Flavour Physics – Lecture 3

Chris Sachrajda

School of Physics and Astronomy
University of Southampton
Southampton SO17 1BJ
UK

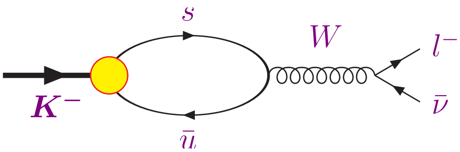
New Horizons in Lattice Field Theory,
Natal, Rio Grande do Norte, Brazil
March 13th -27th 2013

UNIVERSITY OF Southampton
School of Physics and Astronomy

1. Lecture 1: Introduction to Flavour Physics
2. Lecture 2: Lattice Computations in Flavour Physics
3. Lecture 3: Light-quark physics
   1. Determination of V_{us}
   2. Determination of B_K
   3. $K \to \pi\pi$ Decays
4. Lecture 4: Heavy-quark physics

UNIVERSITY OF Southampton
School of Physics and Astronomy

1. Lecture 1: Introduction to Flavour Physics
2. Lecture 2: Lattice Computations in Flavour Physics
3. Lecture 3: Light-quark physics
   1. Determination of V_{us}
   2. Determination of B_K
   3. $K \to \pi\pi$ Decays
4. Lecture 4: Heavy-quark physics



- All QCD effects are contained in a single constant, f_K, the kaon's *(leptonic) decay constant.*

$$\langle 0|\bar{s}\gamma^\mu\gamma^5 u|K(p)\rangle = if_K p^\mu \,. \qquad (f_\pi \simeq 132\,\mathrm{MeV})$$

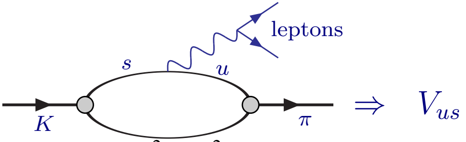
$$\frac{\Gamma(K\to\mu\bar{\nu}(\gamma))}{\Gamma(\pi\to\mu\bar{\nu}(\gamma))} = \frac{|V_{us}|^2}{|V_{ud}|^2}\,\frac{f_K^2}{f_\pi^2}\,\frac{m_K\left(1-\frac{m_\mu^2}{m_K^2}\right)}{m_\pi\left(1-\frac{m_\mu^2}{m_\pi^2}\right)}\times 0.9930(35)$$

- From the experimental ratio of the widths we get:

$$\frac{|V_{us}|^2}{|V_{ud}|^2}\,\frac{f_K^2}{f_\pi^2} = 0.07602(23)_{\mathrm{exp}}(27)_{\mathrm{RC}}\,, \qquad \text{PDG2006}$$

so that a precise determination of f_K/f_π will yield V_{us}/V_{ud} .

- Every collaboration calculates f_K and f_π.



$$\langle\,\pi(p_\pi)\,|\bar{s}\gamma_\mu u\,|\,K(p_K)\,\rangle = f_0(q^2)\,\frac{M_K^2-M_\pi^2}{q^2}q_\mu + f_+(q^2)\,\left[(p_\pi+p_K)_\mu - \frac{M_K^2-M_\pi^2}{q^2}q_\mu\right]$$

where $q \equiv p_K - p_\pi$.

$$\Gamma_{K\to\pi\ell\nu} = C_K^2\,\frac{G_F^2 m_K^5}{192\pi^3}\,I\,S_{\rm EW}[1+2\Delta_{\rm SU(2)}+\Delta_{EM}]\,|V_{us}|^2\,|f_+(0)|^2$$

From the experimental measurement of the width we get:

$$|V_{us}|f_+(0) = 0.2169(9)\,, \qquad \text{PDG2006}$$

so that a precise determination of $f_+(0)$ will yield V_{us}.

UNIVERSITY OF Southampton
School of Physics and Astronomy

FLAG

- We have the two precise results:

$$\left|\frac{V_{us} f_K}{V_{ud} f_\pi}\right| = 0.27599(59) \quad \text{and} \quad |V_{us} f_+(0)| = 0.21661(47)$$

Flavianet – arXiv:0801.1817

- We can view these as two equation for the four unknowns f_K/f_π, $f_+(0)$, V_{us} and V_{ud} .
- Within the Standard Model we also have the unitarity constraint:

$$|V_{ud}|^2 + |V_{us}|^2 \cancel{+ |V_{ub}|^2} = 1$$

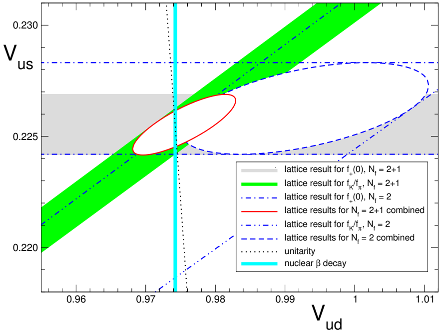
- Thus we now have 3 equations for four unknowns.
- There has been considerable work recently in updating the determination of V_{ud} based on 20 different superallowed transitions. Hardy and Towner, arXiV:0812.1202

$$|V_{ud}| = 0.97425(22)\,.$$

- If we accept this value then we are able to determine the remaining 3 unknowns:

$$|V_{us}| = 0.22544(95), \quad f_+(0) = 0.9608(46), \quad \frac{f_K}{f_\pi} = 1.1927(59)\,.$$

V_{us} from Lattice Simulations



Flavianet Lattice Averaging Group

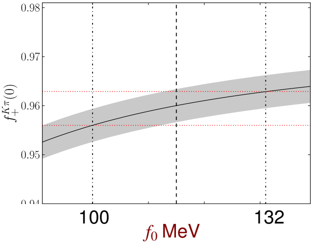
- Lattice calculations of f_K/f_π combined with the experimental widths $\Rightarrow V_{us}/V_{ud}$.
- Following the suggestion of Becirevic et al., precise lattice calculations of the $K_{\ell 3}$ form factor $f_+(0)$ are possible $\Rightarrow V_{us}$.

