ESTIMATORS OF SHARED INFORMATION IN STATIONARY STATES OF STOCHASTIC PROCESSES

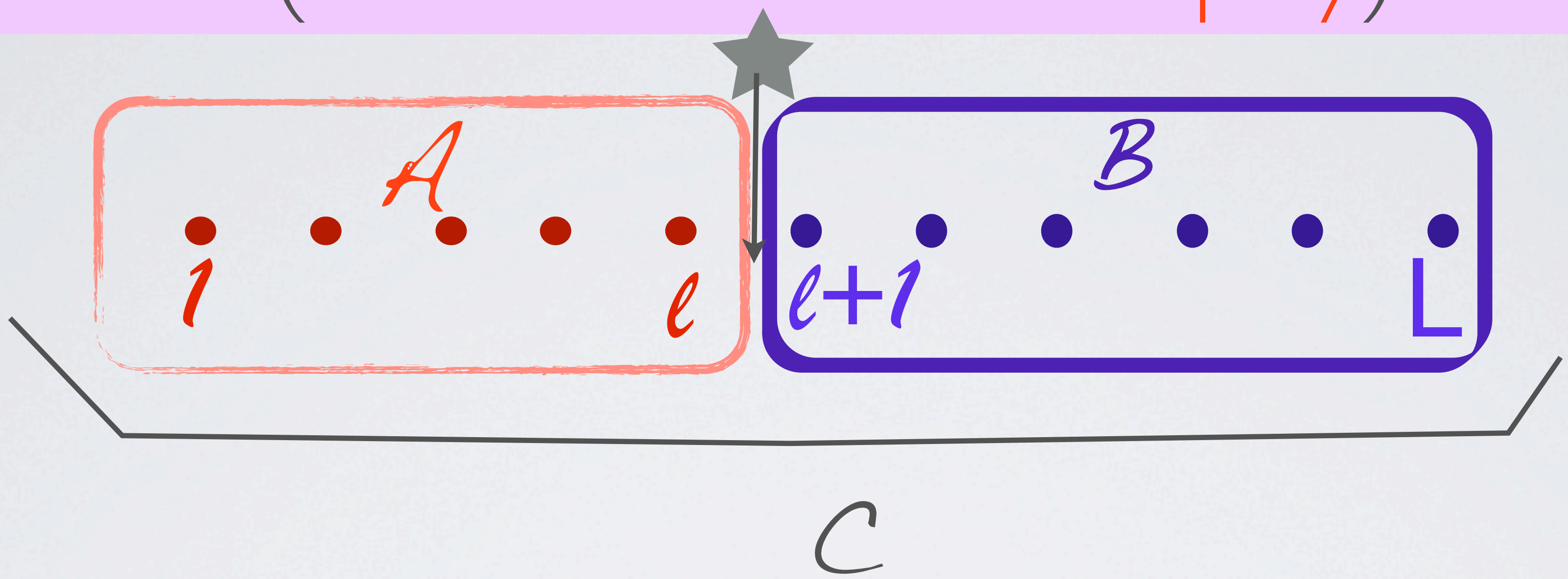
Vladimir Rittenberg

Physikalishes Institut, Bonn University, Bonn, GERMANY

Alcaraz FC, VR and Sierra G - Phys. Rev. E **80**, 030102(R) (2009)
Alcaraz, FC and VR - JSTAT P03024 (2010)

- Introduction and motivation
- Estimators for shared information
- Application 1: Model for polymer adsorption
- Aplication 2: The raise and peel model

Entanglement properties of ground-state wavefunctions of Hermitian Hamiltonians
(1 dimension - to simplify)



$H|\psi\rangle = E_G|\psi\rangle$ ⟵ Ground state (spin basis)

$$\rho = |\psi\rangle\langle\psi|$$

$$S_{vN}(\mathcal{A}) = S_{vN}(\mathcal{B}) = -\mathrm{Tr}(\rho_{\mathcal{A}} \ln \rho_{\mathcal{A}}) \qquad \rho_{\mathcal{A}} = \mathrm{Tr}_{\mathcal{B}}(\rho)$$

Schmidt Decomposition $\rightarrow S_{vN} = -\sum \lambda_i \ln \lambda_i$

$$R_n(l, L) = 1/(1-n) \ln \mathrm{Tr}\rho_A^n \quad (n = 2, 3, \ldots)$$

S_{vN} unchanged only under local transf. of the basis

For large system and subsystems $L >> l >> 1$

$S_{vN}(l,L) \sim constant$ ⟵ non critical (gapped) - area law

$S_{vN} \sim \gamma \ln l + C$ ⟵ critical (gapless)

