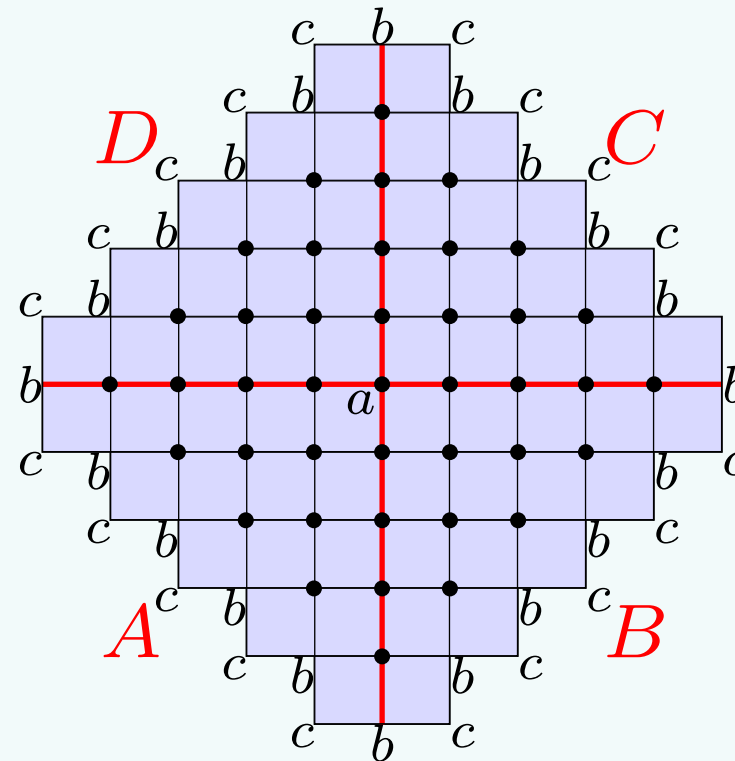


Exact Calculation of Entanglement Entropies

São Carlos, May 2018

Paul A. Pearce
University of Melbourne



- I.Peschel, *Special review: Entanglement in solvable many-particle models*, Brazilian J. Phys. 42, 267 (2012).
- R.J.Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic Press (1982).
- R.Weston, *The entanglement entropy of solvable lattice models*, J. Stat. Mech. L03002 (2006).
- E.Ercolessi, S.Evangelisti, F.Ravanini, *Exact entanglement entropy of the XYZ model and its sine-Gordon limit*, Phys. Lett. A374 2101 (2010).
- A.DeLuca, F.Franchini, *Approaching the RSOS critical points through entanglement: One model for many universalities*, Phys. Rev. B87, 045118 (2013).
- D.Bianchini, F.Ravanini, *Entanglement entropy from CTM in Forrester-Baxter nonunitary RSOS models*, J. Phys. A49, 154005 (2016).
- R.Couvreur, J.L.Jacobsen, H.Saleur, *Entanglement in nonunitary quantum critical spin chains*, Phys. Rev. Lett. 119, 040601 (2017).
- T.Dupic, B.Estienne, Y.Ikhlef, *Entanglement entropies of minimal models from null-vectors*, arXiv:1709.09270.

Outline

Section 1: Fundamental Concepts

Section 2: Eight-Vertex/XYZ Chain

Section 3: RSOS Models

Section 4: Nonunitary Models

- These lectures were presented in November 2017 at the Mini-School on Entanglement and Quantum Phase Transitions at the Asia Pacific Center for Theoretical Physics (APCTP) as part of the activities of the ICTP Asian Network on Condensed Matter Physics and Complex Systems.
- The idea was for me to learn about the interplay between Yang-Baxter integrable systems and quantum entanglement.
- The exact calculation of various Rényi entanglement entropies of the minimal models is an ongoing research project with Francisco Alcaraz.

Quotes/Abstract

1919 Niels Bohr: “If [quantum mechanics](#) hasn't profoundly shocked you, you haven't understood it yet.”

1947 Albert Einstein: Quantum mechanics “cannot be reconciled with the idea that physics should represent reality in time and space, free from [spooky action at a distance](#)”.

1985 John Bell: “There is a way to escape the inference of superluminal speeds and spooky action at a distance. But it involves absolute determinism in the universe, the [complete absence of free will](#).”

1989 David Mermin (often attributed to Richard Feynman): “[Shut up and calculate!](#)”

Abstract

One-dimensional Hamiltonians, such as the XYZ quantum spin chain, exhibit quantum phase transitions. The universal behaviours at such critical points are described, in the continuum scaling limit, by Conformal Field Theories (CFTs). In the first instance, CFTs are characterized by a central charge c . Entanglement entropy provides a convenient means to determine the central charge either numerically or analytically through entanglement “[Area Laws](#)”. In these lectures, we review the application of Yang-Baxter methods and Corner Transfer Matrices to obtain the exact entanglement entropy and central charge for the unitary XYZ quantum spin chain and its specializations. The extension of these methods to other unitary and nonunitary models will also be discussed. In particular, we consider the minimal models $\mathcal{M}(m, m')$ as the continuum scaling limit of the RSOS(m, m') lattice models.

Section 1: Fundamental Concepts

1. Scaling and Conformal Field Theory (CFT)
2. Central charge, effective central charge and conformal weights
3. von Neumann and Rényi entanglement entropies, Schmidt decomposition

John von Neumann (1903–1957)

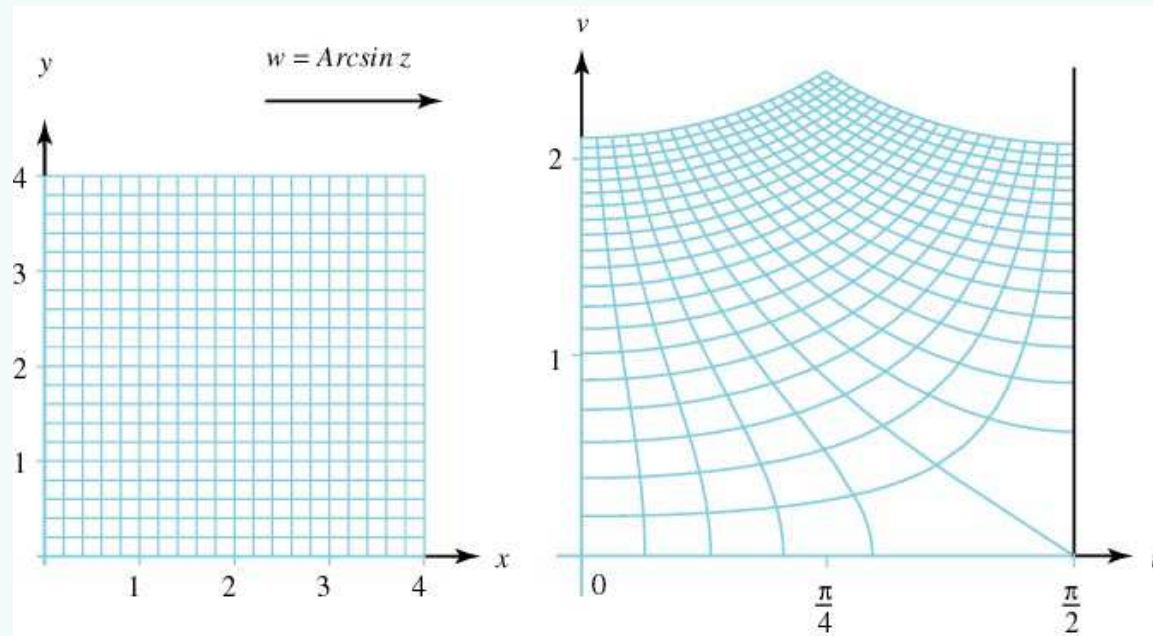


Alfréd Rényi (1921–1970)



- von Neumann: “There’s no sense in being precise when you don’t even know what you’re talking about”.
- Rényi (often attributed to Erdős): “A mathematician is a device for turning coffee into theorems”.

Scaling and CFT



- **Classical versus Quantum Models:** Quantum models in d -dimensions are related to classical models in $(d + 1)$ -dimensions. In this way, properties of 1-dimensional quantum Hamiltonians can be obtained by studying 2-dimensional classical systems (say via transfer matrices).
- **Scaling:** Critical systems (those with correlation length $\xi = \infty$) exhibit translational, rotational and scale invariance. Remarkably, they are also invariant under **local scale transformations** (preserving local angles). In 2-dimensions, these infinitesimal **conformal transformations** form an infinite-dimensional group corresponding to the group of analytic mappings of the plane.
- **Virasoro Algebra:** The Lie algebra (describing the infinitesimal conformal transformations) of the conformal symmetry group is the **Virasoro algebra**

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \quad m, n \in \mathbb{Z}, \quad c \in \mathbb{R}$$

- **Central Charge:** The scalar **central charge** c is a number characterizing the CFT.

Central Charge and Finite-Size Behaviour

● At criticality, the continuum scaling limit ($N \rightarrow \infty$, $a \rightarrow 0$, $Na = x \in \mathbb{R}$) of a 2-d lattice model (and its associated quantum Hamiltonian) is described by a CFT characterized by a set of conformal data:

- The central charge c .
- The conformal weights Δ .
- Conformal characters and partitions functions.

● Traditionally (Blöte, Cardy, Nightingale 1986), the conformal data is calculated from the universal finite-size corrections to the eigenvalues of the periodic transfer matrix $\mathbf{T}(u)$ on the cylinder and double row transfer matrix $\mathbf{D}(u)$ on the strip. The spectral parameter u is essentially the spatial anisotropy.

● The universal finite-size behaviour of eigenvalues for large system size N is

$$\begin{aligned} \log T(u) &\sim -N f_{\text{bulk}}(u) - \frac{2\pi}{N} \left[\left(-\frac{c}{12} + \Delta + \bar{\Delta} + k + \bar{k}\right) \sin \vartheta + \left(-\frac{c}{12} + \Delta - \bar{\Delta} + k - \bar{k}\right) i \cos \vartheta \right] \\ \frac{1}{2} \log D(u) &\sim -N f_{\text{bulk}}(u) - f_{\text{bdy}}(u) - \frac{\pi}{N} \left[\left(-\frac{c}{24} + \Delta + k\right) \sin \vartheta \right] \end{aligned}$$

where $\vartheta = \frac{\pi u}{\lambda}$ and k, \bar{k} are non-negative integer levels. Similar formulas apply to the eigenenergies of the associated 1-d quantum chains, e.g. $\mathcal{H} = \frac{d}{du} \log \mathbf{T}(u) \Big|_{u=0}$.

