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TWO LECTURES ON TWO DIMENSIONAL GAUGE THEORIES

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- **Nontrivial spectra of two dimensional gauge theories**
- **Lattice: partition function and its continuum limit**
- **Adding external charges:**

  Lattice: transfer matrix and spectrum

  continuum limit

  Feynman kernel

  reduced system, hamiltonian and wave functions

  theta states

  screening and effective fractional charge
- **Fractional charges on a lattice and the continuum limit**
- **Nonabelian case**

I. Nontrivial spectra of trivial gauge theories

- **Two dimensional gauge theories are trivial - no transverse degrees of freedom.**
- **True only if we neglect boundary conditions.**

Quantum Maxwell Dynamics in 1+1 dimensions (QMD_2) on a circle

$$E_n^{\Phi} = \frac{e^2}{2} L n^2, \qquad n = 0, \pm 1, \pm 2, ... \qquad\qquad [Manton,' 84]$$

An effective 1DOF hamiltonian

$$H = -\frac{e^2}{2L} \frac{d^2}{dA^2}, \qquad 0 \leq A \; < \; L_A = \frac{2\pi}{L} \tag{1}$$

The spectrum

$$\psi_n(A) = e^{inAL} = e^{ip_n A}, \qquad p_n = n\frac{2\pi}{L_A} = nL, \qquad E_n = \frac{e^2}{2} L n^2 \tag{2}$$

What is A ?

$$A_x(x,t) = A(x,t), \qquad \xrightarrow{\partial_x A(x,t) = 0} A(x,t) = A(t) \neq 0$$

In a periodic (in x) world one cannot set a constant A to 0 by a gauge transformation – 1 DOF left

Why periodicity in A ?

If space is periodic, gauge transformations also have to be periodic

$$g(x) = e^{i\Lambda(x)} = g(x+L), \quad \longrightarrow \quad \Lambda(x+L) = \Lambda(x) + 2\pi n$$

Take $\Lambda(x) = 2\pi \frac{x}{L}$**, then**

$$A \longrightarrow A + \partial_x \Lambda(x) = A + \frac{2\pi}{L}, \quad \textit{are gauge equivalent} \quad \Longrightarrow \quad A \in (0, \frac{2\pi}{L}]$$

Interpretation

- **a string with n units of electric flux winding around a circle**
- **Gauss's law satisfied thanks to the nontrivial topology - topological strings**
- **electric charge even without electrons/sources !**

A generalization: Θ parameter

a)

$$H = -\frac{e^2}{2L}\left(\frac{d}{dA} + i\Theta L\right)^2,$$

$$E_n = \frac{e^2}{2}L(n+\Theta)^2, \quad \psi_n(A) = e^{inAL}$$

b)

$$\tilde{H} = -\frac{e^2}{2L}\frac{d^2}{dA^2},$$

$$E_n = \frac{e^2}{2}L(n+\Theta)^2, \quad \tilde{\psi}_n(A) = e^{i(n+\Theta)AL},$$

$$\tilde{\psi}_n(A) = e^{i\Theta AL}\psi_n(A)$$

Interpretation: $e^2\Theta$ – classic, constant electric field

II. QMD_2 on a lattice

Partition function on a 2x2 lattice

$$Z = \int_0^{2\pi} B(\theta_{12} + \vartheta_{22} - \theta_{11} - \vartheta_{12}) B(\theta_{22} + \vartheta_{12} - \theta_{21} - \upsilon_{22})$$
$$B(\theta_{11} + \vartheta_{21} - \theta_{12} - \vartheta_{11}) B(\theta_{21} + \vartheta_{11} - \theta_{22} - \vartheta_{21})$$
$$d(links)$$

$$B(\phi_P) = e^{\beta \cos(\phi_P)}, \quad d(links) = \Pi_l \frac{d\alpha_l}{2\pi}$$

Change variables from links to plaquettes ϕ_P

- $\#\ links > \#\ plaquettes$
- **One constraint between plaquette angles (PBC)**

$$\sum_P \phi_P = 0$$

$$Z = \int_0^{2\pi} d\phi_1 d\phi_2 d\phi_3 B(\phi_1) B(\phi_2) B(\phi_3) B(\phi_1 + \phi_2 + \phi_3).$$

A character expansion (Fourier analysis on a group)

$$B(\phi) = \Sigma_{n=-\infty}^{\infty} I_n(\beta) \exp{(in\phi)},$$

The partition function "almost" factorizes

$$Z = \Sigma_n I_n(\beta)^4$$

For N_xxN_t lattice

$$Z \; = \; \int d^{N_V-1}\phi_P \left(\Pi_P^{N_V-1} B(\phi_P)\right) B\left(\Sigma_P^{N_V-1}\phi_P\right) = \Sigma_n I_n(\beta)^{N_V}, \quad N_V = N_t * N_x. \quad (3)$$

•• **free boundary conditions (boundary links belong to one plaquette only)**

$$Z \; = \; I_0(\beta)^{N_V}$$

A transfer matrix

$2x4$ **lattice**

• temporal gauge: set all, but one, time like links in each column to 1 .

