The Jaynes-Cummings Model

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The Jaynes-Cummings model is a milestone in the theory of interaction between a two-level system and a single bosonic field mode. This work aims to give a complete description of the model, analyzing the Hamiltonian of the system, its eigenvalues and eigenstates, in order to characterize the dynamics of system. It’s going to start from Classical Mechanics with Maxwell’s equations and their momenta in the Quantum Mechanics system must have a commutator analogous to $\hat{x}$ and $\hat{p}$ as in any simple Quantum Mechanics system. It’s show that quantized fields are nothing more than a system of decoupled harmonic oscillators, at a collection of different wave vectors and wave polarizations. In the end, is possible to see how basic quantum of electromagnetic fields, which receive the name photon, are created and annihilated in discrete process of emission and absorption by atoms and matter in general.

I. INTRODUCTION

The Jaynes-Cummings (JC) model was originally proposed in 1963 by Edwin Jaynes and Fred Cummings [1] in order to study the relationship between the quantum theory of radiation and the semi-classical theory in describing the phenomenon of spontaneous emission. In the semi-classical theory of atom-field interaction, the field is treated as a definite function of time while the atom is quantized. The semi-classical theory can explain many phenomena that are observed in modern optics, for example the existence of Rabi cycles in atomic excitation probabilities for radiation fields with sharply defined energy. The JC model also aims to find how quantization of the radiation field affects the predictions for the evolution of the state of a two-level system, in comparison with semi-classical theory of matter-radiation interaction.

The JC model is still an important and actual topic in quantum physics, since it is used to study different branches of the most recent and outstanding research activities. The natural framework for the implementation and experimental testing of the JC model is cavity quantum electrodynamics (QED), where fundamental experiments has been performed using microwave and optical cavities [2–4]. In the context of solid state systems, semiconductor quantum dots are placed inside photonic crystal, micropillar or microdisk resonators [5, 6], which allows the study of the regime of cavity quantum electrodynamics.

In order to more precisely describe the interaction between an atom and a laser field, the model is generalized in different ways. Some of the generalizations comprehend different initial conditions [7], dissipation and damping in the model [8] and multi-mode description of the field [9].

The physical model is introduced in section II, where the Jaynes-Cummings Hamiltonian will be derivate. With the Hamiltonian in hands will be possible to analyze some features, like the Jaynes-Cummings ladder in section III, the Vacuum-Rabi oscillations in section IV and the phenomena of collapses and revivals in section V.

II. THE MODEL

Considering a physical system composed of an atom coupled to a mode of the electromagnetic field inside an isolated cavity (therefore without interaction with the outside). If the frequency $\omega_k$ of the mode of interest of the field is such that $\hbar\omega_k$ is approximately equal to the energy difference between the ground state $|g\rangle$ and some excited state $|e\rangle$, only these two states will have a significant participation in the dynamics of the system and we consider only these two energy levels of the atom [10].

The JC Hamiltonian will be derived starting from:

$$\hat{H} = \sum_i \left[ \frac{1}{2m_e} \left( \vec{p}_i - \frac{e}{c} \vec{A} \right)^2 + V_{ext}(\vec{r}_i) - \frac{e\hbar}{2m_e c} \sigma_i \cdot \vec{B} + \frac{1}{2} \int (|\vec{E}|^2 + |\vec{B}|^2) d^3r + \frac{1}{2} \sum_j U(\vec{r}_i, \vec{r}_j) \right]$$

which represents a system in a presence of electromagnetic field. The first term is the minimal coupling, while the third, fourth, fifth, respectively, represents the spin-field coupling, the free radiation field, and the interaction between electrons.

At this point, two approximations will be used. The first one is that the distance between electrons is huge, so the interaction between them will be approximately zero and the second one is that photons have low energies, relative to Rydberg energies, implying that spin-field coupling will be approximately zero too.