hep-ph/0403217

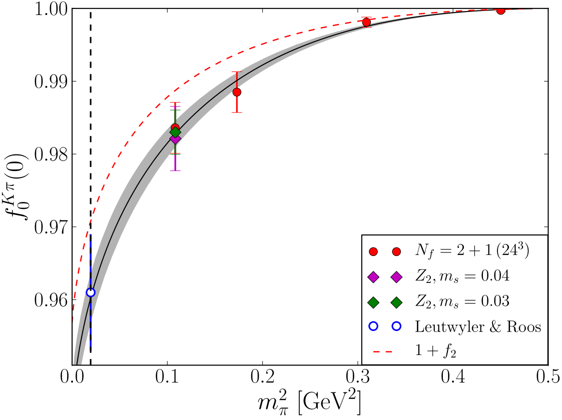
- Results are in remarkable agreement with SM.

- Currently the main uncertainty on $f_+(0)$ is due to the chiral extrapolation.

RBC-UKQCD, arXiv:1004:0886



Chiral Behaviour of $f_0(0)$



P.Boyle et al. arXiv:1004:0886

FLAG

Lattice results are remarkably consistent with the unitarity of the CKM Matrix

- For $N_f = 2+1$ simulations FLAG quotes the following current values:

$$\frac{f_K}{f_\pi} = 1.193(6) \quad \text{and} \quad f_+(0) = 0.9599(34)\left(\begin{smallmatrix}+31\\-47\end{smallmatrix}\right)(14)\,.$$

- Taking the experimental results for $K_{\ell 2}$ and $K_{\ell 3}$ decays and dividing by the $N_f = 2+1$ lattice values of f_K/f_π and $f_+(0)$ gives:

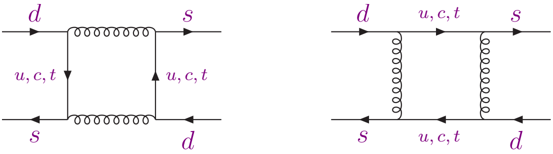
$$V_{ud}^2 + V_{us}^2 = 1.002(16)\,.$$

- If we combine the experimental results with the value of V_{ud} and the lattice values of $f_+(0)$ or f_K/f_π we find:

$$V_{ud}^2 + V_{us}^2 = 1.0000(7) \quad \text{or} \quad V_{ud}^2 + V_{us}^2 = 0.9999(7)\,.$$

- Very significant test of universality of coupling of "W"-like bosons to quarks and leptons.

1. Lecture 1: Introduction to Flavour Physics
2. Lecture 2: Lattice Computations in Flavour Physics
3. Lecture 3: Light-quark physics
    1. Determination of V_{us}
    2. Determination of B_K
    3. $K \to \pi\pi$ Decays
4. Lecture 4: Heavy-quark physics



- The CP-eigenstates (K_1 and K_2) are linear combinations of the two strong-interaction eigenstates:

$$|K_1\rangle = \frac{1}{\sqrt{2}}\left(|K^0\rangle + |\bar{K}^0\rangle\right) \qquad CP|K_1\rangle = |K_1\rangle \quad \text{and}$$
$$|K_2\rangle = \frac{1}{\sqrt{2}}\left(|K^0\rangle - |\bar{K}^0\rangle\right) \qquad CP|K_2\rangle = -|K_2\rangle\,.$$

  - I use the phase convention so that $CP|K^0\rangle = |\bar{K}^0\rangle$.

- Because of the complex phase in the CKM-matrix, the physical states (the mass eigenstates) differ from $|K_1\rangle$ and $|K_2\rangle$ by a small admixture of the other state:

$$|K_S\rangle = \frac{|K_1\rangle + \bar{\varepsilon}\,|K_2\rangle}{(1+|\bar{\varepsilon}|^2)^{\frac{1}{2}}} \quad \text{and} \quad |K_L\rangle = \frac{|K_2\rangle + \bar{\varepsilon}\,|K_1\rangle}{(1+|\bar{\varepsilon}|^2)^{\frac{1}{2}}}\,,$$

  - The parameter $\bar{\varepsilon}$ depends on the phase convention chosen for $|K^0\rangle$ and $|\bar{K}^0\rangle$.

- For $K \to \pi\pi$ and $K \to \pi\pi\pi$ decays, the two pion states are CP-even and the three-pion states are CP-odd $\Rightarrow$ the dominant decays are:

$$K_S \to \pi\pi \quad \text{and} \quad K_L \to 3\pi \; .$$

  - This is the reason why K_L is much longer lived than K_S.

- K_L and K_S are not CP-eigenstates, however $\Rightarrow$ $K_L \to 2\pi$ and $K_S \to 3\pi$ decays may occur.

- CP-violating decays which occur due to the fact that the mass eigenstates are not CP-eigenstates are called *indirect CP-violating decays*.

- A measure of the strength of indirect CP-violation is given by the physical parameter ε_K defined by the ratio:

$$\varepsilon_K \equiv \frac{A\,(K_L \to (\pi\pi)_{I=0})}{A\,(K_S \to (\pi\pi)_{I=0})} = \left(\frac{\mathrm{Im}\, M_{12}}{2\mathrm{Re}\, M_{12}} + \frac{\mathrm{Im}\, A_0}{\mathrm{Re}\, A_0} \right) e^{i\phi_\varepsilon} \sin\phi_\varepsilon \, ,$$

where $\phi_\varepsilon \simeq 43.51 \pm 0.05^\circ$ and $|\varepsilon_K| = (2.228 \pm 0.011) \times 10^{-3}$.