$\Longrightarrow$ 10 angles: two - ϑ_1 and ϑ_2 - on the last vertical links (on the top), and eight horizontal ones, $(\alpha_i, \beta_i), i = 1, 2, 3, 4$

• partition function

$$Z = \int d4\ d3\ d2\ d1\ \ < 4|T|3 >< 3|T|2 >< 2|T|1 >< 1|\Pi|4 >= Tr\left(T^3\Pi\right), \tag{4}$$

where $di = d\alpha_i d\beta_i$ and the states $|i >= |\alpha_i, \beta_i >$.

• elements of transfer matrix are

$$< \alpha', \beta'|T|\alpha, \beta >= B(\alpha' - \alpha)B(\beta' - \beta), \tag{5}$$

while the transition between the last and the first row is described by Π.

$$< \alpha', \beta'|\Pi|\alpha, \beta >= \int d\vartheta_1 d\vartheta_2 B(\alpha + \vartheta_2 - \alpha' - \vartheta_1)B(\beta + \vartheta_1 - \beta' - \vartheta_2) \tag{6}$$

Diagonalizing the transfer matrix

(5) is simply diagonalized by Fourier components

$$\int_0^{2\pi} d\theta' e^{\beta\cos(\theta-\theta')} e^{in\theta'} = I_n(\beta) e^{in\theta}$$

or

$$T|n>= I_n(\beta)|n>, \quad <\theta|n>= e^{in\theta},$$

• This is the basis of electric fluxes, or E_x, in Manton's language.

Two column system, the eigenstates $|m,n>=|m>|n>$ – tensor products for each x-position .

(5) $\Longrightarrow$ T in the fluxes representation

$$<m,n|T|m',n'>= \delta_{mm'}\delta_{nn'}I_n(\beta)I_m(\beta),$$

- The transfer matrix is diagonal in fluxes representation, and moreover
- T factorizes between individual states (x positions).

Diagonalizing the Π matrix

Three columns lattice

$$< \theta'_1, \theta'_2, \theta'_3|\Pi|\theta_1, \theta_2, \theta_3 >= \int d\vartheta_1 d\vartheta_2 \, d\vartheta_3 B(\phi_1)B(\phi_2)B(\phi_3)$$

$$= \Sigma_{m,n,r} I_m I_n I_r \int_{\vartheta' s} e^{im(\theta_1+\vartheta_2-\theta'_1-\vartheta_1)} e^{in(\theta_2+\vartheta_3-\theta'_2-\vartheta_2)} e^{ir(\theta_3+\vartheta_1-\theta'_3-\vartheta_3)}$$

$$= \Sigma_n I_n^3 e^{in(\theta_1+\theta_2+\theta_3-\theta'_1-\theta'_2-\theta'_3)}$$

In the flux basis

$$< m_1, m_2, m_3|\Pi|n_1, n_2, n_3 >= \delta_{M,N} \Sigma_n \delta_{n,n_1} \delta_{n,n_2} \delta_{n,n_3} I_n^3$$

- Π is diagonal as well and, in addition, it requires all fluxes along a row to be equal.
- It enforces Gauss law along a row.
- Upon taking a trace of $T^3\Pi$ reproduces (3)

The continuum limit

$$Z \;=\; \#\, \Sigma_n \left(\frac{I_n(\beta)}{I_0(\beta)}\right)^{N_x * N_t},$$

$$aN_t = T, \quad aN_x = L \quad \beta = \frac{1}{e^2 a^2}, \quad a \to 0.$$

Asymptotic expansion of modified Bessel function

$$I_n(\beta) \to \frac{e^\beta}{\sqrt{2\pi\beta}}\left(1 - \frac{4n^2-1}{8\beta} + ...\right)$$

gives

$$Z_{LQMD_2} \;\to\; \#\; \Sigma_n \left(1 - \frac{e^2}{2} n^2 a^2\right)^{N_x N_t} = \Sigma_n e^{-E_n T}, \quad E_n = \frac{1}{2} e^2 n^2 L,$$

$\longrightarrow$ **Manton fluxes result in the continuum limit of lattice QMD_2**

Continuum limit of the transfer matrix

Transfer matrix evolves states in time.