With these approximations, the Hamiltonian can be
written as:

\[
\hat{H} = \sum_i \left[ \frac{p_i^2}{2m_e} - \frac{e}{m_e} \vec{A} \cdot \vec{p}_i + V_{ext}(\vec{r}_i) \right] + \sum_i \frac{e^2}{2m_e} \vec{A} \cdot \vec{A} + \frac{1}{2} \int \left( |\vec{E}|^2 + |\vec{B}|^2 \right) d^3r
\]

(2)

where the second and fourth terms represent, respectively, the interaction electron-field and interaction between modes of the field. Since the term field-field is smaller than electron-field, it will be despised.

After all these approximations, we can see that the Hamiltonian of Jaynes-Cummings has 3 terms labeled, respectively, \( \hat{H}_{\text{field}}, \hat{H}_{\text{atom}} \) and \( \hat{H}_{\text{int}} \):

\[
\hat{H} = \frac{1}{2} \int \left( |\vec{E}|^2 + |\vec{B}|^2 \right) d^3r + \sum_i \frac{p_i^2}{2m_e} + \sum_i \frac{e^2}{2m_e} \vec{A} \cdot \vec{A} + \frac{1}{2} \int \left( |\vec{E}|^2 + |\vec{B}|^2 \right) d^3r
\]

(3)

which will be quantized in the follow subsections.

A. Quantization of the free electromagnetic field

Starting from Maxwell’s equations for free space and Coulomb’s gauge, is possible to demonstrate that the magnetic vector potential \( \vec{A}(\vec{r}, t) \) satisfies a wave equation. One possible ansatz as solution is:

\[
\vec{A}(\vec{r}, t) = \sum_k c_k(t) \vec{\alpha}_k(\vec{r})
\]

(4)

where \( k \) represents quantum numbers, which specify cavity modes (the different modes are orthogonal), \( \vec{\alpha}_k(\vec{r}) \) are the vibration modes of cavity, which by symmetry of the system are plane waves and \( c_k(t) \) are the amplitudes of waves.

For the total field in some volume \( V \) of the cavity, a Fourier expansion over a collecting of these nodes is used supposing the follow boundary conditions \( \vec{A}(\vec{r} + \vec{L}, t) = \vec{A}(\vec{r}, t) \). The vector potential is written as:

\[
\vec{A}(\vec{r}, t) = \frac{1}{\sqrt{V}} \sum_{k,\alpha} \phi^\alpha(\vec{r}) c_{k,\alpha}(0) e^{i(k\vec{r} - \omega t)} + \phi^{\alpha\dagger}(\vec{r}) c_{k,\alpha}^\dagger(0) e^{-i(k\vec{r} + \omega t)}
\]

(5)

where \( \alpha = 1, 2 \) are the directions of polarization and \( \phi^\alpha \) is the polarization versor.

The zero divergence of \( \vec{B}(\vec{r}, t) \) and the Faraday’s law allow the introduction of potential vector that give the fields \( \vec{B} = \nabla \times \vec{A} \) and \( \vec{E} = -\frac{1}{\epsilon} \frac{\partial \vec{A}}{\partial t} \). In order to have physical solutions, a substitution of variables will be carried out: \( c_{k,\alpha}(t) \rightarrow \left( \frac{\hbar c^2}{2\omega_k} \right)^{1/2} \hat{a}_{k,\alpha}(t) \) and \( c_{k,\alpha}^\dagger(t) \rightarrow \left( \frac{\hbar c^2}{2\omega_k} \right)^{1/2} \hat{a}_{k,\alpha}^\dagger(t) \).

Replacing \( \vec{A}(\vec{r}, t) \) in the \( \hat{H}_{\text{field}} \) the electromagnetic field Hamiltonian can be written as:

\[
\hat{H}_{\text{field}} = \frac{1}{2} \sum_{k,\alpha} \frac{\hbar \omega_k}{2} [\hat{a}_{k,\alpha}(t)\hat{a}_{k,\alpha}^\dagger(t) + \hat{a}_{k,\alpha}^\dagger(t)\hat{a}_{k,\alpha}(t)]
\]

(6)

Remembering the commutation relation for creation and annihilation operators \( [\hat{a}, \hat{a}^\dagger] = 1 \), the equation above can be written as:

\[
\hat{H}_{\text{field}} = \sum_{k,\alpha} \hbar \omega_k \left[ \hat{a}_{k,\alpha}^\dagger(t)\hat{a}_{k,\alpha}(t) + \frac{1}{2} \right]
\]

(7)

where \( \hat{a}_{k,\alpha}(t) \) and \( \hat{a}_{k,\alpha}^\dagger(t) \) are, respectively, the fermionic annihilation and creation operators in the Heisenberg picture and \( \omega_k \) is the frequency associated to the field mode.