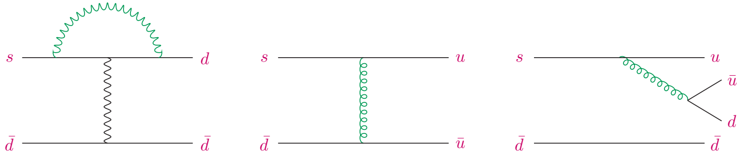
  - A_0 is the amplitude for $K \to (\pi\pi)_{I=0}$ decays.
  - M_{12} is the off diagonal term in the neutral kaon mass matrix.

- Directly CP-violating decays are those in which a CP-even (-odd) state decays into a CP-odd (-even) one: $K_L \;\propto\; K_2 \;+\; \bar{\epsilon} K_1$.

  $K_2 \rightarrow \pi\pi$: Direct (ϵ')

  $\bar{\epsilon} K_1 \rightarrow \pi\pi$: Indirect (ϵ_K)

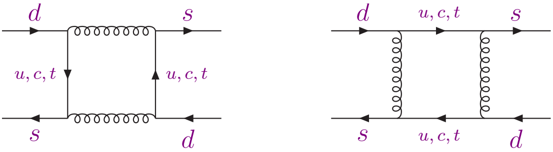
- Consider the following contributions to $K \to \pi\pi$ decays:



$I = 0$, Complex (a) $\quad$ $I = 0$, Real (b) $\quad$ $I = 0$ or 2, Real (c)

  Direct CP-violation in kaon decays manifests itself as a non-zero relative phase between the $I = 0$ and $I = 2$ amplitudes.

- We also have *strong phases*, δ_0 and δ_2 which are independent of the form of the weak Hamiltonian.



- The effective Hamiltonian for $\Delta S = 2$ processes is of the form:

$$\mathscr{H}_{\text{eff}}^{\Delta S=2} = \frac{G_F^2 M_W^2}{16\pi^2}\,\mathscr{F}^0\,Q^{\Delta S=2} \;+\; \text{h.c.}$$

- (In the Standard Model) $Q^{\Delta S=2}$ is the dimension 6 operator:

$$Q^{\Delta S=2} = [\bar{s}\gamma_\mu(1-\gamma_5)d]\,[\bar{s}\gamma^\mu(1-\gamma_5)d]$$

- Important practical simplifications are provided by the Fierz Identity for γ-matrices:

$$[\bar{q}_1\gamma_\mu(1-\gamma_5)q_2]\,[\bar{q}_3\gamma_\mu(1-\gamma_5)q_4] = [\bar{q}_1\gamma_\mu(1-\gamma_5)q_4]\,[\bar{q}_3\gamma_\mu(1-\gamma_5)q_2]$$

- The function $\mathscr{F}^0$ is given by the Inami-Lin functions:

$$\mathscr{F}^0 = \lambda_c^2 S_0(x_c) + \lambda_t^2 S_0(x_t) + 2\lambda_c\lambda_t S_0(x_c, x_t)\,, \quad \text{where}$$

$x_{c,t} = m_{c,t}^2/M_W^2$ and $\lambda_q = V_{qs}^* V_{qd}$. We have used $\lambda_u + \lambda_c + \lambda_t = 0$ and set $x_u = 0$.

- Including QCD effects:

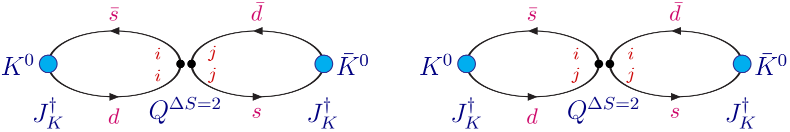
$$\langle \bar{K}^0 | \mathscr{H}_{\text{eff}}^{\Delta S=2} | K^0 \rangle = \frac{G_F^2 M_W^2}{16\pi^2} \left[\lambda_c^2 S_0(x_c)\eta_1 + \lambda_t^2 S_0(x_t)\eta_2 + 2\lambda_c\lambda_t S_0(x_c,x_t)\eta_3 \right]$$
$$\times \left(\frac{g(\mu)^2}{4\pi} \right)^{-\gamma_0/(2\beta_0)} \left\{ 1 + \frac{g(\mu)^2}{(4\pi)^2} \left[\frac{\beta_1\gamma_0 - \beta_0\gamma_1}{2\beta_0^2} \right] \right\} \langle \bar{K}^0 | Q^{\Delta S=2}(\mu) | K^0 \rangle + \text{h.c.} .$$

- In this equation $Q^{\Delta S=2}(\mu)$ is the renormalised operator in some scheme at the renormalization scale μ. This dependence on scheme and scale is cancelled (partially) by the explicit running on the second line above.
- η_i are calculated in perturbation theory.
- It is conventional to define the parameter B_K:

$$B_K(\mu) = \frac{\langle \bar{K}^0 \, | Q^{\Delta S=2}(\mu) | \, K^0 \rangle}{\frac{8}{3} f_K^2 m_K^2} .$$

  - $B_K(\mu)$ contains the non-perturbative effects in neutral kaon mixing and ε_K.
  - When presenting results for B_K, the scheme and μ must be specified.
  - Popular choices are $\overline{\text{MS}} - \text{NDR}$ and NLO renormalisation group independent ($\hat{B}_K$).

