = Z_\zeta^4\,(f_1^{rs})\,, \quad (f_{\rm P}^{rs}(x_0))_{\rm R} = Z_\zeta^2\, Z_{\rm P}\;(f_{\rm P}^{rs}(x_0))$$

- Regular PT (no gauge field zero modes)
- Gap for Dirac operators
- Momentum zero boundary quark fields
  (spatially one takes pbc up to a phase, cf “flavor twisted bc”)
- Schrödinger functional coupling defined with non-trivial V, V'
  β-function known to 3-loops [LNWW; LW; Bode, Weisz, Wolff]

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- We can define Z-factors (schemes) for composite fields, e.g.

$$Z_{\rm P} = \frac{1}{c(a/L)} \frac{\sqrt{f_1^{rs}}}{f_{\rm P}^{rs}(T/2)}\,, \quad c(a/L) = \left.\frac{\sqrt{f_1^{rs}}}{f_{\rm P}^{rs}(T/2)}\right|_{g_0=0}$$

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- There is also a new SF coupling . . .

Exercise

Consider the free Schrödinger functional , i.e. $U(x,\mu)=1$ with pbc in space for the fermions.

- Show that $f_{\rm P}(x_0) = \mathrm{constant}$ for mass-less quarks.
  hints:
  - write down the Wick-contraction in terms of the Schrödinger functional propagator
  - note that it is apropriate to go to momentum space concerning the space components, but to remain in coordinate space concerning the time coordinates
  - what is the the equation for the spatial $\mathbf{p}=0$ contribution to the propagator? note how it splits into $P_{\pm}$ pieces
  - solve the equation by “inspection”, iteration
  - obtain the result for arbitrary quark mass
- Could this result be guessed by dimensional reasoning?

Gradient Flow and SF-coupling

- Gradient flow [Lüscher, 2010; Lüscher & Weisz, 2011]

  new observables
  - UV finite (proven to all orders of PT)
  - excellent numerical precision
  - renormalized coupling in finite volume with pbc [BMW, 2012]
- Flow in finite volume, SF [P. Fritzsch & Ramos, arXiv:1301.4388]
  - lowest order PT to define a new coupling
  - numerical investigation shows excellent precision
- Flow with gauge fields AND quark fields [Lüscher, arXiv:1302.5246]
- General idea

$$x = (x_0, \mathbf{x}) \,, \quad t = \text{ flow time}$$

$$A_\mu(x) = \text{ quantum gauge fields } : \quad \mathcal{Z} = \int \mathrm{D}[A_\mu(x)] \, \ldots$$

$$B_\mu(x,t) = \text{ smoothed gauge fields } , \quad B_\mu(x,0) = A_\mu(x)$$

$$\frac{\mathrm{d}B_\mu(x,t)}{\mathrm{d}t} = D_\nu G_{\nu\mu}(x,t) + \text{gauge fixing}$$

$$\sim -\frac{\delta S_{YM}[B]}{\delta B_\mu}$$

correlation functions of B-fields at arbitrary points are finite

$$\frac{\mathrm{d}B_\mu(x,t)}{\mathrm{d}t} \equiv \dot{B}_\mu(x,t) = \underbrace{D_\nu G_{\nu\mu}(x,t)}_{\uparrow} + \underbrace{D_\mu \partial_\nu B_\nu(x,t)}_{\uparrow} \qquad (*)$$

$$\sim -\frac{\delta S_{YM}[B]}{\delta B_\mu}$$

eliminate by a t-dependent gauge trafo

- it is a continuous form of stout smearing [Morningstar & Peardon]
- in PT: $A_\mu(x) = g_0 \bar{A}_\mu(x)$

$$\begin{aligned} B_\mu(x,t) &= B_{\mu,1}(x,t)g_0 + B_{\mu,2}(x,t)g_0^2 + \ldots \\ & \quad G_{\nu\mu} = [\partial_\nu B_{\mu,1} - \partial_\mu B_{\nu,1}]\, g_0 + \mathrm{O}(g_0^2)\,, \quad D_\nu = \partial_\nu + \mathrm{O}(g_0) \\ \to \quad \dot{B}_{\mu,1}(x,t) &= \partial_\nu \partial_\nu B_{\mu,1}(x,t) \end{aligned}$$