● For the 2-d critical Ising model $\mathcal{M}(3,4)$ and the related 1-d quantum transverse Ising model, $c = \frac{1}{2}$ and $\Delta, \bar{\Delta} \in \{0, \frac{1}{16}, \frac{1}{2}\} = \{\Delta_{r,s}\}$ where r, s are Kac labels. For the [Minimal Models](#) $\mathcal{M}(m, m')$

$$c = 1 - \frac{6(m' - m)^2}{mm'}, \quad \Delta_{r,s} = \frac{(m'r - ms)^2 - (m' - m)^2}{4mm'}, \quad r = 1, 2, \dots, m-1, \quad s = 1, 2, \dots, m'-1$$

Unitary vs Nonunitary

- A CFT is **unitary** if $L_n^\dagger = L_{-n}$ with respect to a positive definite inner product on states. Theories with Hermitian Hamiltonians/transfer matrices are unitary.
- For unitary theories, $\Delta, \bar{\Delta} \geq 0$ and $\Delta = \bar{\Delta} = 0$ for the ground state (vacuum $|0\rangle$). A theory with $\Delta_{\min} < 0$ must be nonunitary. This can happen for theories with real spectra even though the Hamiltonian/transfer matrices are not Hermitian (**CPT/non-Hermitian quantum mechanics**).
- Some examples of nonunitary theories include the Yang-Lee theory $\mathcal{M}(2,5)$, critical dense polymers and critical percolation. In such cases, the **effective central charge** is defined as

$$-\frac{c}{24} + \Delta_{\min} = -\frac{c_{\text{eff}}}{24}, \quad c_{\text{eff}} = c - 24\Delta_{\min}$$

Effectively, the central charge and conformal weights are “shifted” so that $\tilde{\Delta} = \Delta - \Delta_{\min} \geq 0$.

- Note that c always occurs in the combination $-\frac{c}{24} + \Delta$ in the universal finite-size corrections. So such calculations cannot distinguish between c and c_{eff} .
- As we will see, entanglement entropy provides an alternative means to analytically calculate the central charge c .
- It is an interesting question as to whether entanglement entropies can distinguish between the central charge c and the effective central charge c_{eff} for nonunitary theories. The current status is that, for nonunitary theories, one obtains only c_{eff} .

Reduced Density Matrices

- Consider a quantum system with a Hermitian Hamiltonian so that left- and right-eigenvectors coincide and the associated CFT is unitary. A **mixed state** is represented (via spectral decomposition) as a **density matrix**

$$\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|, \quad p_j \geq 0, \quad \sum_j p_j = 1, \quad \text{Tr}\rho = 1, \quad p_j = \text{probability of state } j$$

The one-dimensional projectors have unit trace

$$\text{Tr} |\psi_j\rangle\langle\psi_j| = 1$$

and the eigenstates $|\psi_j\rangle$ are orthonormal. For a **pure state** $\rho = |\psi\rangle\langle\psi|$ and $p_j = \delta_{j,1}$.

- If A, B are two complementary subsystems of $A \cup B$, a **bipartition** of a 1-d quantum system acting on a Hilbert space H is given by

$$H = H_A \otimes H_B, \quad H = \otimes^N \mathbb{C}^2 \text{ for spin systems}$$

- Let $|j\rangle_A, |j\rangle_B$ be orthonormal bases for H_A, H_B respectively. Then, for the normalized ground state $|\psi\rangle \in H_A \otimes H_B$, the **reduced density matrix** is given by the partial trace

$$\rho_A = \text{Tr}_B \rho = \sum_j \langle j|_B (|\psi\rangle\langle\psi|) |j\rangle_B, \quad \rho = |\psi\rangle\langle\psi| = \text{projector onto state } \psi$$

where we trace out on system B . Clearly, for a normalized state, $\text{Tr}_A \rho_A = \text{Tr}\rho = 1$.

- For example, the reduced density matrix for the **entangled state**

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B)$$

is

$$\rho_A = \frac{1}{2} \text{Tr}_B (|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B)(\langle 0|_A \otimes \langle 1|_B - \langle 1|_A \otimes \langle 0|_B) = \frac{1}{2}(|0\rangle_A \langle 0|_A + |1\rangle_A \langle 1|_A)$$

Von Neumann/Rényi Entanglement Entropies

- The von Neumann entanglement entropy is defined by

$$S(\rho_A) = -\text{Tr}_A[\rho_A \log \rho_A], \quad \Rightarrow \quad S(\rho_B) = -\text{Tr}_B[\rho_B \log \rho_B] = S(\rho_A) \quad (\text{Schmidt basis})$$

A pure state has entropy $S = -\lambda \log \lambda|_{\lambda=1} = 0$ and is not entangled. Often \log_2 is used.

- The Rényi Entanglement Entropy is a generalization of the von Neumann entropy given “by taking α independent replicas” (the von Neumann limit follows from l’Hôpital’s rule)

$$S_\alpha(\rho_A) = \frac{1}{1-\alpha} \log \text{Tr}_A(\rho_A)^\alpha, \quad \alpha \geq 0, \alpha \neq 1, \quad S(\rho_A) = \lim_{\alpha \rightarrow 1} S_\alpha(\rho_A) = -\text{Tr}_A[\rho_A \log \rho_A]$$

- Consider a 1-d critical quantum system and let A be an interval of length ℓ and B the rest of the infinite real line. For large ℓ , the von Neumann entropy of the ground state $|0\rangle$ behaves as

$$S_A \sim \frac{c}{3} \log \frac{\ell}{a} + C, \quad \ell \rightarrow \infty$$

- Similarly, if L, R are the left and right-halves of the real line for an off-critical system near criticality, the von Neumann and Rényi entropies of the ground state $|0\rangle$ behave as

$$S(\rho_L) \sim \frac{c}{6} \log \frac{\xi}{a} + C, \quad S_\alpha(\rho_L) \sim \frac{c}{12} \left(\frac{1+\alpha}{\alpha} \right) \log \frac{\xi}{a} + C, \quad \xi \rightarrow \infty$$

- In these “Area Laws”, c is the central charge, ξ is the correlation length, a is the lattice spacing and C is a non-universal constant. They are called “Area Laws” since, for gapped systems in 3-d, S is expected to be proportional to the *area* separating A and B . These (1+1)-d CFT formulas for unitary systems are due to Holzhey, Larsen, Wilczek 1994 and Calabrese, Cardy 2004.

Schmidt Decomposition

- Suppose a bipartite system AB is in the (normalized) state $|\psi\rangle$. Then there exist orthonormal states $|j\rangle_A$ of A and $|j\rangle_B$ of B such that $|\psi\rangle$ is given by the [Schmidt decomposition](#)

$$|\psi\rangle = \sum_j \lambda_j |j\rangle_A |j\rangle_B, \quad \lambda_j \in [0, 1], \quad \sum_j \lambda_j^2 = 1$$

The proof uses the Singular Value Decomposition (SVD) of matrices (see Wikipedia). The SVD is a generalization of the spectral decomposition theorem to non-diagonalizable matrices.

- The number of strictly positive $\lambda_j > 0$ gives the [Schmidt rank](#). A state is entangled if the Schmidt rank is greater than one.
- In the Schmidt basis, the spectral decomposition of the reduced density matrices is

$$\rho_A = \sum_j \lambda_j^2 |j\rangle_A \langle j|_A, \quad \rho_B = \sum_j \lambda_j^2 |j\rangle_B \langle j|_B$$

The reduced density matrices have common nonzero eigenvalues λ_j^2 so they have the same von Neumann entropy

$$S_A = S_B = - \sum_j \lambda_j^2 \log \lambda_j^2$$

- The set of Schmidt eigenvalues $\{-\log \lambda_j^2\}$ is referred to as the “[entanglement spectrum](#)”. The entanglement spectrum provides more information than the entanglement entropy.

Section 2: Eight-Vertex Model/XYZ Chain

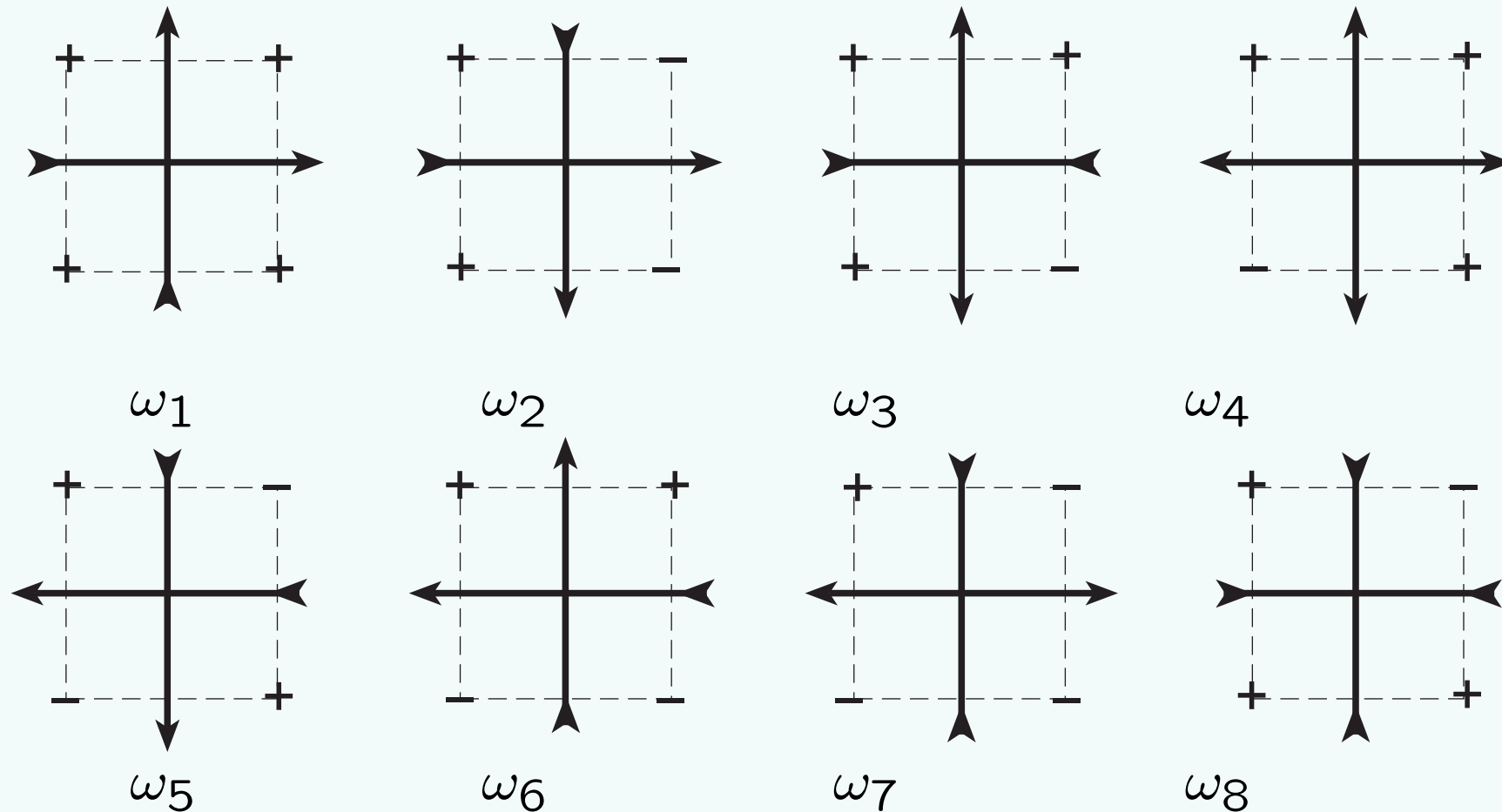
1. Definition, parametrization, integrability, XYZ chain
2. Corner Transfer Matrices (CTMs)
3. Eight-vertex entanglement entropy

Rodney James Baxter (1940–)



- R.Weston, *The entanglement entropy of solvable lattice models*, J. Stat. Mech. L03002 (2006).
- E.Ercolessi, S.Evangelisti, F.Ravanini, *Exact entanglement entropy of the XYZ model and its sine-Gordon limit*, Phys. Lett. A374 2101 (2010).