Lattice Evaluation of B_K

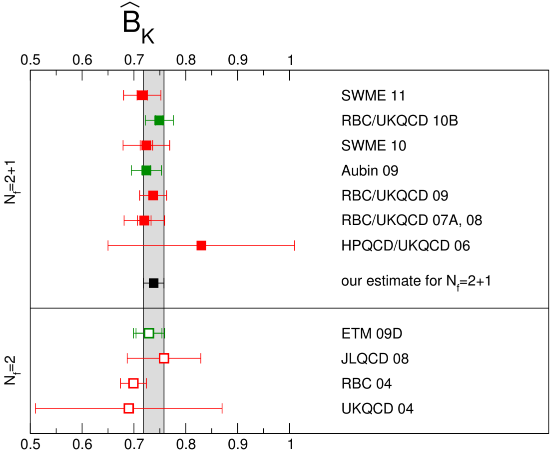


- Each of the two black dots represents one of the currents in $Q^{\Delta S}=2$. They are split for convenience.
- Recalling the Fierz Identity, we see that there are two contributions to the correlation function

$$\langle 0\,|\,J_K^\dagger(t_f)\,Q^{\Delta S=2}(t_Q)\,J_K^\dagger(t_f)\,|\,0\rangle\,,$$

  corresponding to the two diagrams above.
    - i,j are colour labels.
- It is natural to place the sources for the propagator at t_Q.
    - We try to pick sources to get as much *volume-averaging* as possible.
- In this way we get the matrix element $\langle\bar{K}^0\,|Q^{\Delta S=2}_{\rm Latt}(a)\,|\,K^0\rangle$.
- Finally we have to perform the normalization to obtain the matrix element in some continuum scheme or to obtain $\hat{B}_K$.

UNIVERSITY OF Southampton
School of Physics and Astronomy



- For $N_f = 2+1$ FLAG quote arXiv:1011.4408v2 (1 June 2011)

$$\hat{B}_K = 0.738 \pm 0.020\,.$$

- Note the small error!!!

- Beyond the standard model there are in general 5 independent operators which contribute neutral Kaon mixing:

$$\mathcal{H}^{\Delta S=2} = \sum_{i=1}^{5} C_i(\mu)\, Q_i^{\Delta S=2}(\mu)\,.$$

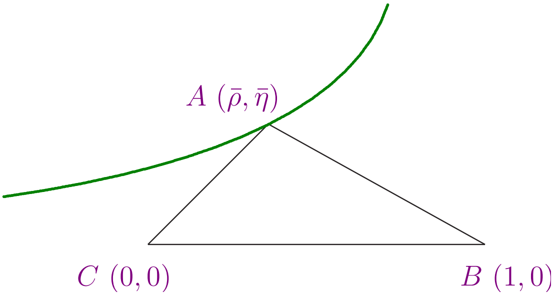
- The five operators are:

$$\begin{aligned}
Q_1^{\Delta S=2} &= [\bar{s}^i \gamma_\mu (1-\gamma_5) d^i]\,[\bar{s}^j \gamma_\mu (1-\gamma_5) d^j] \\
Q_2^{\Delta S=2} &= [\bar{s}^i (1-\gamma_5) d^i]\,[\bar{s}^j (1-\gamma_5) d^j] \\
Q_3^{\Delta S=2} &= [\bar{s}^i (1-\gamma_5) d^j]\,[\bar{s}^j (1-\gamma_5) d^i] \\
Q_4^{\Delta S=2} &= [\bar{s}^i (1-\gamma_5) d^i]\,[\bar{s}^j (1+\gamma_5) d^j] \\
Q_5^{\Delta S=2} &= [\bar{s}^i (1-\gamma_5) d^j]\,[\bar{s}^j (1+\gamma_5) d^i]
\end{aligned}$$

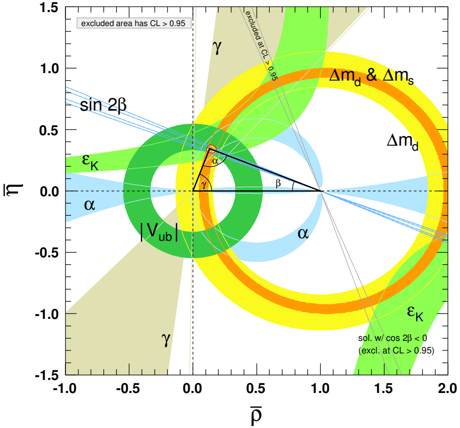
  - i,j are colour indices.
- The matrix elements can be calculated in a similar way to B_K. For a recent study and references to the original literature see Boyle, Garron and Hudspith.

  arXiv:1206.5737
- $Q_1^{\Delta S=2}$ transforms as (27,1) under SU(3)$_L\times$SU(3)$_R$, $Q_2^{\Delta S=2}$ and $Q_3^{\Delta S=2}$ as $(6,\bar{6})$ and $Q_4^{\Delta S=2}$ and $Q_5^{\Delta S=2}$ as (8,8) $\Rightarrow$ Renormalization matrix is block diagonal.

- A precise determination of ε_K would fix the vertex A to lie on a hyperbola.



PDG2012 Unitarity Triangle



UNIVERSITY OF Southampton
School of Physics and Astronomy

Direct Evaluation of $K \to \pi\pi$ Decays

- $K \to \pi\pi$ decays are a very important class of processes for standard model phenomenology.
- Bose Symmetry $\Rightarrow$ the two-pion state has isospin 0 or 2.
- Among the interesting issues are the origin of the $\Delta I = 1/2$ rule (ReA_0/Re$A_2 \simeq 22.5$) and an understanding of the experimental value of ε'/ε, the parameter which was the first experimental evidence of direct CP-violation.
- The evaluation of $K \to \pi\pi$ matrix elements requires an extension of the standard computations of $\langle 0\,|\,O(0)\,|\,h\rangle$ and $\langle h_2\,|\,O(0)\,|\,h_1\rangle$ matrix elements with a single hadron in the initial and/or final state.