As the term \( \frac{1}{2} \hbar \omega_k \), present in all field modes, is a constant, we can shift the zero of the field energy, that is, we will eliminate these contributions in all the modes of the field. So, the Hamiltonian of the quantized electromagnetic field is:

\[
\hat{H}_{\text{field}} = \sum_{k,\alpha} \hbar \omega_k \hat{a}_{k,\alpha}^\dagger(t)\hat{a}_{k,\alpha}(t)
\]

(8)

It is possible to see that quantized fields are a system of decoupled harmonic oscillators.

B. Quantization of matter

The possible states for two-level system will be denoted \( |e\rangle \) and \( |g\rangle \), which represents, respectively, excited and ground states. The Hamiltonian can then be written as:

\[
\hat{H}_{\text{atom}} = E_+|e\rangle\langle e| + E_-|g\rangle\langle g|
\]

\[
= \frac{1}{2} (E_+ + E_g) \hat{I} + \frac{1}{2} (E_+ - E_g) \hat{\sigma}_z
\]

(9)

Writing the energy difference as \( \hbar \omega_0 = E_+ - E_- \) where \( \omega_0 \) is the atomic transition frequency and shifting our zero of energy to \( E_+ + E_- \), because we only care about energy differences, the atomic Hamiltonian can be written as:

\[
\hat{H}_{\text{atom}} \approx \frac{1}{2} \hbar \omega_0 \hat{\sigma}_z
\]

(10)

where \( \hat{\sigma}_z \) is one of the Pauli’s matrix:

\[
\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

(11)

C. Quantization of interaction

Assuming that the wave functions for this system are as follows:

\[
\Psi_{ij}(\vec{r}) = \sum_{j=1,2} \Phi_{ij}(\vec{r}) \hat{b}_j
\]

(12)

where \( \hat{b}_j \) is a field operator and \( \Phi_{ij} \) represents the wave function of a two-level system.

The interaction of an atom with one of the electromagnetic’s mode, in the dipole approximation, can be expressed through:

\[
\hat{H}_{\text{int}} = \frac{e}{mc} \vec{A} \cdot \vec{p}_i \approx -e\vec{r}_i \cdot \vec{E}
\]

(13)
Then, we can determine \( \langle \hat{H}_{\text{int}} \rangle \), which is going to be:

\[
\langle \hat{H}_{\text{int}} \rangle = \int d^3r \psi_i(r) [-e\hat{r}\cdot\hat{E}] \psi_j(r) = \sum_{i,j} \sum_{k,\alpha} \hbar\hat{b}_i^\dagger \hat{b}_j [\Omega \hat{a}_{k,\alpha}(t) + \Omega^* \hat{a}_{k,\alpha}^\dagger(t)]
\]

(14)

where \( \Omega = \Omega^* \) is the Rabi-frequency in the dipole approximation.

Expanding the sum \( \sum_j \), another substitution of variables will be carried out: \( \hat{b}_1^\dagger \hat{b}_1 \rightarrow \hat{\sigma}_+ \) and \( \hat{b}_2^\dagger \hat{b}_2 \rightarrow \hat{\sigma}_- \). The matrix \( \hat{\sigma}_+ \) takes the ground state to the excited and the matrix \( \hat{\sigma}_- \) do the reverse process. They are written as:

\[
\hat{\sigma}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \hat{\sigma}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\]

(15)

The \( \hat{\sigma}_+ \hat{a}^\dagger \) and \( \hat{\sigma}_- \hat{a}^\dagger \) terms vary much more rapidly than the other terms, so the energy for them won’t be conservative. The rotating wave approximation (RWA) will be used for them to be neglected.

In the resonance (\( w_0 = w_k \)), the detune will be equal to zero and it will remove the time dependence. One of the model’s prepositions is the interaction between an atom with only one mode of the cavity, then the \( \sum_{k,\alpha} \) will be dropped.