- heat equation

- in PT: $A_\mu(x) = g_0 \bar{A}_\mu(x)$

$$B_\mu(x,t) = B_{\mu,1}(x,t) g_0 + B_{\mu,2}(x,t) g_0^2 + \dots$$

$$G_{\nu\mu} = [\partial_\nu B_{\mu,1} - \partial_\mu B_{\nu,1}]\, g_0 + \mathrm{O}(g_0^2)\,, \quad D_\nu = \partial_\nu + \mathrm{O}(g_0)$$

$$\rightarrow \quad \dot{B}_{\mu,1}(x,t) = \partial_\nu \partial_\nu B_{\mu,1}(x,t)$$

- heat equation

$$B_{\mu,1}(x,t) = \int \mathrm{d}^D p\, \mathrm{e}^{ipx}\, b_\mu(p,t)$$

$$\dot{b}_\mu = -p^2 b_\mu \quad \rightarrow \quad b_\mu(p,t) = b_\mu(p,0) \mathrm{e}^{-p^2 t}$$

$$B_{\mu,1}(x,t) = \int \mathrm{d}^D y\, K_t(x-y)\, \bar{A}_\mu(y)\,, \quad K_t(z) = (4\pi t)^{-D/2} \mathrm{e}^{-z^2/(4t)}$$

- smoothing over a radius of $\sqrt{8t}$
- gaussian damping of large momenta

- in PT: $A_\mu(x) = g_0 \bar{A}_\mu(x)$

$$\begin{aligned} B_\mu(x,t) &= B_{\mu,1}(x,t) g_0 + B_{\mu,2}(x,t) g_0^2 + \ldots \\ & \quad G_{\nu\mu} = [\partial_\nu B_{\mu,1} - \partial_\mu B_{\nu,1}]\, g_0 + \mathrm{O}(g_0^2)\,, \quad D_\nu = \partial_\nu + \mathrm{O}(g_0) \\ \to \quad \dot{B}_{\mu,1}(x,t) &= \partial_\nu \partial_\nu B_{\mu,1}(x,t) \end{aligned}$$

- heat equation

$$\begin{aligned} B_{\mu,1}(x,t) &= \int \mathrm{d}^D p\, \mathrm{e}^{ipx}\, b_\mu(p,t) \\ & \quad \dot{b}_\mu = -p^2 b_\mu \quad \to \quad b_\mu(p,t) = b_\mu(p,0)\mathrm{e}^{-p^2 t} \\ B_{\mu,1}(x,t) &= \int \mathrm{d}^D y\, K_t(x-y)\, \bar{A}_\mu(y)\,, \quad K_t(z) = (4\pi t)^{-D/2} \mathrm{e}^{-z^2/(4t)} \end{aligned}$$

- smoothing over a radius of $\sqrt{8t}$
- gaussian damping of large momenta
- all correlation functions of B_μ are finite $(t > 0)$ [Lüscher & Weisz, 2011]

in particular $\langle E(t) \rangle\,, \quad E(t) = -\frac{1}{2}\,\mathrm{tr}\, G_{\mu\nu} G_{\mu\nu}$

- order by order iteration:

$$
\begin{aligned}
B_\mu(x,t) &= \sum_k B_{\mu,n}(x,t) g_0^k \\
\dot B_{\mu,k}(x,t) - \partial_\nu \partial_\nu B_{\mu,k}(x,t) &= R_{\mu,k} \\
R_{\mu,1} &= 0, \qquad B_{\mu,1}(x,t) = \int \mathrm{d}^D y\; K_t(x-y)\; \bar A_\mu(y) \\
R_{\mu,2} &= 2[B_{\nu,1}, \partial_\nu B_{\mu,1}] - [B_{\nu,1}, \partial_\mu B_{\nu,1}], \\
R_{\mu,3} &= 2[B_{\nu,2}, \partial_\nu B_{\mu,1}] + 2[B_{\nu,1}, \partial_\nu B_{\mu,2}] \\
&\quad - [B_{\nu,2}, \partial_\mu B_{\nu,1}] - [B_{\nu,1}, \partial_\mu B_{\nu,2}] + [B_{\nu,1}, [B_{\nu,1}, B_{\mu,1}]], \\
\ldots \\
B_{\mu,k}(t,x) &= \int_0^t \mathrm{d}s \int \mathrm{d}^D y\, K_{t-s}(x-y) R_{\mu,k}(s,y) \quad k > 1
\end{aligned}
$$