Effective Hamiltonian for $K \to \pi\pi$ Decays

$$\mathcal{H}_{\rm eff}^{\Delta S=1} = \frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \sum_{i=1}^{10} \left[z_i(\mu) + \tau\, y_i(\mu)\right] Q_i\,, \text{ where } \tau = -\frac{V_{ts}^* V_{td}}{V_{us}^* V_{ud}} \text{ and}$$

Current – Current Operators

$$Q_1 = (\bar{s}d)_L(\bar{u}u)_L \qquad Q_2 = (\bar{s}^i d^j)_L(\bar{u}^j u^i)_L$$

QCD Penguin Operators

$$Q_3 = (\bar{s}d)_L \textstyle\sum_{q=u,d,s}(\bar{q}q)_L \qquad Q_4 = (\bar{s}^i d^j)_L \sum_{q=u,d,s}(\bar{q}^j q^i)_L$$

$$Q_5 = (\bar{s}d)_L \textstyle\sum_{q=u,d,s}(\bar{q}q)_R \qquad Q_6 = (\bar{s}^i d^j)_L \sum_{q=u,d,s}(\bar{q}^j q^i)_R$$

Electroweak Penguin Operators

$$Q_7 = \tfrac{3}{2}(\bar{s}d)_L \textstyle\sum_{q=u,d,s} e_q(\bar{q}q)_L \qquad Q_8 = \tfrac{3}{2}(\bar{s}^i d^j)_L \sum_{q=u,d,s} e_q(\bar{q}^j q^i)_L$$

$$Q_9 = \tfrac{3}{2}(\bar{s}d)_L \textstyle\sum_{q=u,d,s} e_q(\bar{q}q)_R \qquad Q_{10} = \tfrac{3}{2}(\bar{s}^i d^j)_L \sum_{q=u,d,s} e_q(\bar{q}^j q^i)_R$$

This 10 operator basis is very natural but over-complete:

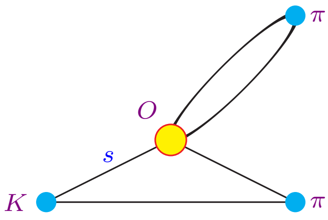
$$\begin{aligned} Q_{10} - Q_9 &= Q_4 - Q_3 \\ Q_4 - Q_3 &= Q_2 - Q_1 \\ 2Q_9 &= 3Q_1 - Q_3\,. \end{aligned}$$

The original material on this topic is taken from two RBC-UKQCD papers:

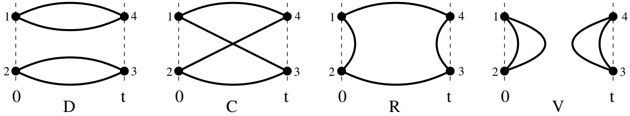
1. "The $K \to (\pi\pi)_{I=2}$ Decay Amplitude from Lattice QCD,"
T.Blum, P.A.Boyle, N.H.Christ, N.Garron, E.Goode, T.Izubuchi, C.Jung, C.Kelly, C.Lehner, M.Lightman, Q.Liu, A.T.Lytle, R.D.Mawhinney, C.T.Sachrajda, A.Soni and C.Sturm,
Phys. Rev. Lett. 108 (2012) 141601, (arXiv:1111.1699 [hep-lat]).

2. "Lattice Determination of the $K \to (\pi\pi)_{I=2}$ Decay Amplitude A_2,"
T.Blum, P.A.Boyle, N.H.Christ, N.Garron, E.Goode, T.Izubuchi, C.Jung, C.Kelly, C.Lehner, M.Lightman, Q.Liu, A.T.Lytle, R.D.Mawhinney, C.T.Sachrajda, A.Soni and C.Sturm,
Phys.Rev. D86 (2012) 074513, (arXiv:1206.5142 [hep-lat]).

$K \to (\pi\pi)_{I=2}$ decay amplitudes

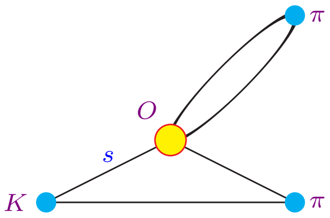


- We need to evaluate correlation functions as in the diagram above.
- In order to divide by $\langle 0|J_\pi J_\pi|\pi\pi\rangle$, we also need to evaluate the two-pion correlation functions.



- For I=2 $\pi\pi$ states the correlation function is proportional to D-C.

$K \to (\pi\pi)_{I=2}$ decay amplitudes (Cont.)



- In the physical decay, in the centre-of-mass frame, $E_{\pi\pi} = m_K$.
- In lattice calculations, in order to eliminate excited states we do not integrate over time, and so, in general, energy is not conserved.
- In the centre-of-mass frame the ground-state is the two-pion state with $E_{\pi\pi} \simeq 2m_\pi$.
- Therefore the correlation function is dominated by the unphysical transition of a kaon at rest into two pions at rest. Maiani-Testa Problem
- The Lellouch-Lüscher solution is to tune the volume so that one of the excited states corresponds to $E_{\pi\pi} = m_K$. (Loss of precision.) hep-lat/0003023

- The operators whose matrix elements have to be calculated are:

$$\begin{aligned}
O^{3/2}_{(27,1)} &= (\bar s^i d^i)_L \{(\bar u^j u^j)_L - (\bar d^j d^j)_L\} + (\bar s^i u^i)_L (\bar u^j d^j)_L \\
O^{3/2}_{7} &= (\bar s^i d^i)_L \{(\bar u^j u^j)_R - (\bar d^j d^j)_R\} + (\bar s^i u^i)_L (\bar u^j d^j)_R \\
O^{3/2}_{8} &= (\bar s^i d^j)_L \{(\bar u^j u^i)_R - (\bar d^j d^i)_R\} + (\bar s^i u^j)_L (\bar u^j d^i)_R
\end{aligned}$$

- It is convenient to use the Wigner-Eckart Theorem: (Notation - $O^{\Delta I}_{\Delta I_z}$)

$${}_{I=2}\langle \pi^+(p_1)\pi^0(p_2)\,|O^{3/2}_{1/2}|K^+\rangle = \frac{\sqrt{3}}{2}\langle \pi^+(p_1)\pi^+(p_2)\,|O^{3/2}_{3/2}|K^+\rangle\,,$$

  where
  - $O^{3/2}_{3/2}$ has the flavour structure $(\bar s d)\,(\bar u d)$.
  - $O^{3/2}_{1/2}$ has the flavour structure $(\bar s d)\,(\,(\bar u u) - (\bar d d)\,) + (\bar s u)\,(\bar u d)$.