Eight-Vertex Model



- The eight distinct vertex configurations of the eight-vertex model showing one of the two corresponding configurations of the related Ising model. The spins and arrows are related by

$$\alpha = ab, \quad \beta = bc, \quad \gamma = cd, \quad \delta = da$$

- The model is Yang-Baxter integrable in the symmetric case, $\omega_1 = \omega_5$, $\omega_2 = \omega_6$, $\omega_3 = \omega_7$, $\omega_4 = \omega_8$, when the Boltzmann weights are equal in pairs under arrow reversal.
- The partition function is

$$Z = \sum_{\text{arrow states}} \prod_{\text{faces}} W \begin{pmatrix} \delta & \gamma \\ \alpha & \beta \end{pmatrix}, \quad \alpha, \beta, \gamma, \delta = \pm 1$$

Eight-Vertex Ising Model

- The eight vertex model on the square lattice can be formulated as an Ising model with spins at the corners of the elementary faces and Boltzmann face weights

$$W \begin{pmatrix} d & c \\ a & b \end{pmatrix} = R \exp(Kac + Lbd + Mabcd) = \begin{array}{|c|c|} \hline d & c \\ \hline a & b \\ \hline \end{array} \quad a, b, c, d = \pm 1$$

- The four independent vertex weights are related to R, K, L, M by

$$\begin{aligned} \omega_1 = \omega_5 &= Re^{K+L+M}, & \omega_2 = \omega_6 &= Re^{-K-L+M} \\ \omega_3 = \omega_7 &= Re^{K-L-M}, & \omega_4 = \omega_8 &= Re^{-K+L-M} \end{aligned}$$

- This is not the usual rectangular Ising model since it involves four-spin interactions in addition to two-spin interactions. The case $M = 0$ corresponds to two decoupled Ising models. This mapping is one-to-two since we can arbitrarily fix one spin somewhere on the lattice.

It follows that

$$Z_{\text{Ising}}^2 = 2Z_{\text{Vertex}} \Big|_{M=0}$$

- In general, the eight-vertex model is not critical ($\xi < \infty$). The critical case corresponds to the six-vertex model ($\omega_4 = \omega_8 = 0$). On the critical manifold, the correlation length ξ diverges ($\xi = \infty$).
- The six-vertex model was studied extensively by Elliott Lieb and others using the Bethe ansatz.

Eight-Vertex Parametrization

- The face weights of the (symmetric) eight-vertex model can be parametrized in terms of elliptic functions

$$\omega_1 = R \operatorname{snh} \lambda, \quad \omega_2 = R k \operatorname{snh} \lambda \operatorname{snh} u \operatorname{snh}(\lambda - u), \quad \omega_3 = R \operatorname{snh}(\lambda - u), \quad \omega_4 = R \operatorname{snh} u$$

- Here $k = k(q)$ is the elliptic modulus, q is the elliptic nome and

$$\begin{aligned} \operatorname{snh} u &= \operatorname{snh}(u, k) = -i \operatorname{sn}(iu, k) \\ \operatorname{sn}(u, k) &= 2q^{1/4} k^{-1/2} \sin \frac{\pi u}{2I} \prod_{n=1}^{\infty} \frac{1 - 2q^{2n} \cos \frac{\pi u}{I} + q^{4n}}{1 - 2q^{2n-1} \cos \frac{\pi u}{I} + q^{4n-2}} \\ k(q) &= 4\sqrt{q} \prod_{n=1}^{\infty} \left(\frac{1 + q^{2n}}{1 + q^{2n-1}} \right)^4, \quad I(q) = \frac{\pi}{2} \prod_{n=1}^{\infty} \left(\frac{1 + q^{2n-1}}{1 - q^{2n-1}} \frac{1 - q^{2n}}{1 + q^{2n}} \right)^2 \end{aligned}$$

- The four face weights depend on the 4 variables

R = overall constant normalization = $R' / \operatorname{snh} \lambda$
 u = the spectral parameter ($0 < u < \lambda$)
 λ = the crossing parameter = constant
 q = the departure-from-criticality variable = constant ($0 < q < 1$)

- The critical six-vertex manifold corresponds to setting $q = k = 0$. In this case, the (ratios of) elliptic functions reduce to trigonometric functions

$$\operatorname{snh}(u, 0) \mapsto \sin iu$$

Yang-Baxter Integrability

- In the basis $\{+, +\}, \{+, -\}, \{-, +\}, \{-, -\}$, the quantum \check{R} -matrix (acting from $\{\gamma, \beta\}$ to $\{\delta, \alpha\}$) adding a single face to the lattice is:

$$W\left(\begin{array}{c|c} \delta & \gamma \\ \hline \alpha & \beta \end{array}\right) : \check{R}(u) = R \left(\begin{array}{cccc} \sinh(\lambda - u) & 0 & 0 & k \sinh \lambda \sinh u \sinh(\lambda - u) \\ 0 & \sinh \lambda & \sinh u & 0 \\ 0 & \sinh u & \sinh \lambda & 0 \\ k \sinh \lambda \sinh u \sinh(\lambda - u) & 0 & 0 & \sinh(\lambda - u) \end{array} \right)$$

This action extends to a length N chain of spins by taking tensor products. For example,

$$\check{R}_2(u) = I \otimes \check{R}(u) \otimes I \otimes \dots \otimes I$$

- The \check{R} -matrix satisfies an **inversion relation** and the **Yang-Baxter Equation (YBE)**

$$\begin{aligned} \check{R}_j(u) \check{R}_j(-u) &= R^2 \sinh(\lambda - u) \sinh(\lambda + u) I \\ \check{R}_j(u) \check{R}_{j+1}(u+v) \check{R}_j(v) &= \check{R}_{j+1}(v) \check{R}_j(u+v) \check{R}_{j+1}(u) \end{aligned}$$

- The YBE implies **commutation** of the row transfer matrices

$$\langle \alpha | \mathbf{T}(u) | \gamma \rangle = \sum_{\beta_1, \beta_2, \dots, \beta_N = \pm 1} \prod_{j=1}^N W\left(\begin{array}{c|c} \gamma_j & \beta_{j+1} \\ \hline \beta_j & \alpha_j \end{array} \middle| u \right) = \beta_1 \begin{array}{ccccccc} \gamma_1 & \gamma_2 & \gamma_3 & \dots & \dots & \dots & \gamma_1 \\ \hline \alpha_1 & \alpha_2 & \alpha_3 & \dots & \dots & \dots & \alpha_1 \end{array} \beta_1$$

$$[\mathbf{T}(u), \mathbf{T}(v)] = 0$$

Commuting Periodic Row Transfer Matrices

$$\text{YBE} + \text{Inversion} \Rightarrow [\mathbf{T}(u), \mathbf{T}(v)] = 0 \Rightarrow \text{Integrable}$$

$$\begin{aligned}
 \mathbf{T}(u)\mathbf{T}(v) &= \begin{array}{c} \begin{array}{|c|c|c|c|c|} \hline v & v & v & v & v \\ \hline \end{array} & \begin{array}{|c|c|} \hline v-u & u-v \\ \hline \end{array} \\ \hline \begin{array}{|c|c|c|c|c|} \hline u & u & u & u & u \\ \hline \end{array} & \begin{array}{|c|c|} \hline v-u & u-v \\ \hline \end{array} \\ \hline \end{array} \\
 &= \begin{array}{c} \begin{array}{|c|} \hline v-u \\ \hline \end{array} & \begin{array}{|c|c|c|c|c|} \hline u & u & u & u & u \\ \hline \end{array} & \begin{array}{|c|} \hline u-v \\ \hline \end{array} \\ \hline \begin{array}{|c|c|c|c|c|} \hline v & v & v & v & v \\ \hline \end{array} & \begin{array}{|c|c|} \hline v-u & u-v \\ \hline \end{array} \\ \hline \end{array} \\
 &= \begin{array}{c} \begin{array}{|c|c|c|c|c|} \hline u & u & u & u & u \\ \hline \end{array} & \begin{array}{|c|c|} \hline u-v & v-u \\ \hline \end{array} \\ \hline \begin{array}{|c|c|c|c|c|} \hline v & v & v & v & v \\ \hline \end{array} & \begin{array}{|c|c|} \hline u-v & v-u \\ \hline \end{array} \\ \hline \end{array} \\
 &= \mathbf{T}(v)\mathbf{T}(u)
 \end{aligned}$$

- Commuting (normal) transfer matrices have a common set of u -independent eigenvectors and are therefore simultaneously diagonalizable. The eigenvalue spectra can be found by solving certain [functional equations](#) in the form of Baxter's T - Q equation (Bethe Ansatz).