Matrix element of $\Pi \equiv$ kernel (propagator) of this evolution

$$< \theta_1', \theta_2', \theta_3' | \Pi | \theta_1, \theta_2, \theta_3 >= \Sigma_n I_n(\beta)^3 e^{in(\theta_1'+\theta_2'+\theta_3'-\theta_1-\theta_2-\theta_3)}$$

Gauss's low at each vertex singles out only the *sum* of all angles ($\theta_S = \theta_1 + \theta_2 + \theta_3$) as a relevant variable.

In the large β limit

$$\exp\left(-\frac{\beta}{2*3}(\theta_S' - \theta_S)^2\right) \tag{7}$$

For N_x rows this becomes

$$\exp\left(-\frac{\beta}{2*N_x}(\theta_S' - \theta_S)^2\right) = \exp\left(-\frac{1}{2}\frac{L}{e^2}\frac{(A'-A)^2}{\epsilon}\right) = K(A', A, \epsilon) \tag{8}$$

where we have identified:

$$L = N_x * a, \theta_S = L * A, \theta_i = aA, a = \epsilon.$$

This is nothing but the heat kernel for propagation of a free particle with mass $m = L/e^2$ by a time ϵ. Its Hamiltonian reads

$$H = -\frac{1}{2}\frac{e^2}{L}\frac{d^2}{dA^2}$$

which is Manton's Hamiltonian.

Three comments:

• It is important to realize that although the sum of all θ's can vary over the interval

$$0 < \theta_S = L * A < N_x 2\pi,$$

the relevant interval is $(0, 2\pi)$ only, since the lattice kernel, as well as the eigenfunctions $e^{in\theta_S}$, are periodic, in LA, over $(0, 2\pi)$.

Therefore our free particle indeed lives on a circle $(0, 2\pi/L)$.

•This emergence of a compact interval is essentially different from what happens in the continuum limit of standard (x dependent) theory/fields.

There a local potential associated with each link

$$0 < A_i < 2\pi/a \quad \longrightarrow \quad 0 < A(x) < \infty,$$

while here the global variable A remains still bounded even in the continuum limit.

• (7,8) contains only the contributions from the first winding sector. Complete result is given by the Jacobi theta function:

$$K(A', A, \epsilon) = \theta_3 \left(\frac{L}{2}(A' - A), e^{\frac{e^2 L}{2}\epsilon} \right)$$

• Volume reduction

Another derivation - Coulomb gauge on a lattice

A single row of $N_x = 3$ horizontal links $\theta_1, \theta_2, \theta_3$

A local gauge transformation specified by $\alpha_1, \alpha_2, \alpha_3$

$$\begin{aligned}\theta_1 &\to {}^g\theta_1 = \theta_1 + \alpha_1 - \alpha_2 \\ \theta_2 &\to {}^g\theta_2 = \theta_2 + \alpha_2 - \alpha_3 \\ \theta_3 &\to {}^g\theta_3 = \theta_3 + \alpha_3 - \alpha_1\end{aligned}$$

or

$${}^g\theta_i = \theta_i + \beta_i, \quad \Sigma_{i=1}^3 \beta_i = 0$$

If we choose

$$\begin{aligned}\beta_1 &= \frac{1}{3}(\theta_1 + \theta_2 + \theta_3) - \theta_1 \\ \beta_2 &= \frac{1}{3}(\theta_1 + \theta_2 + \theta_3) - \theta_2 \\ \beta_3 &= \frac{1}{3}(\theta_1 + \theta_2 + \theta_3) - \theta_3\end{aligned}$$

then all new link angles are equal

$$^g\theta_1 =^g \theta_2 =^g \theta_3 = \frac{1}{3}(\theta_1 + \theta_2 + \theta_3) \equiv \theta_{row}.$$

$\Longrightarrow$**Only one degree of freedom remains**

Now the transfer matrix reads

$$< \theta|\Pi|\theta' >= \Sigma_n I_n(\beta)^{N_x} e^{inN_x(\theta-\theta')}$$

Continuum limit $N_x\theta \rightarrow LA$

$$\beta = \frac{1}{e^2a^2}, \quad aN_x = L, \quad \theta = aA$$

repeating earlier steps gives

$$< \theta|\Pi|\theta' >\longrightarrow \Sigma_n e^{-E_n a} e^{inL(A-A')} = K(A, A', \epsilon = a)$$

which is nothing but a spectral representation of the Feynman kernel propagating the system (1-2) through a time lapse $\epsilon = a$.