- We can then use antiperiodic boundary conditions for the u-quark say, so that the $\pi\pi$ ground-state is $\langle \pi^+(\pi/L)\,\pi^+(-\pi/L)\,|$. C-h Kim, Ph.D. Thesis
  - • Do not have to isolate an excited state. •
  - Size (L) needed for physical $K \to \pi\pi$ decay halved.

UNIVERSITY OF Southampton
School of Physics and Astronomy

- The main theoretical ingredients of the *infrared* problem with two-pions in the s-wave are understood.
- Two-pion quantization condition in a finite-volume

$$\delta(q^*) + \phi^P(q^*) = n\pi\,,$$

where $E^2 = 4(m_\pi^2 + q^{*\,2})$, δ is the s-wave $\pi\pi$ phase shift and ϕ^P is a kinematic function.

M.Lüscher, 1986, 1991, $\cdots$.

- The relation between the physical $K \to \pi\pi$ amplitude A and the finite-volume matrix element M

$$|A|^2 = 8\pi V^2\,\frac{m_K E^2}{q^{*\,2}}\left\{\delta'(q^*) + \phi^{P\,\prime}(q^*)\right\}|M|^2\,,$$

where $'$ denotes differentiation w.r.t. q^* .

L.Lellouch and M.Lüscher, hep-lat/0003023; C.h.Kim, CTS and S.Sharpe, hep-lat/0507006;
N.H.Christ, C.h.Kim and T.Yamazaki hep-lat/0507009

- Computation of $K \to (\pi\pi)_{I=2}$ matrix elements does not require the subtraction of power divergences or the evaluation of disconnected diagrams.
- In 2011-2012, we evaluate the $\Delta I = 3/2$ $K \to \pi\pi$ matrix elements for the first time and at physical kinematics.

$K \to (\pi\pi)_{I=2}$ decay amplitudes (Cont.)

- The calculations were performed on a $32^3 \times 64 \times 32$ ($L = 4.58$ fm, $a^{-1} = 0.14$ fm lattice using Domain Wall Fermions and the IDSDR gauge action.

Systematic Error Budget	$\mathrm{Re}A_2$	$\mathrm{Im}A_2$
lattice artefacts	15%	15%
finite-volume corrections	6.0%	6.5%
partial quenching	3.5%	1.7%
renormalization	1.8%	5.6%
unphysical kinematics	0.4%	0.8%
derivative of the phase shift	0.97%	0.97%
Wilson coefficients	6.6%	6.6%
Total	18%	19%

- The dominant error is due to lattice artefacts and the fact that out lattice is coarse. This will be eliminated when the calculation is repeated at a second lattice spacing.
- The 15% estimate, intended to be conservative, is obtained by
  1. Studying the dependence on a of quantities which have been calculated at several lattice spacings.
  2. In particular by determining the a dependence of B_K, which is also given by the matrix element of a (27,1) operator.

UNIVERSITY OF Southampton
School of Physics and Astronomy

Our results for the amplitude A_2 are:

$$\mathrm{Re}A_2 = (1.381 \pm 0.046_{\mathrm{stat}} \pm 0.258_{\mathrm{syst}})\, 10^{-8}\, \mathrm{GeV}$$
$$\mathrm{Im}A_2 = -(6.54 \pm 0.46_{\mathrm{stat}} \pm 1.20_{\mathrm{syst}})\, 10^{-13}\, \mathrm{GeV}.$$

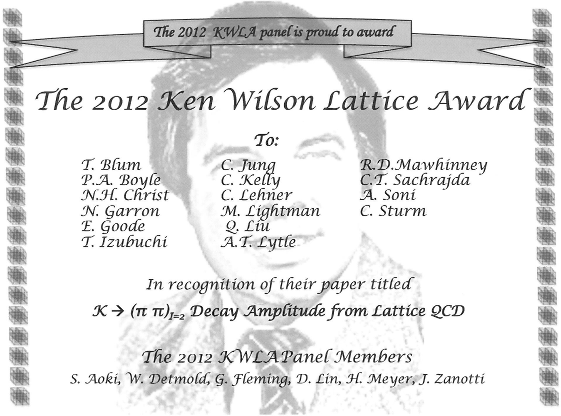
- The result for Re A_2 agrees well with the experimental value of $1.479(4) \times 10^{-8}$ GeV obtained from K^+ decays and $1.573(57) \times 10^{-8}$ GeV obtained from K_S decays .
- Im A_2 is unknown so that our result provides its first direct determination.
- For the phase of A_2 we find Im A_2/Re$A_2 = -4.42(31)_{\mathrm{stat}}(89)_{\mathrm{syst}}\, 10^{-5}$.
- Combining our result for Im A_2 with the experimental results for Re A_2, Re $A_0 = 3.3201(18) \cdot 10^{-7}$ GeV and ε'/ε we obtain:

$$\frac{\mathrm{Im}A_0}{\mathrm{Re}A_0} = -1.61(19)_{\mathrm{stat}}(20)_{\mathrm{syst}} \times 10^{-4}\,.$$

(Of course, we wish to confirm this directly.)

$$\frac{\mathrm{Im}A_0}{\mathrm{Re}A_0} = \frac{\mathrm{Im}A_2}{\mathrm{Re}A_2} - \frac{\sqrt{2}\,|\varepsilon|}{\omega}\frac{\varepsilon'}{\varepsilon}$$

$$-1.61(19)_{\mathrm{stat}}(20)_{\mathrm{syst}} \times 10^{-4} = -4.42(31)_{\mathrm{stat}}(89)_{\mathrm{syst}} \times 10^{-5} - 1.16(18) \times 10^{-4}\,.$$