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\end{aligned}
$$

- For $\langle E \rangle$, $\;E = -\frac{1}{2}\,\mathrm{tr}\, G_{\mu\nu} G_{\mu\nu}$

$$
\begin{aligned}
\langle E \rangle &= E_0\, g_0^2 + E_0\, g_0^4 + \ldots \\
E_0 &= \langle\, \mathrm{tr}\, [\partial_\mu B_{\nu,1} \partial_\mu B_{\nu,1} - \partial_\mu B_{\nu,1} \partial_\nu B_{\mu,1}] \rangle \\
&\sim \int_p \mathrm{e}^{-p^2 2t} [p^2 \delta_{\mu\nu} - p_\mu p_\nu]\, D_{\mu\nu}(p)
\end{aligned}
$$

finite (also with cutoff reg'n)!

use the flow in SF: $T \times L^3$ world with Dirichlet BC in time, $T = L$
define

$$\begin{aligned} \langle E(t) \rangle &\equiv -\tfrac{1}{2}\langle \,\mathrm{tr}\, G_{\mu\nu} G_{\mu\nu}(x,t)\rangle_{x_0=T/2} = \frac{\mathcal{N}}{t^2}\,\bar{g}^2_{\mathrm{MS}}(\mu)\,(1 + c_1 \bar{g}^2_{\mathrm{MS}} + \ldots) \\ \bar{g}^2_{\mathrm{GF}}(L) &\equiv \mathcal{N}^{-1}\, t^2 \langle E(t) \rangle\Big|_{t=c^2L^2/8} \end{aligned}$$

This is a family of schemes characterized by c (dimensionless)

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$$\bar{g}^2_{\mathrm{GF}}(L) \equiv \mathcal{N}^{-1}\, t^2 \langle E(t) \rangle \Big|_{t=c^2 L^2/8}$$

This is a family of schemes characterized by c (dimensionless)

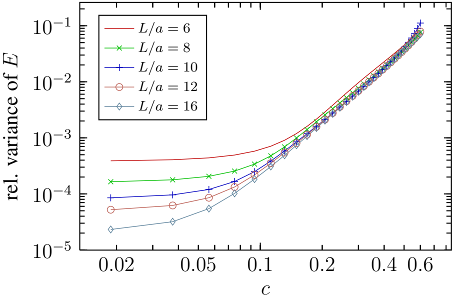
$$\mathcal{N}(c) = \frac{c^4(N^2-1)}{128} \sum_{\mathbf{n},n_0} e^{-c^2\pi^2(\mathbf{n}^2+\frac{1}{4}n_0^2)} \times \frac{2\mathbf{n}^2 s^2_{n_0}(T/2) + (\mathbf{n}^2+\frac{3}{4}n_0^2) c^2_{n_0}(T/2)}{\mathbf{n}^2+\frac{1}{4}n_0^2}$$

- the lattice version is known (and needed)

$$\text{relative variance} = \frac{\langle E^2 \rangle - \langle E \rangle^2}{\langle E \rangle^2}$$

should be finite as $a \to 0$, $L/a \to \infty$

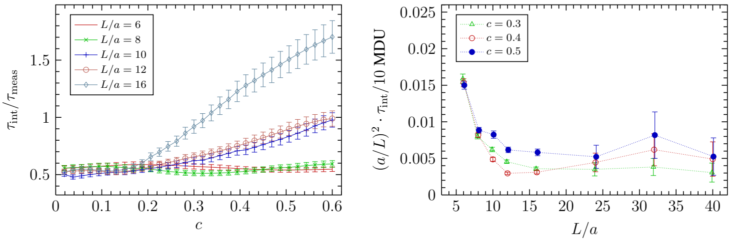
Numerically, Fritzsch & Ramos:



Gradient Flow and SF-coupling

statistical precision

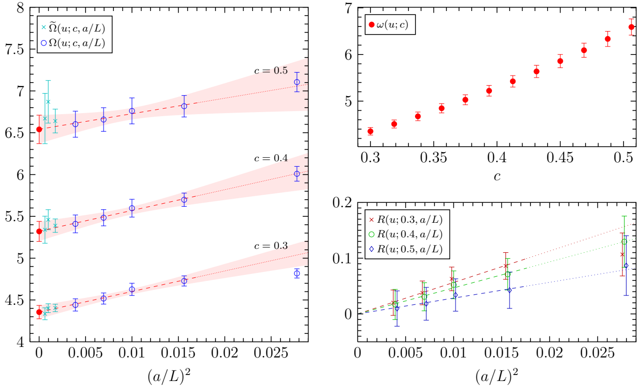
autocorrelations scale as expected: $\tau_{\rm int} \propto a^{-2}$