III. Adding external charges

Wilson loops - a tailing trick

$$W[\Gamma] = \Pi_{l\in\Gamma} e^{i\theta_l} = \Pi_{p\in in(\Gamma)} e^{i\phi_p}$$

$$\begin{aligned} Z\langle W\rangle &= \int d^{N_V-1}\phi_p \left(\Pi_{p\in in(\Gamma)} e^{i\phi_p} B(\phi_p)\right)\left(\Pi_{p\in out(\Gamma)} B(\phi_p)\right) B\left(\Sigma_p^{N_V-1}\phi_p\right) \\ &= \Sigma_n I_n(\beta)\left(\Pi_{in(\Gamma)}\int_{\phi_p} e^{i(n+1)\phi_p} B(\phi_p)\right)\left(\Pi_{out(\Gamma)}\int_{\phi_{p'}} e^{in\phi_{p'}} B(\phi_{p'})\right) \\ &= \Sigma_n I_n(\beta)^{N_x * N_t - n_x * n_t} I_{n+1}(\beta)^{n_x * n_t}. \end{aligned} \quad (9)$$

•• show that tailing outside of the loop gives *the same* result.

Wilson loops - directly from link variables

$$
\begin{aligned}
Z\langle W\rangle &= \int_0^{2\pi} d(links)\left(\Pi_{l\in\Gamma}e^{i\theta_l}\right)\left(\Pi_p^{N_V}B(\phi_p)\right) \\
&= \Sigma_{m_1,m_2,\ldots,m_{N_V}} I_{m_1}\ldots I_{m_{N_V}}\int_{links}\left(\Pi_{l\in\Gamma}e^{i\theta_l}\right)\left(\Pi_p^{N_V}e^{im_p\phi_p}\right) \\
&= \Sigma_{m_1,m_2,\ldots,m_{N_V}} I_{m_1}\ldots I_{m_{N_V}}\Pi_l^{2N_V}\Delta(m_{P_L(l)},m_{P_R(l)}), \\
\Delta(m_{P_L(l)},m_{P_R(l)}) &= \begin{cases}\delta_{m_L(l),m_R(l)} & l\notin\Gamma \\ \delta_{m_L(l),m_R(l)+1} & l\in\Gamma\end{cases}
\end{aligned}
$$

$$\Longrightarrow (9)\ .$$

Space like Polyakov loops

$$Z < P(n_t)^\dagger P(0) >= \Sigma_n \left(I_n^{N_x}\right)^{N_t-n_t} \left(I_{n+1}^{N_x}\right)^{n_t}, \qquad (10)$$

From the transfer matrix ($2x4$ lattice again)

$$Z < P^\dagger(3)P(2) >= Tr\left(\Pi P^\dagger \Pi P \Pi^2\right). \qquad (11)$$

Polyakov loop operator is diagonal in the angular basis,

$$< \theta_1, \theta_2|P|\theta_1', \theta_2' >= \delta_{\theta_1,\theta_1'}\delta_{\theta_2,\theta_2'}e^{i\theta_1}e^{i\theta_2}$$

hence it just a creates a unit of flux at each link.

$$< n, m|P|n', m' >= \delta_{n,n'+1}\delta_{m,m'+1}$$

with a unit overlap.
Calculating the trace (11) in the flux basis one obtains

$$Z < P(3)P(2) >= \Sigma_n I_n^6 I_{n+1}^2,$$

which goes into (10) for general sizes.
●● Notice that (10) is symmetric with respect to the time reflection $n_t \to N_t - n_t$. Why ?

Time like Polyakov loops

As before

$$Z < P^{\dagger}(1)P(n_x+1) >= \Sigma_n I_n(\beta)^{N_t*(N_x-n_x)} I_{n+1}(\beta)^{N_t*n_x}, \tag{12}$$

Transfer matrix approach

Polyakov lines are just additional projection operators.

The numerator of the $< PP >$ as in (4) (still for (3x4) lattice)

$$Z < P(1)^{\dagger} P(3) >= Tr\left((\Pi^{PP})^4\right) \tag{13}$$

where Π^{PP} is the projection operator similar to (6)

$$< \alpha, \beta, \gamma | \Pi^{PP} | \alpha', \beta', \gamma' >= \int d\vartheta_1 d\vartheta_2 d\vartheta_3$$
$$e^{-i\vartheta_1} B(\alpha' + \vartheta_2 - \alpha - \vartheta_1) B(\beta' + \vartheta_3 - \beta - \vartheta_2) e^{i\vartheta_3} B(\gamma' + \vartheta_1 - \gamma - \vartheta_3) \tag{14}$$

but with additional $U(1)$ elements from Polyakov lines at $i_x = 1$ and $i_x = 3$.