Thus, the \( \hat{H}_{\text{int}} \) can be written as:

\[
\hat{H}_{\text{int}} = \hbar \Omega [\hat{\sigma}_+ \hat{a} + \hat{\sigma}_- \hat{a}^\dagger]
\]

(16)

where \( \hat{\sigma}_+ \hat{a} \) represents a photon being absorbed and the atom is excited from the ground state to the excited state and \( \hat{\sigma}_- \hat{a}^\dagger \) represents a photon being emitted and the atom is de-excited.

Finally, the Hamiltonian for the Jaynes-Cummings Model (\( \hat{H}_{JC} \)) is:

\[
\hat{H}_{JC} = \hbar w_0 \hat{a}^\dagger \hat{a} + \frac{1}{2} \hbar w_0 \hat{\sigma}_z + \hbar \Omega [\hat{\sigma}_+ \hat{a} + \hat{\sigma}_- \hat{a}^\dagger]
\]

(17)

### III. JAYNES-CUMMINGS LADDER

The interaction Hamiltonian can only cause transitions of the type \( |e\rangle |n\rangle \leftrightarrow |g\rangle |n+1\rangle \), where these product states are referred to be as the bare states of the Jaynes-Cummings model. For a fixed \( n \), the dynamics of the system are confined to the two dimensional space of product states \( \{|e\rangle |n\rangle, |g\rangle |n+1\rangle \} \).

The Hamiltonian can be written as:

\[
\hat{H}_{JC} = \begin{pmatrix} n\hbar w_k + \frac{1}{2} \hbar w_0 & \hbar \Omega \sqrt{n+1} \\ \hbar \Omega \sqrt{n+1} & (n+1)\hbar w_k - \frac{1}{2}\hbar w_0 \end{pmatrix}
\]

(18)

where the eigenvalues are given by:

\[
E_{\pm} = \left(n + \frac{1}{2}\right) \hbar w_0 \pm \hbar \sqrt{(w_0 - w_k)^2 + 4\Omega^2(n+1)}
\]

(19)

On resonance (\( w_0 = w_k \)) and relabeling \( g_0 = 2\Omega \):

\[
E_{\pm} = \left(n + \frac{1}{2}\right) \hbar w_0 \pm g_0 \hbar \sqrt{n + 1}
\]

(20)

With figure 1, it is possible to visualize how occurs the coupling between an atom and field states. It can be noted that with the increase in the number of photons, the energy between the dressed states increases, that is, transitions with higher energy can be made.

![FIG. 1: Justaposition of bare states (uncoupled) and dressed states (coupled). Picture of a two level atom coupled to a single mode field.](image)

**IV. VACUUM-RABI OSCILLATIONS**

The Jaynes-Cummings Hamiltonian may be separated into two commuting parts:

\[
\hat{H}_{JC} = \hat{H}_0 + \hat{H}_{\text{int}}
\]

(21)

where \( \hat{H}_0 \) represents the Hamiltonian for the field plus the atom and \( \hat{H}_{\text{int}} \) represents the interaction between field-atom. All the dynamics of the system are contained in the second part.

Let the initial state of the field-atom system be \( |i\rangle = |e, n\rangle \) and the final state be \( |f\rangle = |g, n+1\rangle \), thus the state vector may be written as:

\[
|\Psi(t)\rangle = C_i|i\rangle + C_f|f\rangle
\]

(22)

The Schrodinger equation in the interact picture is given by:

\[
i\hbar \frac{d}{dt} |\Psi_I(t)\rangle = \hat{H}_{\text{int}} |\Psi_I(t)\rangle
\]

(23)

and this allows to write down differential equations for the coefficients \( C_i \) and \( C_f \):

\[
C_i = -i\sqrt{n+1}C_f
\]

\[
C_f = -i\sqrt{n+1}C_i
\]

(24)

The result of the pair of harmonic-oscillator-looking equations, with the following initial conditions \( C_i(0) = 1 \)
and $C_f(0) = 0$, is:

$$C_i(t) = \cos(\Omega \sqrt{n + 1} t)$$
$$C_f(t) = -i \sin(\Omega \sqrt{n + 1} t) \quad (25)$$