Statistical precision is good and theoretically understood.
There will be no surprises on the way to the continuum limit.

keeping old SF-coupling $\bar{g}_{\rm SF}(L)$ fixed (defines L), compute

$$\Omega(u;c,a/L) \quad = \left[\hat{\mathcal{N}}^{-1}(c,a/L)\cdot t^2\langle E(t,T/2)\rangle\right]^{\bar{g}^2_{\rm SF}=u,\, m=0}_{t=c^2L^2/8}$$



- small cutoff effects $\longrightarrow$ ready for applications $\longrightarrow$ ...
  $\rightarrow$ precise Λ-parameter

Flow equation for the quarks $\chi = \chi(x,t)$

$$\partial_t \chi = \Delta\chi, \qquad \partial_t \overline{\chi} = \overline{\chi}\overleftarrow{\Delta},$$
$$\Delta = D_\mu D_\mu, \qquad D_\mu = \partial_\mu + B_\mu,$$

with initial conditions

$$\chi|_{t=0} = \psi, \qquad \overline{\chi}|_{t=0} = \overline{\psi},$$

This is very similar to “Gaussian smearing”, “Wuppertal smearing” [S. Güsken, U. Löw, K. H. Mütter, RS, A. Patel, K. Schilling, 1989] except that it is continuous and Δ depends on t through $B_\mu(x,t)$ and is the 4-d cov. Laplacian.

It might be useful to consider (approximate) continuity and $B_\mu(x,t)$ also for Gaussian smearing.

For a perturbative analysis add terms which can be removed by a gauge transformation

$$\partial_t \chi = [\Delta - \partial_\nu B_\nu]\chi, \qquad \partial_t \overline{\chi} = \overline{\chi}[\overleftarrow{\Delta} + \partial_\nu B_\nu],$$
$$\Delta = D_\mu D_\mu, \qquad D_\mu = \partial_\mu + B_\mu,$$

Lowest order in the coupling (remember $B_\mu = B_{\mu,1} g_0 + \ldots$)

$$\Delta = \partial_\mu \partial_\mu$$

solved again by the heat kernel

$$\chi(x,t) = \int \mathrm{d}^D y \, K_t(x-y)\psi(y) + \mathrm{O}(g_0)$$
$$K_t(z) = \frac{\mathrm{e}^{-z^2/4t}}{(4\pi t)^{D/2}}$$

A smooth, smeared, field. But not a local field.

The “quark” propagator at leading order, at finite t

$$\langle \chi(t,x)\overline{\chi}(s,y)\rangle = \int \frac{\mathrm{d}^D p}{(2\pi)^D}\, \mathrm{e}^{ip(x-y)}\, \frac{\mathrm{e}^{-(t+s)p^2}}{M_0 + i\not{p}} + \mathrm{O}(g_0^2)$$

The one-loop self-energy graphs have a divergence which is cancelled by wavefunction renormalization

$$\chi = Z_\chi^{-1/2}\chi_\mathrm{R}, \qquad \overline{\chi} = \overline{\chi}_\mathrm{R} Z_\chi^{-1/2},$$

$$Z_\chi = 1 + \frac{3C_\mathrm{F}}{16\pi^2\epsilon} g^2 + \mathrm{O}(g^4), \qquad C_\mathrm{F} = \frac{N^2-1}{2N},$$

At $t, s > 0$ the quark propagator is now non-singular for

$$(t, x) \;\to\; (s, y)$$

In particular

$$\Sigma_{\mathrm{R},t}^{rr} = Z_\chi \Sigma_t^{rr}$$

No additive renormalization!

Correlation functions of the t-dependent fields can be written in terms of a 4+1 dimensional *local* field theory [Zinn-Justin, 1986; Zinn-Justin & Zwanziger, 1988], in dim-reg.