In the flux basis this transition operator reads $M = (m_1, m_2, m_3)$, $N = (n_1, n_2, n_3)$.

$$< m_1, m_2, m_3 | \Pi^{PP} | n_1, n_2, n_3 >= \delta_{MN} \Sigma_n \delta_{n_1,n+1} \delta_{n_2 n+1} \delta_{n_3,n} I_{n_1}(\beta) I_{n_2}(\beta) I_{n_3}(\beta), \tag{15}$$

so the fluxes between two Polyakov lines are the same, likewise fluxes outside, however the common two values differ by one unit.

The general case of N_x sites and loops separated by n_x units.

$$< M|\Pi^{PP}|N >= \delta_{MN}\Sigma_n I_n(\beta)^{N_x-n_x} I_{n+1}(\beta)^{n_x}\Pi^{n_x}_{interior}\delta_{m_i,n+1}\Pi^{N_x-n_x}_{exterior}\delta_{m_j,n} \tag{16}$$

Now taking the trace of $N_t - th$ power reproduces readily (12).

Continuum limit

As earlier, introduce the dimensionful lattice constant, use the asymptotic form of Bessel functions and express (12) in terms of physical distances (in particular the distance between sources, $an_x = R$) to obtain

$$Z < P(0)^\dagger P(R) >= \Sigma_n e^{-E_n^{PP}T}, \tag{17}$$

with

$$E_n^{PP} = \frac{e^2}{2}\left(n^2(L-R) + (n+1)^2 R\right), \quad n = 0, \pm1, \pm2, ... \tag{18}$$

An exercise

- space Polyalov lines

$$Z < P_s(0)^\dagger P_s(t) >= \Sigma_n e^{-\frac{e^2}{2} n^2 L(T-t)} e^{-\frac{e^2}{2}(n+1)^2 Lt}, \tag{19}$$

- time Polyalov lines

$$Z < P_t(0)^\dagger P_t(R) >= \Sigma_n e^{-\frac{e^2}{2}\left(n^2(L-R)+(n+1)^2 R\right)T}, \tag{20}$$

- Wilson loops

$$Z < W_{R,t} >= \Sigma_n e^{-\frac{e^2}{2}\left(n^2(LT-Rt)+(n+1)^2 Rt\right)}, \tag{21}$$

show that (21) admits the interpretation, in terms of time evolution, analogous to (19) and (20)

A straightforward interpretation:

$$E_n^{PP} = \frac{e^2}{2}\left(n^2(L-R) + (n+1)^2 R\right), \quad n = 0, \pm 1, \pm 2, ... \tag{22}$$

- Time like Polyakov lines modify Gauss's low at spatial points 0 and R - they introduce external unit charges at these positions.
- Such charges cause additional unit of flux extending over distance R.
- Hence the two contributions to the eigenenergies: an "old" flux over the distance $L-R$ and the new one, bigger by one unit (fluxes are additive !) , over R.
- Interesting special cases:

  → at large T comes the lowest, $n = 0$ and $n = -1$, states dominate. Then we just have standard (unit flux) strings of length R and L-R ,

  → $R = 0$ – old topological flux with charge n.

  → $R = L$ – when external charges meet at the "end point" of a circle, they annihilate ($e^+\delta_P(0) + e^-\delta_P(L) = 0$)and leave behind a topological string with length L and charge bigger by one unit.
- Varying R interpolates between integer valued topological fluxes.

Equivalent form

$$E_n^{PP} = \frac{e^2}{2} L(n+\rho)^2 + const.(L,R), \quad \rho = \frac{R}{L}, \quad const. = \frac{e^2}{2} L\rho(1-\rho) \tag{23}$$

- Indeed $e\frac{R}{L}$ is the electric field, generated by two sources, *averaged* over the whole volume.
- The system does not see any distances, $A_x(x) = const.$, hence averaging over the volume.
- Changing R allows to mimic arbitrary real charge $q = e(n+\rho)$.
- Only $[\rho]$ is relevant.