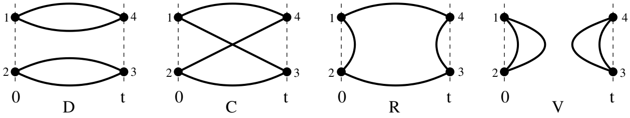


- For this work we received the 2012 Ken Wilson Lattice award at Lattice 2012.
- Criteria: The paper must be important research beyond the existing state of the art. $\cdots$

The material on this topic is taken from the thesis of Qi Liu and the two RBC-UKQCD papers:

1. "K to $\pi\pi$ Decay Amplitudes from Lattice QCD,"
   T.Blum, P.A.Boyle, N.H.Christ, N.Garron, E.Goode, T.Izubuchi, C.Lehner, Q.Liu, R.D.Mawhinney, C.T.Sachrajda, A.Soni, C.Sturm, H.Yin and R. Zhou,
   Phys.Rev. D84 (2011) 114503, (arXiv:1106.2714 [hep-lat]).
2. "Emerging understanding of the $\Delta I = 1/2$ rule from Lattice QCD,"
   P.A.Boyle, N.H.Christ, N.Garron, E.J.Goode, T.Janowski, C.Lehner, Q.Liu, A.T.Lytle, C.T.Sachrajda, A.Soni and D.Zhang
   Phys.Rev.Lett. (to appear), (arXiv:1212.1474 [hep-lat]).

- The $I = 0$ final state has vacuum quantum numbers.
- Vacuum contribution must be subtracted; disconnected diagrams require statistical cancelations to obtain the $e^{-2m_\pi t}$ behaviour.
- Consider first the two-pion correlation functions, which are an important ingredient in the evaluation of $K \to \pi\pi$ amplitudes.

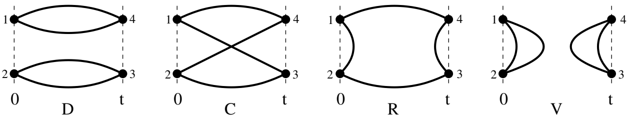


- For I=2 $\pi\pi$ states the correlation function is proportional to D-C.
- For I=0 $\pi\pi$ states the correlation function is proportional to 2D+C-6R+3V.

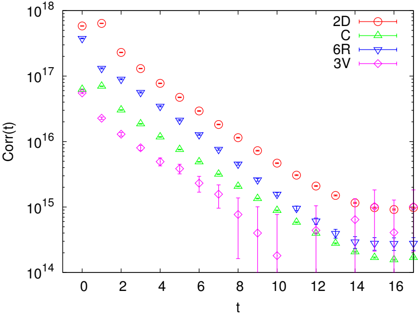
The major practical difficulty is to subtract the vacuum contribution with sufficient precision.

- In the paper we report on high-statistics experiments on a $16^3 \times 32$ lattice, $a^{-1} = 1.73$ GeV, $m_\pi = 420$ MeV, with the propagators evaluated from each time-slice.

Diagrams contributing to two-pion correlation functions



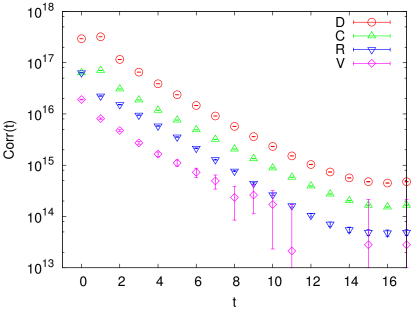
- For I=2 $\pi\pi$ states the correlation function is proportional to D-C.
- For I=0 $\pi\pi$ states the correlation function is proportional to 2D+C-6R+3V.

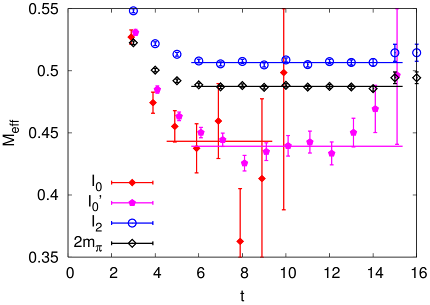


RBC/UKQCD, Qi Liu – Lattice 2010

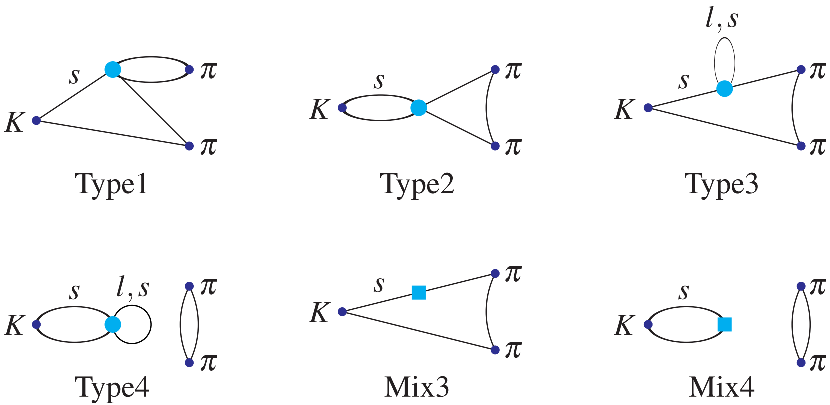
Two-pion Correlation Functions

RBC/UKQCD, arXiv:1106.2714





- $M_{\rm eff} = \log C(t)/C(t+1)$.



- There are 48 different contractions and we classify the contributions into the 6 different types illustrated above.
- Mix3 and Mix4 are needed to subtract the power divergences which are proportional to matrix elements of $\bar{s}\gamma_5 d$.