Action

$$S_{\rm tot} = S_{\rm 4d} + S_{\rm G,fl} + S_{\rm FP,fl} + S_{\rm F,fl} \qquad \alpha_0 = 1$$

$$S_{\rm G,fl} = -2\int_0^\infty {\rm d}t \int {\rm d}^D x \; {\rm tr}\left\{ L_\mu(t,x)\big(\partial_t B_\mu - D_\nu G_{\nu\mu} - \alpha_0 D_\mu \partial_\nu B_\nu\big)(t,x)\right\},$$

$$S_{\rm F,fl} = \int_0^\infty {\rm d}t \int {\rm d}^D x \,\{\bar\lambda(t,x)(\partial_t - \Delta + \alpha_0 \partial_\nu B_\nu)\chi(t,x) + \overline{\chi}(t,x)(\overleftarrow{\partial}_t - \overleftarrow{\Delta} - \alpha_0\partial_\nu B_\nu)\lambda(t,x)\}$$

- $L_\mu,\ \lambda,\ \bar\lambda$: Lagrange multipliere fields

  Variation wrt $L_\mu,\ \lambda,\ \bar\lambda$: flow equations for the fields $B, \chi, \overline{\chi}$
- $S_{\rm 4d}$: determines the initial conditions at $t = 0$ to be the quantum fields of the $4d$ theory
- the exact equivalence is seen by considering the Wick contractions

This *local* formulation helps to identify possible additional (to the 4-d theory) counterterms.

The only one is (note: $[\lambda] = 5/2$)

$$\int \mathrm{d}^D x \, \{\bar{\lambda}(0,x)\psi(x) + \overline{\psi}(x)\lambda(0,x)\}$$

corresponding to the field renormalizations

$$\chi = Z_\chi^{-1/2}\chi_\mathrm{R}, \qquad \lambda = Z_\chi^{1/2}\lambda_\mathrm{R}$$

The propagators contain θ-functions

$$\theta(s-t)$$

therefore Feynman diagrams have loops only for $t = 0$ and trees going to $t > 0$. This means there are no divergencies for the fields at positive t (apart from the above?)

a tree

An all-order perturbative proof of the renormalization properties has been given by Lüscher and Weisz (pure gauge theory).

The Grassmann integrals over the fermion fields $\bar{\lambda}\ \ldots\ \psi$ yield (relatively) straight forwardly the Wick contractions

$$\contraction{}{\psi}{(x)}{\overline{\psi}}\psi(x)\overline{\psi}(y)\,\det(\ldots) \equiv \int_{\text{Grassmann}} \psi(x)\overline{\psi}(y)$$

$$\contraction{}{\psi}{(x)}{\overline{\psi}}\psi(x)\overline{\psi}(y) = S(x,y), \qquad (\not{D} + M_0)S(x,y) = \delta(x-y),$$

$$\contraction{}{\lambda}{(t,x)}{\bar{\lambda}}\lambda(t,x)\bar{\lambda}(s,y) = 0,$$

$$\contraction{}{\chi}{(t,x)}{\bar{\lambda}}\chi(t,x)\bar{\lambda}(s,y) = \theta(t-s)K(t,x;s,y)$$

$$\{\partial_t - \Delta + \alpha_0 \partial_\nu B_\nu\}K(t,x;s,y) = 0 \quad \text{if} \quad t \geq s,$$

$$\lim_{t\to s} K(t,x;s,y) = \delta(x-y),$$

Using the Wick contractions, not (as usual) field transformations, Lüscher shows that (generalized) Ward identities hold, e.g.

$$\langle \{\partial_\mu A_\mu^{rs}(x) - (m_{0,r} + m_{0,s})P^{rs}(x) + \tilde{P}^{rs}(x)\}\,\phi_1(t_1,x_1)\ldots\phi_n(t_n,x_n)\rangle$$
$$= 0 \quad \text{if} \quad (x,0) \neq (x_i,t_i)$$

with

$$\tilde{P}^{rs}(x) = \bar{\lambda}_r(0,x)\gamma_5\psi_s(x) + \overline{\psi}_r(x)\gamma_5\lambda_s(0,x)$$

Leading Low Energy Constants of the chiral effective theory

We have the matrix elements

$$\langle \pi^{rs}(\mathbf{p})|i\hat{P}_{\mathrm{R}}^{ud}(\mathbf{x})|0\rangle = \frac{G_\pi}{\sqrt{m_\pi}} \mathrm{e}^{-i\mathbf{px}}$$
$$\langle \pi^{rs}(\mathbf{p})|\hat{A}_{\mathrm{R},0}^{ud}(\mathbf{x})|0\rangle = \frac{F_\pi m_\pi}{\sqrt{m_\pi}} \mathrm{e}^{-i\mathbf{px}}$$

They are obtained from correlation functions by e.g.