XYZ Chain

- We regard the *crossing parameter* λ as constant, u as a variable and write the transfer matrix as $\mathbf{T}(u)$. Since $[\mathbf{T}(u), \mathbf{T}(v)] = 0$, $\mathbf{T}(u)$ is a one-parameter family of commuting transfer matrices.
- The integrable **XYZ quantum spin chain** belongs to the commuting family $\mathbf{T}(u)$. Explicitly, the logarithmic derivative of the eight vertex transfer matrix yields a **Hermitian** Hamiltonian

$$\left. \frac{d}{du} \log \mathbf{T}(u) \right|_{u=0} = \mathcal{H}_{\text{XYZ}} = - \sum_{j=1}^N (J_x \sigma_j^x \sigma_{j+1}^x + J_y \sigma_j^y \sigma_{j+1}^y + J_z \sigma_j^z \sigma_{j+1}^z), \quad [\mathbf{T}(u), \mathcal{H}_{\text{XYZ}}] = 0$$

where $\sigma_j^x, \sigma_j^y, \sigma_j^z$ are Pauli matrices acting on site j and N is the length of the chain. The constants J_x, J_y and J_z allow for anisotropic interactions. If $J_x = J_y$, that is $k = q = 0$, the resulting XXZ model is critical. When $J_x = J_y = J_z = J$, the model reduces to the **Heisenberg spin chain**

$$\mathcal{H}_{\text{XXX}} = -J \sum_{j=1}^N \boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_{j+1}$$

- Without loss of generality, we fix the ratios as

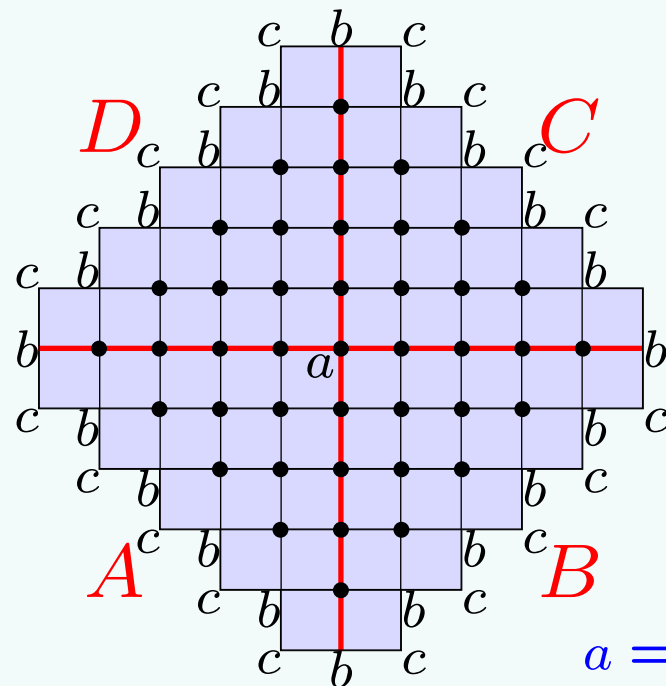
$$J_x : J_y : J_z = 1 : \Gamma : \Delta$$

In the ferromagnetic regime ($\Delta > 1$), the elliptic parametrization is

$$\Gamma = \frac{1 + k \operatorname{sn}^2(i\lambda)}{1 - k \operatorname{sn}^2(i\lambda)}, \quad \Delta = -\frac{\operatorname{cn}(i\lambda) \operatorname{dn}(i\lambda)}{1 - k \operatorname{sn}^2(i\lambda)}, \quad 0 < k < 1, \quad 0 < \lambda < I(k')$$

where $\operatorname{sn}(z) = \operatorname{sn}(z, k)$, $\operatorname{cn}(z) = \operatorname{cn}(z, k)$, $\operatorname{dn}(z) = \operatorname{dn}(z, k)$ are standard Jacobi elliptic functions. $I(k')$ is the complete elliptic integral of the first kind of conjugate modulus $k' = \sqrt{1 - k^2}$.

Corner Transfer Matrices



- The one-point functions of Yang-Baxter integrable lattice models are obtained using Baxter's Corner Transfer Matrices (CTMs). The idea is to build the square lattice quadrant-by-quadrant. The partition function and one-point function of the eight-vertex Ising model are

$$Z = \text{Tr } ABCD, \quad \langle \sigma_1 \rangle = \frac{\text{Tr } SABCD}{\text{Tr } ABCD}, \quad A(u) = C(u) = B(\lambda - u) = D(\lambda - u)$$

- Baxter showed the CTMs commute for $N \rightarrow \infty$ with eigenvalues that are simple exponentials! The diagonalized CTM Hamiltonian is

$$A_d(u) = \text{Tr } z^{2\mathcal{H}_{\text{CTM}}}, \quad \mathcal{H}_{\text{CTM}} = \frac{1}{2} \sum_{j \geq 1} j(1 - \sigma_j^z \sigma_{j+2}^z), \quad z = \exp(-\pi u / 2I(k))$$

Fixing $\mu_j = \sigma_j \sigma_{j+2}$ and the half-rows $\mu = \{\mu_1, \mu_2, \mu_3, \dots\}$ gives $S_{\mu, \mu} = \sigma_1 = \mu_1 \mu_3 \mu_5 \dots$ and

$$S_d = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \dots, \quad [A_d(u)]_{\mu, \mu} = \exp \left[-\frac{\pi u}{2I} \sum_{j \geq 1} j(1 - \mu_j) \right]$$

Eight-Vertex Magnetization/Polarization

- The diagonalized matrices are direct products of 2×2 matrices

$$\begin{aligned}
 \mathbf{S}_d &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \dots \\
 \mathbf{A}_d(u) &= \mathbf{C}_d(u) = \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & s^2 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & s^3 \end{pmatrix} \otimes \dots \\
 \mathbf{B}_d(u) &= \mathbf{D}_d(u) = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & t^2 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & t^3 \end{pmatrix} \otimes \dots \\
 s &= \exp[-\pi u/2I(k)], \quad t = \exp[-\pi(\lambda - u)/2I(k)]
 \end{aligned}$$

- It follows that the *magnetization* is

$$\langle \sigma_1 \rangle = \prod_{n=1}^{\infty} \frac{1 - x^{2n-1}}{1 + x^{2n+1}} = (k')^{1/4} = (1 - k^2)^{1/8}, \quad x = (st)^2 = e^{-\epsilon}, \quad \epsilon = \pi\lambda/I(k)$$

where $k' = k'(x)$ is the *conjugate* elliptic modulus of the (low-temperature) nome x . The terms with even powers of x cancel in the numerator and denominator.

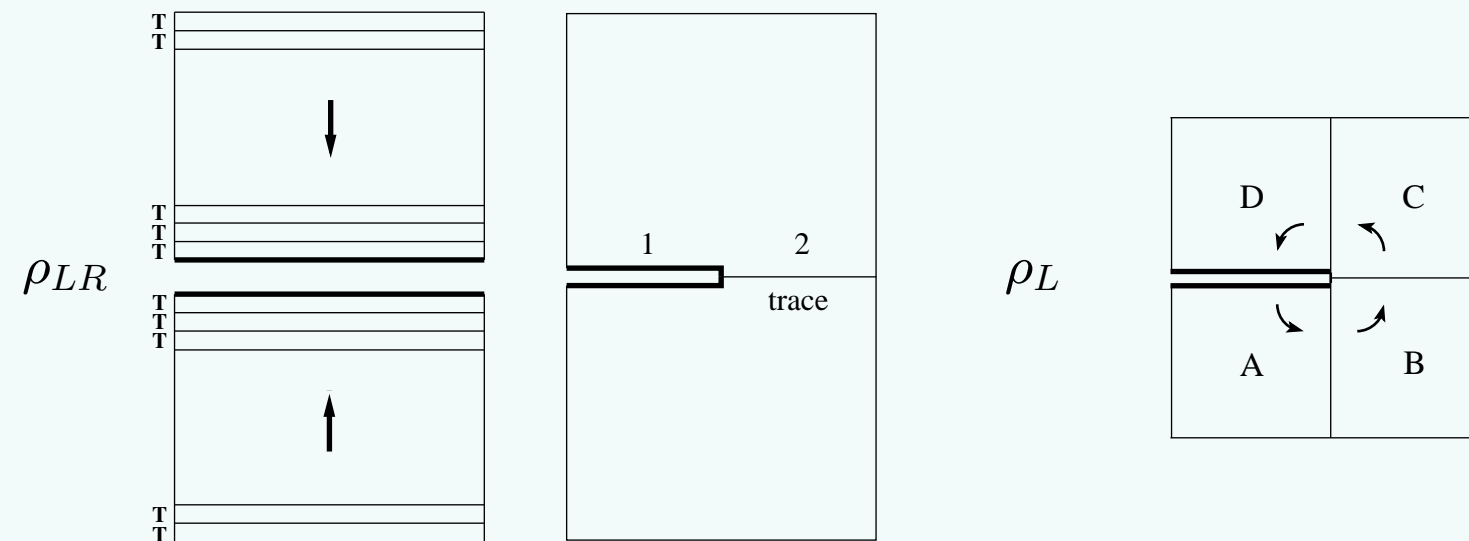
- The *polarization* of the eight-vertex model is

$$\langle \alpha \rangle = \langle \sigma_1 \sigma_2 \rangle = \prod_{n=1}^{\infty} \left(\frac{1 - x^n}{1 + x^n} \frac{1 + q^n}{1 - q^n} \right)^2$$

This cannot be obtained by a direct application of CTMs but was conjectured by [Baxter and Kelland](#) and subsequently derived by [Jimbo, Miwa and Nakayashiki](#) using difference equations.

- Note the notation here is $x = x_{\text{Baxter}}^2$ of Baxter's book.

Eight-Vertex Reduced Density Matrix



- Decompose the Hilbert space $H = H_L \otimes H_R$ where L, R refer to the semi-infinite spin chains on the left and the right. The reduced density matrix of the ground state is

$$\rho_L = \text{Tr}_R \rho_{LR}$$

- Now use CTMs with fixed boundary conditions. Since a trace is invariant under a change of basis/diagonalization

$$[\rho_L]_{\sigma, \sigma'} = Z^{-1} (ABCD)_{\sigma, \sigma'} = Z^{-1} (AB)_{\sigma, \sigma'}^2$$

$$(\rho_L)_d = Z^{-1} \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & x^2 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & x^3 \end{pmatrix} \otimes \dots \dots, \quad x = (st)^2 = e^{-\epsilon}, \quad \epsilon = \pi\lambda/I(k)$$

- The normalizing partition function is

$$Z = Z(\epsilon) = \text{Tr}_L ABCD = \text{Tr}_L x^{\mathcal{H}_{\text{CTM}}} = \prod_{j=1}^{\infty} (1 + x^j) = \prod_{j=1}^{\infty} (1 + e^{-j\epsilon})$$

Eight-Vertex Entanglement

- Observe that ρ_L is a function of λ and k only. Moreover, ρ_L can be written as

$$\rho_L = Z^{-1}(\mathbf{AB})^2 = Z^{-1}e^{-\epsilon\mathcal{N}}, \quad Z = \text{Tr} e^{-\epsilon\mathcal{N}}, \quad \epsilon = \pi\lambda/I(k)$$

where \mathcal{N} is a diagonal operator with integer eigenvalues.