Hamiltonian and wave functions

Transfer matrix: transform (16) to the angular representation, in Coulomb gauge

$$< \theta|\Pi^{PP}|\theta' > \; = \; \Sigma_n I_n(\beta)^{N_x-n_x} I_{n+1}(\beta)^{n_x} e^{in(N_x-n_x)(\theta-\theta')} e^{i(n+1)n_x(\theta-\theta')} \tag{24}$$

$$\equiv K_L^{PP}(\theta,\theta') \; = \; \Sigma_n I_n(\beta)^{N_x-n_x} I_{n+1}(\beta)^{n_x} e^{inN_x(\theta-\theta')} e^{in_x(\theta-\theta')} \tag{25}$$

In the continuum limit , $N_x\theta = LA, n_x\theta = RA$**, we get**

$$K_L^{PP}(\theta,\theta') \longrightarrow K^{PP}(A,A',\epsilon) = \Sigma_n e^{-\frac{e^2L}{2}\left((n+\rho)^2+\rho(1-\rho)\right)\epsilon} e^{i(n+\rho)L(A-A')}. \tag{26}$$

which is the momentum expansion of the Feynman kernel describing 1DOF QM with above spectrum. Now we can identify eigenfunctions and the hamiltonian

$$H = -\frac{e^2L}{2}\frac{d^2}{d\chi^2} + \frac{e^2L}{2}\rho(1-\rho), \quad \psi_n(\chi) = e^{i(n+\rho)\chi}. \tag{27}$$

Or, in another basis

$$\bar{K}^{PP}(A, A', \epsilon) \equiv e^{-i\rho(A-A')L} K^{PP}(A, A', \rho).$$

$$\bar{H} = -\frac{e^2L}{2}\left(\frac{d}{d\chi} + i\rho\right)^2 + \frac{e^2L}{2}\rho(1-\rho), \quad \chi = LA, \quad \bar{\psi}_n(\chi) = e^{in\chi},$$

with the spectrum (23) and corresponding, *periodic* eigenfunctions.

- **Θ parameter acquires now a straightforward interpretation**

$$\Theta_{Manton} = \rho = \frac{R}{L},$$

- **A new constant term.**

Θ-vacua

- The transformation $A \longrightarrow A + \frac{2\pi}{L}$ is a large gauge transformation, $\Lambda(x) = \frac{2\pi x}{L}$, $\Lambda(x + L) = \Lambda(x) + 2\pi$
- Full analogy 4D YM and/or the crystal : many classical configurations around which we can quantize
- Θ vacua: $|\Theta\rangle = \Sigma_n e^{i\Theta n}|n\rangle$
- The wave function of a Θ-state $\psi_\Theta(x) = \langle x|\Theta\rangle$ satisfies $\psi_\Theta(x - d) = e^{i\Theta}\psi_\Theta(x)$
- The solution (Bloch theorem) : $\psi_\Theta(x) = e^{i\Theta x/d}u_\Theta(x)$, with periodic $u_\Theta(x)$
- Our case: $\psi_n(A) = e^{i(n+\rho)AL} = e^{i\rho AL}e^{inAL}$ is exactly of Bloch type upon identification $x \to A$, $d \to 2\pi/L$, $\Theta \to 2\pi\rho$

- Introducing external charges fixes the Θ-vacuum in QMD_2.

- D=4 : in a Θ-vacuum some field configurations acquire electric charge [Witten '76].

More, different charges

R_2 - distance between doubly charged sources
R_1 - distance between singly charged ones

$$Z < P(i)^\dagger P(j)^{2\dagger} P^2(j+n_2) P(i+n_1) >=$$

$$\Sigma_n I_n(\beta)^{N_t(N_x-n_1)} I_{n+1}(\beta)^{N_t(n_1-n_2)} I_{n+3}(\beta)^{N_t n_2},$$

- eigenenergies in the continuum limit

$$\begin{aligned} E_n^{PPPP} &= \frac{e^2}{2}\left(n^2(L-R_1)+(n+1)^2(R_1-R_2)+(n+3)^2R_2\right) \\ &= \frac{e^2}{2}L\left((n+\rho_1+2\rho_2)^2+\rho_1(1-\rho_1)+4\rho_2(2-\rho_1-\rho_2)\right) \end{aligned}$$

etc. 1 DOF quantum mechanical systems can be also readily constructed.

- This time $\Theta = (R_1+2R_2)/L$, i.e. it is again equal to the external field averaged over the whole volume.

IV. Arbitrary charges on a lattice

Why? To learn about screening

Massive Schwinger model

$$\sigma_q = m\, e \left(1 - \cos\left(2\pi \frac{q}{e}\right)\right) \qquad m/e << 1, \qquad [Coleman\ et\ al.,\ '75]$$

$\Rightarrow$ generalizations for large N QCD_2.

$\Rightarrow$ How to put arbitrary (noncongruent with e) charges on a lattice?