$$\int \mathrm{d}^3\mathbf{x}\, \langle P_{\mathrm{R}}^{ud}(x) P_{\mathrm{R}}^{du}(0)\rangle = -\frac{G_\pi G_{\pi,t}}{m_\pi} \mathrm{e}^{-m_\pi x_0} \{1 + \mathrm{O}(\mathrm{e}^{-\Delta E x_0})\}$$

The two lowest order LEC's are given by

$$F = \lim_{m_{\mathrm{R}} \to 0} F_\pi \qquad \Sigma = \lim_{m_{\mathrm{R}} \to 0} F_\pi G_\pi$$

(remember PCAC: $2 m_{\mathrm{R}} G_\pi = m_\pi^2 F_\pi$)

Include correlation functions at positve t

Properties such as

$$\int \mathrm{d}^3\mathbf{x}\, \langle P_{\mathrm{R}}^{ud}(x) P_{\mathrm{R},t}^{du}(0)\rangle = -\frac{G_\pi G_{\pi,t}}{m_\pi} \mathrm{e}^{-m_\pi x_0} \{1 + \mathrm{O}(\mathrm{e}^{-\Delta E x_0})\}$$

together with the generalized WI's allow to show

$$\Sigma_{\mathrm{R},t} = -\frac{m_\pi^2 F_\pi}{2G_\pi} \int \mathrm{d}^4 x\, \langle P_{\mathrm{R}}^{ud}(x) P_{\mathrm{R},t}^{du}(0)\rangle$$

$$\Sigma = \lim_{m_{\mathrm{R}}\to 0} F_\pi G_\pi = \lim_{m_{\mathrm{R}}\to 0} \frac{\Sigma_{\mathrm{R},t} G_\pi}{G_{\pi,t}}$$

$$F = \lim_{m_{\mathrm{R}}\to 0} F_\pi = \lim_{m_{\mathrm{R}}\to 0} \frac{\Sigma_{\mathrm{R},t}}{G_{\pi,t}}$$

- Σ **without** contact terms, a^{-3} subtraction

Include correlation functions at positive t

Properties such as

$$\int \mathrm{d}^3\mathbf{x}\, \langle P_{\mathrm{R}}^{ud}(x) P_{\mathrm{R},t}^{du}(0)\rangle = -\frac{G_\pi G_{\pi,t}}{m_\pi} \mathrm{e}^{-m_\pi x_0} \{1 + \mathrm{O}(\mathrm{e}^{-\Delta E x_0})\}$$

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- Σ **without** contact terms, a^{-3} subtraction

- On the history
  - Smearing has been around for a while starting with APE smearing, Wuppertal smearing,... stout smearing
  - There was always the suspicion that the smeared observables (smeared to physical smearing radii) have no continuum limit
  - A first statement that there is a continuum limit of smeared Wilson loops if the smearing parameters are properly adjusted was in a paper by Narayanan & Neuberger, 2006.
  - Now we have a full understanding that the flow-observables are finite
  - Old and not very well known papers (1986, 1988) have been influential in the all-order proof
- It is remarkable that after so many years of QCD and lattice QCD we have a whole new class of finte observables.
- New elements in the tool kit, mainly for LGT?
- The flow equations are not unique, eg. for the quarks: $-D_\mu D_\mu \to \not{D}\not{D}^\dagger$
- There is an oportunity for a new generation to develop new methods with the (and enlarge the) tool kit (see Fritzsch & Ramos)

Summary I

- The Standard Model is (many feel: too) alive
- We need to push it to its limits in energy **and precision**
- Somewhat provocative but true: If we want a non-perturbative result, we need it renormalized non-perturbatively.
- The perturbative series is divergent, asymptotic (well understood! I recommend 't Hooft Erice lectures).
  When one uses it $\alpha(\mu)$ better is small.
- For scale dependent renormalizations, $\alpha(\mu)\; m_{\mathrm{R}}(\mu)$, $Z_{\mathrm{LL}}(\mu)$

  step scaling with finite volume schemes

  can be used to go to very large μ and connect to

  Renormalization Group Invariants
- On the other hand, RI-sMOM is more genaral (automatic) is mostly used and dominant discretization errors can be removed perturbatively

  Can the question of NP gauge fixing be better understood?

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- **There is a New Horizon**

  The Gradient flow

- **A New Horizon: the Gradient flow**
  - it is remarkable that after so many years of QCD and lattice QCD we have a whole new class of finte observables.
  - little explored
  - waiting for additional ideas
  - New elements in the tool kit (mainly for LGT?)