- The von Neumann entropy S is given by

$$S = -\text{Tr}_L \rho_L \log \rho_L = \epsilon Z^{-1} \text{Tr} \mathcal{N} e^{-\epsilon\mathcal{N}} + \log Z = -\epsilon \frac{\partial \log Z}{\partial \epsilon} + \log Z$$

We thus obtain an exact analytic expression for the entanglement entropy of the XYZ model valid for general λ and k

$$S_{\text{XYZ}} = \sum_{j=1}^{\infty} \frac{j\epsilon}{(1 + e^{j\epsilon})} + \sum_{j=1}^{\infty} \log(1 + e^{-j\epsilon})$$

- Using Euler-Maclaurin in the scaling limit $\epsilon \rightarrow 0$, this is approximated by

$$S_{\text{XYZ}} \sim \frac{1}{\epsilon} \int_0^{\infty} \left[\frac{y}{1 + e^y} + \log(1 + e^{-y}) \right] dy + C \sim \frac{\pi^2}{6\epsilon} + C, \quad \epsilon \rightarrow 0$$

After setting $a = e^{-y}$ in the integral with $x = 1$, it becomes

$$\int_0^x \left(\frac{\log(1+a)}{a} - \frac{\log a}{1+a} \right) da = 2L_+(x) = 2L\left(\frac{x}{1+x}\right), \quad 2L\left(\frac{x}{1+x}\right) \Big|_{x=1} = 2L\left(\frac{1}{2}\right) = \frac{\pi^2}{6}$$

where $L(x)$ is the [Rogers dilogarithm](#).

Eight-Vertex Entanglement Asymptotics

- The eight-vertex correlation length was calculated by [Johnson, Krinsky, McCoy 1973](#)

$$\xi^{-1} = -\frac{1}{2} \log k'(x), \quad x = e^{-\epsilon} \rightarrow 0 \quad \text{as} \quad \xi \rightarrow \infty$$

where the elliptic moduli are

$$k(q) = 4\sqrt{q} \prod_{n=1}^{\infty} \left(\frac{1 + q^{2n}}{1 + q^{2n-1}} \right)^4, \quad k'(q) = \prod_{n=1}^{\infty} \left(\frac{1 - q^{2n-1}}{1 + q^{2n-1}} \right)^4, \quad k^2 + (k')^2 = 1$$

- To obtain the limiting behaviour as $\epsilon \rightarrow 0$, we perform a [conjugate modulus transformation](#) using the fact that $k' = k(x')$

$$\xi^{-1} = -\frac{1}{4} \log[1 - k(x')^2] \sim \frac{1}{4} k(x')^2 \sim 4x', \quad x' = \exp(-\frac{\pi^2}{\epsilon}) \rightarrow 0 \quad \text{as} \quad \xi \rightarrow \infty$$

to find

$$\log \xi \sim \frac{\pi^2}{\epsilon} + C + \dots, \quad \epsilon \rightarrow 0$$

- Combining this with the previous expression for S_{XYZ} gives the [von Neumann entropy](#)

$$S_{XYZ} \sim \frac{c}{6} \log \frac{\xi}{a} + C + \dots, \quad \xi \rightarrow \infty, \quad c = 1$$

- On specializing, this result applies to the XXX, XXZ models and free fermion (Ising) models. For the Ising case ($M = 0$ at the decoupling point), $c = \frac{1}{2}$ since there are two independent Ising models on the two sublattices ($c = 1 = \frac{1}{2} + \frac{1}{2}$).

Section 3: RSOS Models

1. Definition of $RSOS(m, m')$ models
2. Unitary reduced density matrices and Rényi entropies
3. Correlation length and entropy asymptotics

George Andrews



Rodney J. Baxter



Peter J. Forrester



- A.DeLuca, F.Franchini, *Approaching the RSOS critical points through entanglement: One model for many universalities*, Phys. Rev. B87, 045118 (2013).

Minimal Models $RSOS(m, m') / \mathcal{M}(m, m')$

- The Restricted Solid-on-Solid (RSOS) lattice models ([Andrews, Baxter, Forrester 1984/1985](#)) are defined on a square lattice with heights $a = 1, 2, \dots, m' - 1$. Nearest neighbour heights differ by ± 1 so the heights live on the $A_{m'-1}$ Dynkin diagram. The nonzero Boltzmann weights are

$$\begin{aligned}
 W\left(\begin{array}{cc|c} a \pm 1 & a & u \\ a & a \mp 1 & \end{array}\right) &= \frac{\vartheta_1(\lambda - u)}{\vartheta_1(\lambda)} \\
 W\left(\begin{array}{cc|c} a & a \pm 1 & u \\ a \mp 1 & a & \end{array}\right) &= \frac{g_{a \mp 1} \vartheta_1((a \pm 1)\lambda) \vartheta_1(u)}{g_{a \pm 1} \vartheta_1(a\lambda) \vartheta_1(\lambda)} \\
 W\left(\begin{array}{cc|c} a & a \pm 1 & u \\ a \pm 1 & a & \end{array}\right) &= \frac{\vartheta_1(a\lambda \pm u)}{\vartheta_1(a\lambda)}
 \end{aligned}$$

- Here $\vartheta_1(u) = \vartheta_1(u, p)$ is a standard elliptic theta function

$$\vartheta_1(u, p) = 2p^{1/4} \sin u \prod_{n=1}^{\infty} (1 - 2p^{2n} \cos 2u + p^{4n})(1 - p^{2n}), \quad 0 < p < 1$$

u is the spectral parameter and the elliptic nome $p = e^{-\epsilon}$ is a temperature-like variable, with p^2 measuring the departure from criticality. The inessential gauge factors g_a can be set to 1.

The crossing parameter λ is

$$\lambda = \frac{(m' - m)\pi}{m'}, \quad 2 \leq m < m', \quad m, m' \text{ coprime}$$

- Integrability follows since the local face weights satisfy the Yang-Baxter equation. The off-critical quantum Hamiltonians are given in [BEPR2015](#). The continuum scaling limit is associated with the minimal model $\mathcal{M}(m, m')$. [The unitary cases are given by \$m' = m + 1\$.](#)

Unitary RSOS($m, m+1$) Reduced Density Matrix

- Using commuting Corner Transfer Matrices, the partition function and Local Height Probabilities (LHPs) of the RSOS models are

$$Z = \text{Tr } ABCD, \quad P_a = \langle \delta_{l_1, a} \rangle = \frac{\text{Tr } S_a ABCD}{\text{Tr } ABCD}, \quad A(u) = C(u) = B(\lambda - u) = D(\lambda - u)$$

- It follows that

$$\rho_L = \text{Tr}_R \rho_{LR}, \quad [\rho_L]_{l, l'} = Z^{-1} (ABCD)_{l, l'} = Z^{-1} (AB)_{l, l'}^2$$

where $l = \{l_1, l_2, \dots, l_{N+1}, l_{N+2}\}$ and $l_1 = a$. Fixing $l_{N+1} = b$, $l_{N+2} = c = b \pm 1$ determines the $2(m-1)$ ground state boundary conditions in which the bulk heights alternate between b and c .

- After diagonalization, the normalizing partition function is

$$Z = \text{Tr } ABCD = \sum_l E(x^{l_1}, x^{m'}) x^{\phi[l]}, \quad \phi[l] = [\mathcal{H}_{\text{CTM}}]_{l, l} = \frac{1}{2} \sum_{j=1}^N j |l_j - l_{j+2}|$$

The low-temperature nome x is related to the conjugate nome p' of p by

$$x = \exp\left(-\frac{4\pi^2}{m'\epsilon}\right), \quad x^{m'} = \exp\left(-\frac{4\pi^2}{\epsilon}\right) = p', \quad p = e^{-\epsilon}, \quad m' = m+1$$

At criticality

$$p \rightarrow 0, \quad \epsilon \rightarrow \infty, \quad x \rightarrow 1, \quad p' \rightarrow 1$$

Unitary RSOS($m, m+1$) Rényi Entropies

- The RSOS($m, m+1$) Rényi entropy for α replicas is given by

$$S_\alpha(\rho_L) = \frac{1}{1-\alpha} \log \text{Tr}_L(\rho_L)^\alpha = \frac{1}{1-\alpha} [\log Z_\alpha - \alpha \log Z_1]$$

where

$$Z_\alpha^{(N)} = \text{Tr}(\mathbf{ABCD})^\alpha = \sum_{a=1}^m [E(x^a, x^{m'})]^\alpha X_N(a, b, c; x^{2\alpha}), \quad X_N(a, b, c; q) = \sum_{l_2, \dots, l_N} q^{\phi[l]}$$

The elliptic function $E(z, q)$ is

$$E(z, q) = \sum_{k=-\infty}^{\infty} (-1)^k q^{k(k-1)/2} z^k = \prod_{n=1}^{\infty} (1 - q^{n-1}z)(1 - q^n z^{-1})(1 - q^n)$$

- From [Andrews, Baxter, Forrester 1984](#) it follows that in the thermodynamic limit

$$Z_\alpha = \lim_{N \rightarrow \infty} Z_\alpha^{(N)} = x^{\frac{\alpha bc}{2}} (q)_\infty^{-1} \sum_{a=1}^m [E(x^a, x^{m'})]^\alpha \Gamma(a, d; q)$$

where $d = \frac{b+c-1}{2}$, $q = x^{2\alpha}$, $(q)_\infty = \prod_{n=1}^{\infty} (1 - q^n)$ and

$$\Gamma(a, d; q) = q^{\frac{a(a-1)}{4}} \left\{ q^{-\frac{ad}{2}} E(-q^{m(m'-a)+m'd}, q^{2mm'}) - q^{\frac{ad}{2}} E(-q^{m(m'+a)+m'd}, q^{2mm'}) \right\}$$

Unitary RSOS($m, m+1$) Asymptotics

- Fixing the central height a and moving to the conjugate nome gives

$$Z_\alpha^{(a)} = \frac{\vartheta_3\left(\frac{\pi d}{2m} - \frac{\pi a}{2m'}, p^{\frac{1}{8\alpha m}}\right) - \vartheta_3\left(\frac{\pi d}{2m} + \frac{\pi a}{2m'}, p^{\frac{1}{8\alpha m}}\right)}{p^{\frac{m'}{48\alpha}} \sqrt{2mm'} \vartheta_4\left(\frac{im'\epsilon}{8\alpha}, p^{\frac{3m'}{4\alpha}}\right)}, \quad p = e^{-\epsilon}$$

where $p \rightarrow 0$ at criticality and

$$\begin{aligned} \vartheta_3(u, t) &= \prod_{n=1}^{\infty} (1 + 2t^{2n-1} \cos 2u + t^{4n-2})(1 - t^{2n}) = 1 + 2 \sum_{n=1}^{\infty} t^{n^2} \cos 2nu \\ \vartheta_4(u, t) &= \prod_{n=1}^{\infty} (1 - 2t^{2n-1} \cos 2u + t^{4n-2})(1 - t^{2n}) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n t^{n^2} \cos 2nu \end{aligned}$$