- One way: as above $q = e(n + R/L)$
- Another way: new observables

Wilson loops with arbitrary charge

$$Z\langle W_Q\rangle = \int (W[\Gamma])^Q \, e^{-S}, \qquad Q = q/e$$

Contras:

gauge invariance – not if you carefully/consistently deal with multivaluedness

dependence on the boundaries in angular variables – not if you do loops

Pros:

Results are consistent $(MC \leftrightarrow TH)$

New structure appears QMD_2

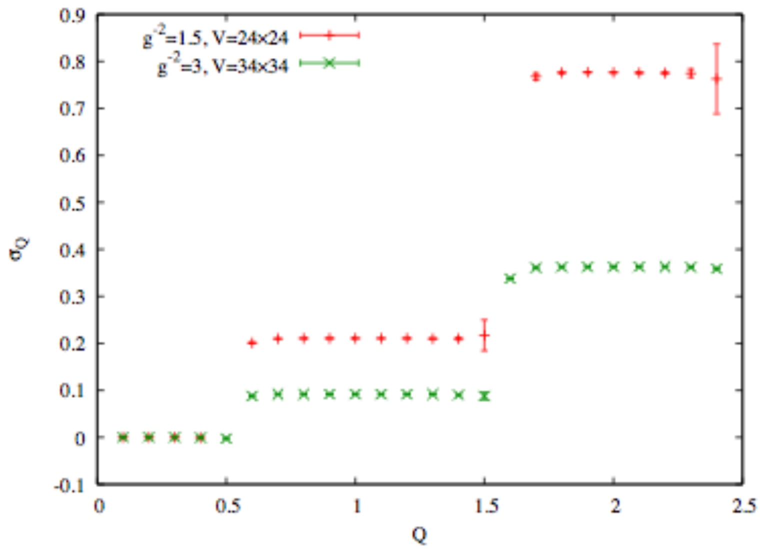
Why not !

Q-loops theoretically

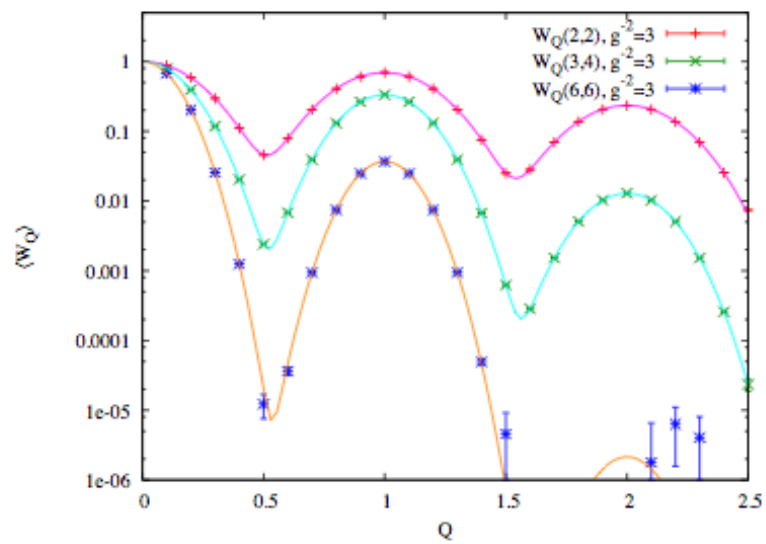
$$
\begin{aligned}
Z\langle W_Q\rangle &= \int_0^{2\pi} d(links) \left(\Pi_{l\in\Gamma} e^{iQ\theta_l}\right) \left(\Pi_p^{N_V} B(\phi_p)\right) \\
&= \Sigma_{m_1,m_2,\ldots,m_{N_V}} I_{m_1}\ldots I_{m_{N_V}} \int_{links} \left(\Pi_{l\in\Gamma} e^{iQ\theta_l}\right) \left(\Pi_p^{N_V} e^{im_p\phi_p}\right) \\
&= \Sigma_{m_1,m_2,\ldots,m_{N_V}} I_{m_1}\ldots I_{m_{N_V}} \left(\Pi_{l\notin\Gamma} \delta_{m_L(l),m_R(l)}\right) \left(\Pi_{l\in\Gamma} \bar{S}(Q - m_L(l) + m_R(l))\right) \\
&= \Sigma_{m,n} I_n^{N_x N_t - n_x n_t} I_m^{n_x n_t} S(Q - m + n)^{n_x + n_t},
\end{aligned}
$$

$$
\bar{S}(x) = \frac{\sin \pi x}{\pi x}, \qquad S(x) = \left(\frac{\sin \pi x}{\pi x}\right)^2
$$

and "experimentally" [P. Korcyl, M. Koren]