- These results are for the $K \to \pi\pi$ (almost) on-shell amplitudes with 420 MeV pions at rest: RBC/UKQCD arXiv:1106.2714

Re A_0	$(3.80 \pm 0.82)\,10^{-7}$ GeV	Im A_0	$-(2.5 \pm 2.2)\,10^{-11}$ GeV
Re A_2	$(4.911 \pm 0.031)\,10^{-8}$ GeV	Im A_2	$-(5.502 \pm 0.0040)\,10^{-13}$ GeV

- This was an exploratory exercise in which we are learning how to do the calculation.
- We, along with the rest of the world, continue to develop techniques with the aim of enhancing the signal for disconnected diagrams.
- The exploratory results for $K \to (\pi\pi)_{I=0}$ decays are very encouraging.
- For $(\pi\pi)_I = 0$ states the Wigner-Eckart theorem and the use of antiperiodic boundary conditions for the d-quark does not help.

  C.Sachrajda and G.Villadoro hep-lat/0411033

  We are currently developing and testing the use of G-parity boundary conditions.

  C.-h Kim, hep-lat/0311003

  $\Rightarrow$ a quantitative understanding of the $\Delta I = 1/2$ rule and the value of ε'/ε.

- The evaluation of disconnected diagram has allowed us to study the η and η' mesons and their mixing. RBC-UKQCD – arXiV:1002.2999 arXiv:1212.1474

- In his thesis Qi Liu extended the above study to the $24^3 \times 64$ ensembles.
  - Larger $T \Rightarrow$ suppression of around-the-world effects.
  - Two-pion sources separated in time $\Rightarrow$ better plateaus.
  - Faster algorithm for the inversions.

1. $16^3 \times 32$ ensembles; 877 MeV kaon decaying into two 422 MeV pions at rest:

$$\frac{\mathrm{Re}\,A_0}{\mathrm{Re}\,A_2} = 9.1 \pm 2.1\,.$$

2. $24^3 \times 64$ ensembles; 662 MeV kaon decaying into two 329 MeV pions at rest:

$$\frac{\mathrm{Re}\,A_0}{\mathrm{Re}\,A_2} = 12.0 \pm 1.7\,.$$

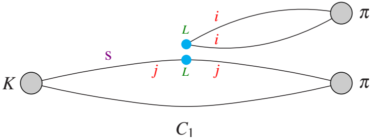
- Whilst both these results are obtained at unphysical kinematics and are different from the physical value of 22.5, it is nevertheless interesting to understand the origin of these enhancements.
- 99% of the contribution to the real part of A_0 and A_2 come from the matrix elements of the current-current operators.

Contributions from Individual Matrix Elements

i	Q_i^{lat} [GeV]	$Q_i^{\overline{\text{MS}}\text{-NDR}}$ [GeV]
1	8.1(4.6) 10^{-8}	6.6(3.1) 10^{-8}
2	2.5(0.6) 10^{-7}	2.6(0.5) 10^{-7}
3	-0.6(1.0) 10^{-8}	5.4(6.7) 10^{-10}
4	–	2.3(2.1) 10^{-9}
5	-1.2(0.5) 10^{-9}	4.0(2.6) 10^{-10}
6	4.7(1.7) 10^{-9}	-7.0(2.4) 10^{-9}
7	1.5(0.1) 10^{-10}	6.3(0.5) 10^{-11}
8	-4.7(0.2) 10^{-10}	-3.9(0.1) 10^{-10}
9	–	2.0(0.6) 10^{-14}
10	–	1.6(0.5) 10^{-11}
ReA_0	3.2(0.5) 10^{-7}	3.2(0.5) 10^{-7}

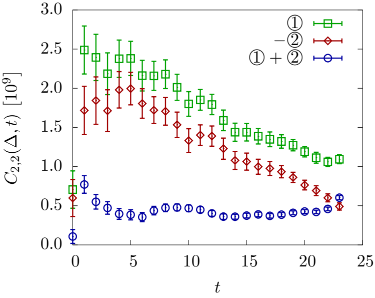
- Contributions from each operator to ReA_0 for $m_K = 662$ MeV and $m_\pi = 329$ MeV. The second column contains the contributions from the 7 linearly independent lattice operators with $1/a = 1.73(3)$ GeV and the third column those in the 10-operator basis in the $\overline{\text{MS}}$-NDR scheme at $\mu = 2.15$ GeV. Numbers in parentheses represent the statistical errors.

Emerging understanding of the $\Delta I = 1/2$ Rule (Cont.)

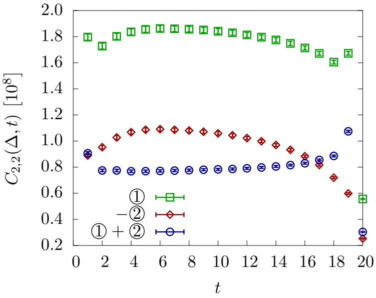


- Re A_2 is proportional to $C_1 + C_2$.
- The contribution to Re A_0 from Q_2 is proportional to $2C_1 - C_2$ and that from Q_1 is proportional to $C_1 - 2C_2$ with the same sign.
- Colour counting might suggest that $C_2 \simeq \frac{1}{3} C_1$.
  - Much continuum phenomenology has been done in the vacuum insertion hypothesis.
- We find instead that $C_2 \approx -C_1$ so that A_2 is significantly suppressed!
- A_2 has a larger kinematic dependence than A_0.
- **We believe that the strong suppression of Re A_2 and the (less-strong) enhancement of Re A_0 is a major factor in the $\Delta I = 1/2$ rule.**
  - Of course before claiming a quantitative understanding of the $\Delta I = 1/2$ rule we need to compute Re A_0 at physical kinematics and reproduce the experimental value of 22.5.

Evidence for the Suppression of Re A_2



Physical Kinematics



$m_\pi \simeq 330$ MeV at threshold.

- Notation ⓘ $\equiv C_i,\ i = 1,2$.

- Evaluation of long-distance effects in ΔM_K and ε_K.
- Development and testing of G-parity boundary conditions with the primary aim of computing the $K\to(\pi\pi)_{I=0}$ decay amplitude A_0.
- Beginning to perform the exploratory work to study the rare kaon decays $K\to\pi\ell^+\ell^-$ and $K\to\pi\nu\bar{\nu}$.