- We use the known result for the correlation length (O'Brien, Pearce 1997)

$$\xi^{-1} = -\log k'(p^\nu) \sim 8p^\nu, \quad p^\nu \sim \xi^{-1}, \quad \nu = \frac{m'}{4} = \text{correlation length exponent}$$

- Expanding everything for small p and collecting together gives

$$\log Z_\alpha^{(a)} = -\frac{c_m}{12\alpha} \log p^\nu + C'_{adm'} + \dots, \quad c_m = 1 - \frac{6}{m(m+1)}$$

where the boundary entropy of Affleck-Ludwig 1991 is

$$C'_{adm'} = \log \left(\frac{4}{\sqrt{2mm'}} \sin \frac{\pi d}{m} \sin \frac{\pi a}{m'} \right)$$

- The leading term of the Renyi entropy is thus of the expected form

$$S_\alpha^{(a)} = \frac{1}{1-\alpha} \left[\log Z_\alpha^{(a)} - \alpha \log Z_1^{(a)} \right] \sim \frac{c_m}{12} \left(\frac{1+\alpha}{\alpha} \right) \log \xi + C'_{adm'} + \dots, \quad c = c_m = 1 - \frac{6}{m(m+1)}$$

One Point Functions of Nonunitary RSOS Models

- The RSOS one point functions have been calculated using CTMs (Forrester-Baxter85)

$$P_r(s) = \langle \delta(\sigma_0, s) \rangle_b = \lim_{N \rightarrow \infty} \frac{\text{Tr}SABCD}{\text{Tr}ABCD}$$

where S fixes the center height σ_0 to the value s . The $m-1$ groundstate boundary conditions are labelled by $b = b(r) = \lfloor \frac{rm'}{m} \rfloor$ with $r = 1, 2, \dots, m-1$ and $c = b+1$. Explicitly,

$$\begin{aligned} P_r(s) &= \frac{q^{\frac{c}{24} - \Delta_{r,s}^{m,m'} + \frac{(s-r)(s-r-1)}{4} - \frac{3}{16}} E(q^{\frac{s}{2}}, q^{\frac{m'}{2(m'-m)}})(q)_\infty}{E(-q^{\frac{1}{2}}, q^2) E(-q^{\frac{r}{2}}, q^{\frac{m}{2(m'-m)}})} \text{ch}_{r,s}^{m,m'}(q) \\ &= \sqrt{\frac{2m}{m'}} \frac{\eta(t^{\frac{m'}{m'-m}}) \vartheta_1\left(\frac{s\pi(m'-m)}{m'}, t\right)}{\vartheta_4\left(0, t^{\frac{m'}{m'-m}}\right) \vartheta_1\left(\frac{\pi r(m'-m)}{m}, t^{\frac{m'}{m}}\right)} \sum_{(r',s') \in \mathcal{J}} \mathcal{S}_{rs;r's'} \text{ch}_{r',s'}^{m,m'}\left(t^{\frac{m'}{m'-m}}\right) \end{aligned}$$

where the low-temperature nome is $q = e^{-4\pi\lambda/\epsilon}$, the critical nome is $t = e^{-\epsilon}$ and

$$E(x, q) = \sum_{k=-\infty}^{\infty} (-1)^k q^{k(k-1)/2} x^k = \prod_{n=1}^{\infty} (1 - q^{n-1}x)(1 - q^n x^{-1})(1 - q^n)$$

Here $\eta(t) = t^{1/24}(t)_\infty$ is the Dedekind eta function and the modular matrix \mathcal{S} is

$$\mathcal{S}_{rs;r's'} = \sqrt{\frac{8}{mm'}} (-1)^{(r'+s')(r+s)} \sin \frac{\pi(m'-m)rr'}{m} \sin \frac{\pi(m'-m)ss'}{m'}$$

- There are $\frac{1}{2}(m-1)(m'-1)$ independent OPFs $P_r(s)$ with the Kac table symmetry $P_{m-r}(m'-s) = P_r(s)$ so we restrict (r, s) to the (bottom-left) fundamental domain

$$\mathcal{J} = \{(r, s) \mid 1 \leq r \leq m-1, 1 \leq s \leq m'-1, m'r + ms \leq mm'\}$$

Critical Exponents of Generalized Order Parameters

- The non-unitary OPFs all diverge at criticality $t \rightarrow 0$

$$P_r(s) \sim (t^{\frac{\pi}{\lambda}})^{\Delta_{r_0, s_0}^{m, m'}}, \quad \Delta_{r_0, s_0}^{m, m'} = \frac{1 - (m' - m)^2}{4mm'} = \text{Min}_{(r, s) \in \mathcal{J}} \Delta_{r, s}^{m, m'} < 0, \quad m' - m > 1$$

- This divergence cannot happen for a probabilistic distribution but can occur for the non-unitary minimal models because these models are unphysical in the sense that some Boltzmann weights are negative. Forrester-Baxter remark that, [it is not possible to define exponents of the order parameters in the usual sense.](#)
- Generalizing Huse (1984), [\(Pearce-Seaton2012\)](#) introduced Generalized Order Parameters (GOPs)

$$R_{r'', s''} = \sum_{(r, s) \in \mathcal{J}} \mathcal{S}_{r'' s''; rs} \frac{\sin \frac{\pi r (m' - m)}{m}}{\sin \frac{\pi s (m' - m)}{m'}} P_r(s)$$

The ratio of sines cancels out the trigonometric factors from the ϑ_1 functions in $P_r(s)$. Since $\mathcal{S}^2 = I$, the modular matrix \mathcal{S} effectively undoes the modular \mathcal{S} matrix introduced by the conjugate modulus transformation.

- Defining new observables $\mathcal{O}_{r, s}$ as ratios of the GOPs yields

$$\mathcal{O}_{r, s} = \frac{R_{r, s}}{R_{1, 1}} \sim (t^{\frac{\pi}{\lambda}})^{\Delta_{r, s}^{m, m'}} + (t^{\frac{\pi}{\lambda}})^{\Delta_{r_0, s_0}^{m, m'}} O(t^2), \quad \mathcal{O}_{r, s} \sim (t^2)^{\beta_{r, s}}$$

where t^2 measures the departure from criticality. The coefficient of the leading term is precisely 1. The second term is the correction relative to $R_{1, 1}$.

Minimal Model Kac Tables

- The associated critical exponents corresponding to the $\varphi_{1,3}$ perturbation are

$$\beta_{r,s} = (2 - \alpha) \Delta_{r,s}^{m,m'} = \frac{(rm' - sm)^2 - (m' - m)^2}{8m(m' - m)}, \quad 2 - \alpha = \frac{\pi}{2\lambda} = \frac{m'}{2(m' - m)}$$

We have thus constructed observables with associated critical exponents for all conformal weights satisfying

$$\Delta_{r,s}^{m,m'} < \Delta_{r_0,s_0}^{m,m'} + \frac{2(m' - m)}{m'} = \frac{1 + (m' - m)(9m - m')}{4mm'}$$

- These occur for (r, s) in the Kac table satisfying $(m'r - ms)^2 < 1 + 8m(m' - m)$:

s
4
3
2
1
r

0
$-\frac{1}{5}$
$-\frac{1}{5}$
0

s
6
5
4
3
2
1
r

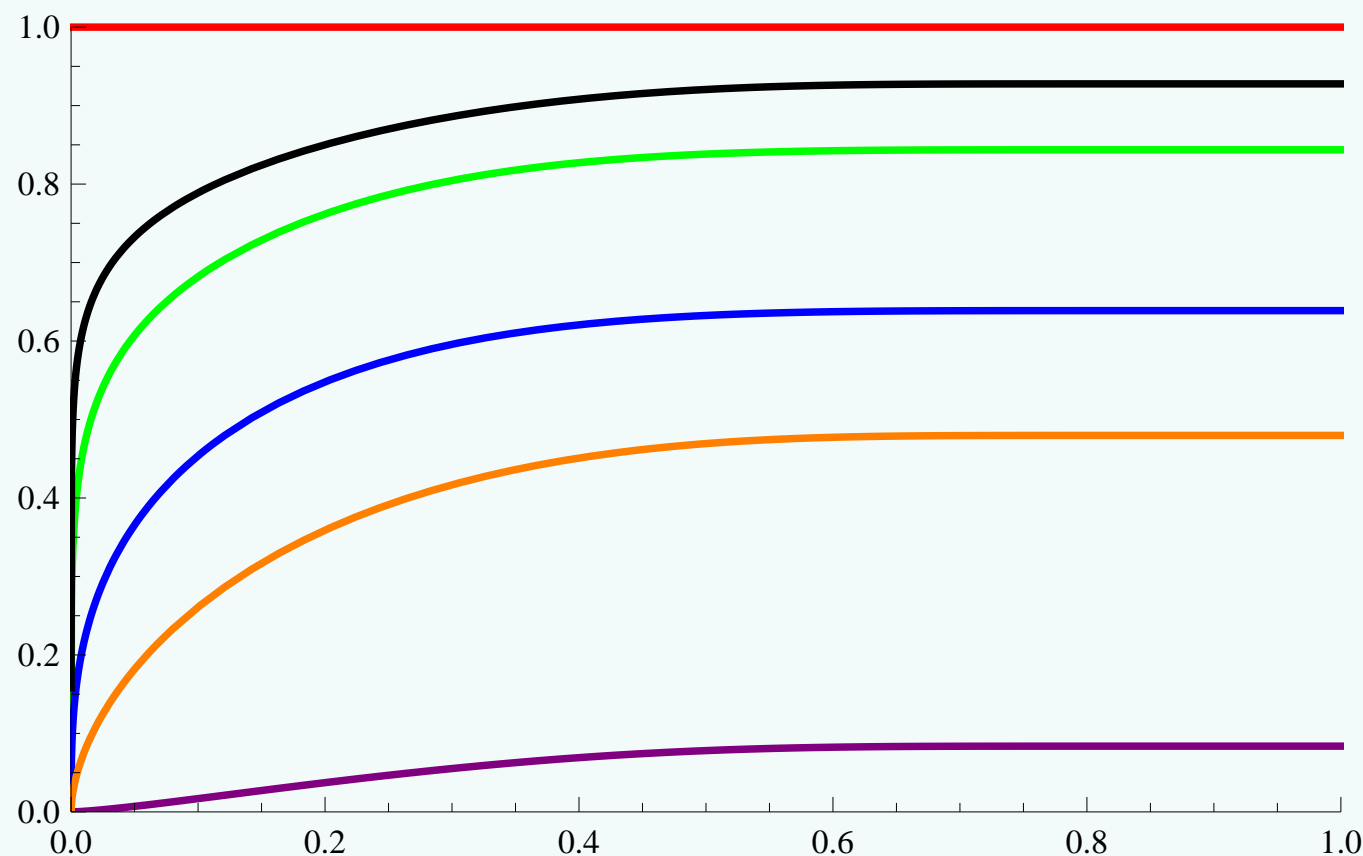
$\frac{5}{2}$	$\frac{13}{16}$	0
$\frac{10}{7}$	$\frac{27}{112}$	$-\frac{1}{14}$
$\frac{9}{14}$	$-\frac{5}{112}$	$\frac{1}{7}$
$\frac{1}{7}$	$-\frac{5}{112}$	$\frac{9}{14}$
$-\frac{1}{14}$	$\frac{27}{112}$	$\frac{10}{7}$
0	$\frac{13}{16}$	$\frac{5}{2}$