(a) String tension as a function of Q



(b) Ansatz vs. data (not a fit)

Figure 1:

- Q-loops can be defined on a lattice - MC agrees with TH
- They do not create states with arbitrary charge
  – they excite the only existing quantum states with integer charges

Continuum limit

$$Z\langle W_Q\rangle = \Sigma_{m,n} I_n^{N_x N_t - n_x n_t} I_m^{n_x n_t} S(Q-(n-m))^{n_t+n_x} =$$

$$\Sigma_{m,n} \exp\left(-\frac{e^2}{2} n^2 L(T-t)\right) \exp\left(-\frac{e^2}{2}\left(n^2(L-R)+m^2 R\right) t\right)$$
$$S(Q-(n-m))^{(t+R)/a}$$

does not exist at fixed, not integer Q.

$\Longrightarrow$ However the *classical* limit:

$Q \to \infty$, with $q = Qe - fixed$, on a fixed lattice $(a, N's, const.)$

does exist!

Then $\beta \equiv b^2 = 1/e^2a^2 \to \infty$, but not because $a \to \infty$,
but because $e \to 0$.
The spectrum of fluxes becomes continuous: $n \to u = n/b, m \to v = n/b$

Therefore $(Q = q/e = \sqrt{\beta/\kappa} = b/g, g = 1/qa)$

$$ZK_{\Pi QQ} = \beta \int dudv \exp\left(-\frac{1}{2}(u^2(N_x - n_x) + v^2 n_x)\right)$$
$$S\left(b(g^{-1} - (u-v))\right)^2 e^{ibu(\Theta_{L-R} - \Theta'_{L-R})} e^{ibv(\Theta_R - \Theta'_R)}$$

using

$$S(b\Delta) \xrightarrow{b\to\infty} \frac{1}{b}\delta(\Delta)$$

gives

$$ZK_{\Pi QQ} = \sqrt{\beta} \int du \exp\left(-\frac{1}{2}(u^2(N_x - n_x) + (u - g^{-1})^2 n_x)\right)$$
$$e^{ibu(\Theta_{L-R} - \Theta'_{L-R})} e^{ib(u-g^{-1})(\Theta_R - \Theta'_R)}$$

Now, do the gaussian integral, take the continuum limit to obtain

$$ZK_{\Pi QQ} = \sqrt{\beta}\sqrt{\frac{2\pi a}{L}} \exp\left(-\frac{L}{2}\frac{(A-A')^2}{a}\right) \exp\left(-\frac{q^2}{2}\rho(1-\rho)La\right)$$

$\Longrightarrow$ a free particle propagating over a time a, but in a constant background potential

$$V = \frac{q^2}{2}\rho(1-\rho)L$$

with arbitrary, real value of a classical charge q.

- The classical energy with a continuous charge q results from the contribution of many microscopic states with discrete charges.
- the structure (zeroes of the string tension)

V. Nonabelian case: YM_2 on a circle

• Continuum: problem reduces to N constant in space, but constrained, angles θ_i, $\Sigma_i \theta_i = 0$.

Hamiltnian is again quadratic and the spectrum is known explicitly [Hetrick and Hosotani '89]

$$E_{\{n\}} = \frac{g^2 L}{4}\left(\Sigma_i n_i^2 - \frac{1}{N}\left(\Sigma_i n_i\right)^2\right), \quad i = 1, ..., N-1$$

• Continuum: different spectrum was obtained by Rajeev: $E_R = \frac{g^2 L}{2} C_2(R)$
• Discrepancy comes from the Casimir energy due to the curvature of the group manifold [Hetrick '93, Witten '91,'92]
• Lattice: continuum spectrum $\Longleftarrow$ the large β behaviour of the character expansion of Boltzman factor.

It is given by the Casimir plus, the N dependent, constant curvature correction/Casimir energy, and agrees with Hetrick and Hosotani .
• External charges in YM_2 – studied by many [Semenoff et al. '97] but above connection with Θ-vacuum not.

INNOVATIVE ECONOMY
NATIONAL COHESION STRATEGY
FNP Foundation for Polish Science
EUROPEAN REGIONAL DEVELOPMENT FUND

EU grant (via Foundation for Polish Science)

Jagellonian University International PhD Studies on Physics of Complex Systems

- 1 M Euro
- 4 years
- 14 PhD students (1/2 - 2 years abroad)
- 9 Local Supervisors
- 17 Foreign Partners: J. Ambjorn, J.P. Blaizot, H. Nicolai, S. Sharpe, ...