With this coefficients, is possible to determinate the probability of the system remains in the excited state:

$$P_i(t) = |C_i(t)|^2 = \cos^2(\Omega \sqrt{n + 1} t) \quad (26)$$

and the probability it makes a transition to the ground state:

$$P_f(t) = |C_f(t)|^2 = \sin^2(\Omega \sqrt{n + 1} t) \quad (27)$$

The atomic inversion then is given by:

$$W(t) = |C_i(t)|^2 - |C_f(t)|^2 = \cos(2\Omega \sqrt{n + 1} t) \quad (28)$$

Is interesting to see that in the absence of light ($n=0$), there is still a non-zero transition probability. This phenomena is known as spontaneous emission.

V. COLLAPSE AND REVIVAL OF ATOMIC OSCILLATIONS

In the atom-field interaction, one of the most interesting phenomena can be collapse-revival effect in population inversion of atomic levels[11].

To calculate the atomic inversion, will be considered the initial condition in which the atom is in a superposition of ground and excited states and the field in a coherent state:

$$|\Psi(0)\rangle = |\Psi_{atom}(0)\rangle \otimes |\Psi_{field}(0)\rangle \quad (29)$$

where

$$|\Psi_{atom}(0)\rangle = C_g|g\rangle + C_e|e\rangle$$

$$|\Psi_{field}(0)\rangle = \sum_{n=0}^{\infty} e^{-\frac{1}{2}|\alpha|^2} \frac{\alpha^n}{\sqrt{n!}} |n\rangle = \sum_{n=0}^{\infty} C_n|n\rangle$$

The solution to Schrodinger’s equation is:

$$|\Psi(t)\rangle = \sum_{n=0}^{\infty} \{A(n)|e\rangle + B(n)|g\rangle\} |n\rangle \quad (30)$$

where:

$$A(n) = C_e C_n \cos(\Omega \sqrt{n + 1}) - i C_g C_{n+1} \sin(\Omega \sqrt{n + 1})$$
$$B(n) = i C_e C_{n+1} \sin(\Omega \sqrt{n}) + C_g C_n \cos(\Omega \sqrt{n})$$

The atomic inversion, using again the follow initial conditions $C_i(0) = 1$ and $C_f(0) = 0$, is given by:

$$W(t) = P_i(t) - P_f(t) = e^{-N} \sum_{n=0}^{\infty} \frac{N_n}{\sqrt{n!}} \cos(2\Omega \sqrt{n + 1} t) \quad (31)$$

where $N = |\alpha|^2$ is the average photon number.

Relabeling $\lambda = 2\Omega$:

$$W(t) = e^{-N} \sum_{n=0}^{\infty} \frac{N_n}{\sqrt{n!}} \cos(\lambda \sqrt{n + 1} t) \quad (32)$$

From figure 2, is possible can see that $W(t)$ quickly collapses, that is, it drops to zero and, as time increases, it revival. This phenomena can be understood from equation 32. Each sum term represents Rabi’s oscillations for a given $n$. At $t = 0$, the atom is prepared in a defined state and therefore all terms in the sum are correlated.

As time passes, the different Rabi’s frequencies interfere with each other, and the atomic inversion quickly collapses. However, as time grows further, the correlation between Rabi’s oscillations is restored and a resurgence occurs. This behavior of collapse and resurgence repeats itself over time.

FIG. 2: Collapse and revival of atomic oscillations for n=35 and n=65.

VI. CONCLUSION

In this work, we formulate the problem of an two-level atom coupled with the radiation field. To demonstrate the JC Hamiltonian, we used the dipole approximation and the rotating wave approximation (RWA), where the Jaynes-Cummings Model is valid.

After the Hamiltonian was derive, we introduce the concept of dressed states, which represents the coupling between states of the atom with states of the field. We also studied 2 interesting situations where the model can be used: Vacuum-Rabi oscillations, which are purely quantum mechanical and are the result of the atom spontaneously emitting a photon and absorbing of it; and the collapse and revival of atomic oscillations, characteristic of interaction of a two-level atom with a cavity prepared in a coherent way.