s
6
5
4
3
2
1
r

$\frac{15}{4}$	$\frac{9}{5}$	$\frac{11}{20}$	0
$\frac{16}{7}$	$\frac{117}{140}$	$\frac{3}{35}$	$\frac{1}{28}$
$\frac{33}{28}$	$\frac{8}{35}$	$-\frac{3}{140}$	$\frac{3}{7}$
$\frac{3}{7}$	$-\frac{3}{140}$	$\frac{8}{35}$	$\frac{33}{28}$
$\frac{1}{28}$	$\frac{3}{35}$	$\frac{117}{140}$	$\frac{16}{7}$
0	$\frac{11}{20}$	$\frac{9}{5}$	$\frac{15}{4}$

- Kac tables of conformal weights $\Delta_{r,s}^{m,m'}$ for $\mathcal{M}(2,5)$, $\mathcal{M}(3,7)$, $\mathcal{M}(4,7)$. The boxes that are accessible by CTMs are shaded in pink. Some observables $\mathcal{O}_{r,s}$ diverge at $t = 0$.

Plot of $\mathcal{M}(4, 7)$ GOPs



s			
6	$\frac{5}{2}$	$\frac{13}{16}$	0
5	$\frac{10}{7}$	$\frac{27}{112}$	$-\frac{1}{14}$
4	$\frac{9}{14}$	$-\frac{5}{112}$	$\frac{1}{7}$
3	$\frac{1}{7}$	$-\frac{5}{112}$	$\frac{9}{14}$
2	$-\frac{1}{14}$	$\frac{27}{112}$	$\frac{10}{7}$
1	0	$\frac{13}{16}$	$\frac{5}{2}$
	1	2	3 r

- Plot of the order parameters $\mathcal{O}_{r,s}$, as a function of t , for the $\mathcal{M}(4, 7)$ RSOS model. From top to bottom, we plot $\mathcal{O}_{1,1}, \frac{1}{\mathcal{O}_{2,3}}, \frac{1}{\mathcal{O}_{1,2}}, \mathcal{O}_{1,3}, \mathcal{O}_{2,2}, \mathcal{O}_{1,4}$ corresponding to $|\Delta_{r,s}| = 0, \frac{5}{112}, \frac{1}{14}, \frac{1}{7}, \frac{27}{112}, \frac{9}{14}$ in increasing order with critical exponents $\beta_{r,s} = \frac{7}{6} \Delta_{r,s}$. The order parameters are non-negative, vanish at criticality and are increasing functions of t (PS2012).

- Remarkably, the expressions for the off-critical order parameters involve the Virasoro characters and the modular S matrix.

- Critical $\mathcal{M}(m, m')$ RSOS models are associated with Temperley-Lieb quantum Hamiltonians and well-defined CFTs. They also admit quantum Hamiltonians off-criticality (BEPR2015).

Section 4: Nonunitary Models

1. Nonunitary RSOS Rényi Entropies
2. $U_q[sl(2)]$ open XXZ spin chain
3. $N = 2$ and numerical entanglement entropies for large N

Romain Couvreur



Jesper L. Jacobsen



Hubert Saleur



- D.Bianchini, F.Ravanini, *Entanglement entropy from CTM in Forrester-Baxter nonunitary RSOS models*, J. Phys. A49, 154005 (2016).
- R.Couvreur, J.L.Jacobsen, H.Saleur, *Entanglement in nonunitary quantum critical spin chains*, Phys. Rev. Lett. 119, 040601 (2017).

Nonunitary RSOS(m, m') Rényi Entropies

- From CTMs (ForresterBaxter85), the partition functions for α “replicas” in the (r, s) sectors are

$$Z_{r,s}^{(\alpha)} = q^{\alpha(s-r)(s-r-1)/4} E(q^{s/2}, q^{m'/2(m'-m)})^\alpha X_{r,s}(q^\alpha)$$

where

$$\begin{aligned} X_{r,s}(q) &= \frac{1}{(q)_\infty} \left[E(-q^{mm'+rm'-sm}, q^{2mm'}) - q^{rs} E(-q^{mm'+rm'+sm}, q^{2mm'}) \right] \\ &= q^{c/24 - \Delta_{r,s}^{m,m'}} \text{ch}_{r,s}^{m,m'}(q) \end{aligned}$$

The elliptic functions are as before and $(q)_\infty = E(q, q^3)$.

- It follows that

$$\frac{Z_{r,s}^{(\alpha)}}{(Z_{r,s}^{(1)})^\alpha} = \frac{X_{r,s}(q^\alpha)}{[X_{r,s}(q)]^\alpha} = \frac{\text{ch}_{r,s}^{m,m'}(q^\alpha)}{[\text{ch}_{r,s}^{m,m'}(q)]^\alpha} = \frac{\sum_{(r,s) \in \mathcal{J}} \mathcal{S}_{r,s;r',s'} \text{ch}_{r',s'}^{m,m'}(t^{\frac{4\nu}{\alpha}})}{\left[\sum_{(r,s) \in \mathcal{J}} \mathcal{S}_{r,s;r',s'} \text{ch}_{r',s'}^{m,m'}(t^{4\nu}) \right]^\alpha}$$

The last form follows from a modular transformation on the characters.

The conjugate modulus form of the character in the numerator is

$$\text{ch}_{r,s}^{m,m'}(q^\alpha) = \sqrt{\frac{3}{2mm'}} \frac{\vartheta_3\left(\frac{\pi}{2}\left(\frac{r}{m} - \frac{s}{m'}\right), t^{\frac{1}{4\alpha m(m'-m)}}\right) - \vartheta_3\left(\frac{\pi}{2}\left(\frac{r}{m} + \frac{s}{m'}\right), t^{\frac{1}{4\alpha m(m'-m)}}\right)}{\vartheta_1\left(\frac{\pi}{3}, t^{\frac{m'}{6\alpha(m'-m)}}\right)}$$

- Clearly, the Renyi entropies are

$$S_{r,s}^{(\alpha)} \sim \frac{c_{\text{eff}}}{12} \left(\frac{1+\alpha}{\alpha}\right) \log \xi + C_{r,s} + \dots, \quad \xi^{-1} \sim 8(t^2)^\nu, \quad \nu = \frac{m'}{4(m'-m)}$$

$U_q[sl(2)]$ Open XXZ Spin Chain

- Consider the critical open XXZ spin chain with boundary terms (AlcarazEtAl87)

$$H = - \sum_{j=1}^{N-1} e_j, \quad e_j = -\frac{1}{2} \left[\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \frac{q+q^{-1}}{2} (\sigma_j^z \sigma_{j+1}^z - 1) + h_j \right]$$

where $q = e^{i\lambda}$, $|q| = 1$ and $\sigma_j^{x,y,z}$ are Pauli matrices. The case $h_j = 0$ describes the critical XXZ chain.

- Let us instead add terms with

$$h_j = \frac{q-q^{-1}}{2} (\sigma_j^z - \sigma_{j+1}^z)$$

This breaks Hermiticity but gives the $U_q[sl(2)]$ quantum group invariant open XXZ spin chain. H is not Hermitian but its eigenvalues are real (Morin-Duchesne Et Al 2016). Significantly, its left- and right-eigenstates differ. Notice that the h_j terms act as a gauge and cancel out in the periodic system.

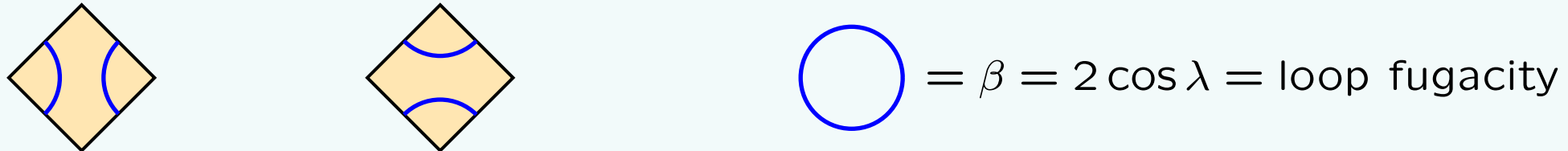
- The e_j give a representation of the Temperley-Lieb (TL) algebra (Temperley, Lieb 1971)

$$e_j^2 = (q + q^{-1})e_j = 2 \cos \lambda e_j, \quad e_j e_{j\pm 1} e_j = e_j, \quad [e_j, e_k] = 0, \quad |j - k| > 1$$

These relations admit a nice diagrammatic interpretation in terms of “monoids”.

Temperley-Lieb Algebra

- The Temperley-Lieb algebra admits a planar diagrammatic representation consisting of “monoid” diagrams (physically this is a loop gas)



- The monoids satisfy



TL \Rightarrow YBE: The YBE is satisfied if the face weights are of the form

$$\begin{array}{|c}
 \text{diamond with } u \\
 \hline
 = \frac{\sin(\lambda - u)}{\sin \lambda} \text{diamond with arcs on left} + \frac{\sin u}{\sin \lambda} \text{diamond with arcs on right}
 \end{array}$$

- There are three known types of TL representations yielding exactly solvable models:
 - (i) quantum spin (vertex models)
 - (ii) height or Restricted Solid-On-Solid (RSOS)
 - (iii) loop (polymers/percolation)

Naïve $N = 2$ XXZ Entanglement Entropy

- Restrict $\lambda \in [0, \pi/2]$ (other values are related by a duality). Consider $N = 2$ and take subsystem A (B) as the left (right) spin. The Hamiltonian is

$$H = -e_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -q^{-1} & 1 & 0 \\ 0 & 1 & -q & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- The lowest energy is $E^{(0)} = -(q + q^{-1}) < 0$. The other eigenenergy is $E^{(1)} = 0$. The right ground state is

$$H|0\rangle = E^{(0)}|0\rangle, \quad |0\rangle = \frac{1}{\sqrt{2}}(q^{-1/2}|\uparrow\downarrow\rangle - q^{1/2}|\downarrow\uparrow\rangle) = \frac{1}{\sqrt{2}}(0, q^{-1/2}, -q^{1/2}, 0)^T$$

- Using the convention that complex numbers are conjugated when calculating the bra associated with a given ket gives $\langle 0|0\rangle = 1$. In the basis $\uparrow\uparrow, \uparrow\downarrow, \downarrow\uparrow, \downarrow\downarrow$, the (normalized) density matrix is

$$\rho = |0\rangle\langle 0| = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -q^{-1} & 0 \\ 0 & -q & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{Tr } \rho = 1$$

- The reduced density matrix and von Neumann entropy are

$$\rho_A = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S_A = -\text{Tr}_A \rho_A \log \rho_A = \log 2$$

- But this does not take into account the proper left- and right-eigenvectors!

Modified $N = 2$ XXZ Entanglement Entropy

- CJS2017 say it is more correct to work with left- and right-eigenstates

$$H|E_R\rangle = E|E_R\rangle, \quad \langle E_L|H = E\langle E_L|, \quad (\text{or } H^\dagger|E_L\rangle = E|E_L\rangle, \text{ since } E \in \mathbb{R})$$

- Restricting to the sector $S^z = 0$, the right eigenstates with energies $E^{(0)}, E^{(1)}$ are

$$\begin{aligned} |0_R\rangle &= \frac{1}{\sqrt{q+q^{-1}}} (q^{-1/2}|\uparrow\downarrow\rangle - q^{1/2}|\downarrow\uparrow\rangle) \\ |1_R\rangle &= \frac{1}{\sqrt{q+q^{-1}}} (q^{1/2}|\uparrow\downarrow\rangle + q^{-1/2}|\downarrow\uparrow\rangle) \end{aligned}$$

The left eigenstates $|0_L\rangle, |1_L\rangle$ are obtained by $q \rightarrow q^{-1}$. The normalizations are $\langle i_L|i_R\rangle = 1$, and $\langle i_L|j_R\rangle = 0$ for $i \neq j$.

- Since $\langle 0_R|1_R\rangle \neq 0$ we use left- and right-eigenstates to build a projector onto the ground state

$$\tilde{\rho} = |0_R\rangle\langle 0_L| = \frac{1}{q+q^{-1}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q^{-1} & -1 & 0 \\ 0 & -1 & q & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{\rho}_A = \text{Tr}_B (q^{-2\sigma_B^z} \tilde{\rho}) = \frac{1}{q+q^{-1}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^z = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Observe that $\tilde{\rho}_A$ is normalized for the modified [Markov trace](#) (note the opposite power of q)

$$\text{Tr}_A (q^{2\sigma_A^z} \tilde{\rho}_A) = \text{tr}_A \tilde{\rho}_A = 1$$

- For such nonunitary systems, the entanglement entropy should perhaps be defined as the [Markov trace](#)

$$\tilde{S}_A = -\text{tr}_A \tilde{\rho}_A \log \tilde{\rho}_A = -\text{Tr}_A (q^{2\sigma_A^z} \tilde{\rho}_A \log \tilde{\rho}_A) = \log(q + q^{-1})$$

This result is more appealing since it depends on q through the combination $q + q^{-1}$ which is the quantum dimension of the spin-1/2 representation of $U_q[sl(2)]$. It also satisfies $\tilde{S}_A = \tilde{S}_B$.

Summary of Exact Area Laws/Entanglement Results

- 2006 Weston: Higher-spin XXZ (level $k = \text{spin} - \frac{k}{2}$)

$$c_k = \frac{3k}{k+2} = \text{WZW central charges}$$

- 2013 De Luca, Franchini: Unitary minimal models $\mathcal{M}(m, m+1)/\text{RSOS}(m, m+1)$ models

$$c = 1 - \frac{6}{m(m+1)}$$

- 2016 Bianchini, Ravanini: Nonunitary minimal models $\mathcal{M}(m, m')/\text{RSOS}(m, m')$ with $m' \neq m+1$

c is replaced by c_{eff} !!

- 2017 Couvreur, Jacobsen, Saleur:

Modifications are perhaps needed for nonunitary theories to see c rather than c_{eff} !!

The suggested modifications are as follows:

- The projector onto the ground state should be $|0_R\rangle\langle 0_L|$ and not $|0\rangle\langle 0|$.
- The trace of the Hamiltonian/transfer matrices should be replaced by the [Markov trace](#).

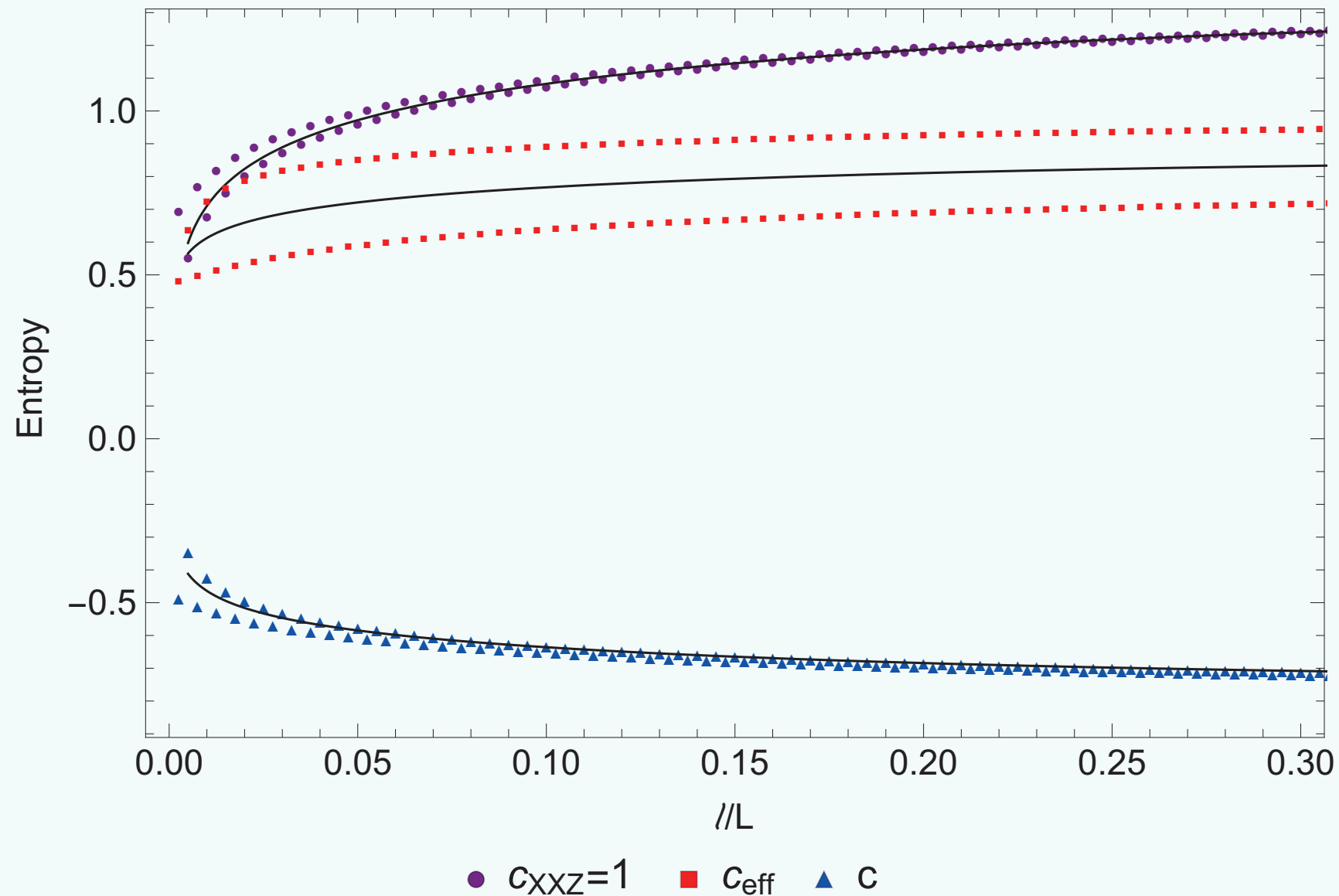
- For RSOS models, the Markov trace is given by

$$\text{tr} \mathbf{D}(u) = \sum_{s \geq 1} [s]_q \text{Tr}_s \mathbf{D}(u), \quad [s]_q = \frac{q^s - q^{-s}}{q - q^{-1}}, \quad q = e^{i\lambda}$$

where the first height is $a_1 = 1$, the last height is $a_N = s$ and $[s]_q$ is the quantum dimension.

Numerical Entanglement for $\mathcal{M}(3,5)$

[Couvreur, Jacobsen, Saleur 2016]



$$\Delta_{r,s} = \begin{pmatrix} \frac{3}{4} & 0 \\ \frac{1}{5} & -\frac{1}{20} \\ -\frac{1}{20} & \frac{1}{5} \\ 0 & \frac{3}{4} \end{pmatrix}$$

- Numerical entanglement entropy for the nonunitary minimal model $\mathcal{M}(3,5)$, versus $\frac{1}{\ell}$, for ℓ up to 400 sites and open boundary conditions. Purple dots show the usual entanglement entropy with the unmodified trace. Red squares show the $\alpha = 2$ Rényi entropy, with the Markov trace giving weight $n_1 = 2 \cos \frac{\pi}{5}$ to non-contractible loops; this scales with $c_{\text{eff}} = \frac{3}{5}$. Blue triangles show $\tilde{S}^{(2)}$, with the Markov trace and $n_1 = 2 \cos \frac{2\pi}{5}$; the scaling then involves the true central charge $c = -\frac{3}{5}$.

Open Research Problems

Use off-critical CTMs to calculate analytically the Rényi entropies and central charges c of nonunitary RSOS models $\mathcal{M}(m, m')$ for different boundary conditions.

Is it possible to use entanglement entropies to obtain the conformal weights $\Delta_{r,s}$?

Is it possible to use to use critical CTMs to calculate entanglement entropies?