Calabrese and Cardy 2004

critical and conformal invariant - central charge c

$\gamma = \frac{c}{6}$ for open systems $\gamma = \frac{c}{3}$ for periodic systems

$$S_{vN}(l,L) = \gamma \ln \tilde{L}_C + C, \quad \boxed{\tilde{L}_C = L \sin(\pi l/L)/\pi}$$

⟵ f.s.s.

$$R_n(l,L) = 1/(1-n) \ln \mathrm{Tr}\rho^n = \tilde{L}_C^{-\frac{c}{6}(n-\frac{1}{n})} \quad n = 2, 3, \ldots$$

Shared information in stationary states of classical systems

$$\frac{d}{dt}|P> = -H|P> \qquad |\Psi> \longrightarrow |P>$$ probability distribution

$$H_{aa} = -\sum_b H_{ab}$$

$$H|P_{stat}\rangle = 0 \qquad \langle 1,1,1,\ldots|H = 0$$

Ground-state energy is zero for any # os sites

H is stochastic in a special basis

Quantum case
(ground state)

$\longleftrightarrow$

Stochastic case
(stationary state)

$|\Psi_{gs}>= \sum c_n|\phi_n>$ $\longrightarrow$ $|P_{stat}>= \sum p_a|\phi_a>$

c_n complex

$|\phi_n>$ basis- Hilbert space $\longrightarrow$

p_a (probabilities)

$|\phi_a>$ config. of the system

H Hermitian $\longrightarrow$ H non-Hermitian

Schmidt decomposition
density matrix ρ
entanglement entropy

$\longrightarrow$ new measures of shared information

critical violates are law $\longrightarrow$ critical violates are law

Most critical systems conformally invariant $S_{vN} \sim c/6 \ln l$ $\longrightarrow$ Variety of critical system including conformal invariant ones

Aim

Produce estimators of the shared information among subsystems

★ Similar properties as in the quantum case

a) Vanish if the subsystems are separated
b) If ξ finite $\longrightarrow$ are finite
c) If we have logarithmic behavior : $E(l,L) \sim \gamma_E \ln \tilde{L}_E(l,L) + C_E$
γ_E and $\tilde{L}_E$ are universal and C_E non universal
d)If we have power-law behavior: $E(l,L) \sim \gamma_E \tilde{L}^{\delta_E} + D_E$
γ_E , δ_E and $\tilde{L}_E$ are universal and D_E non universal

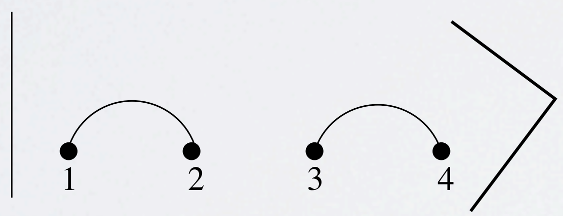
Configuration space ("Hilbert space" for classical model)

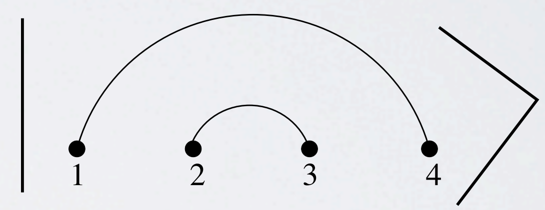
Inspiration: Quantum chains spin 1/2 SU(2) symmetric (Hermitian): Ex. Heisenberg chain (open bound. cond)

$$H = -J \sum_{i=1}^{L-1} \vec{S}_i \vec{S}_{i+1}$$

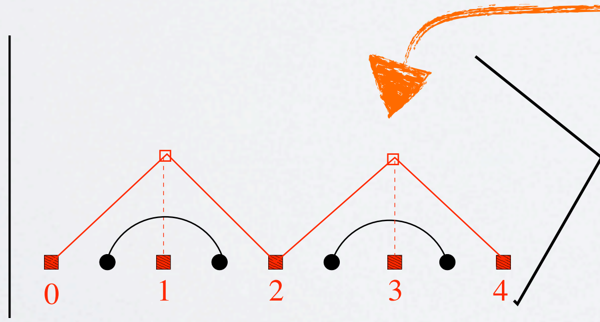
$S = 0$ sector $\rightarrow$ singlet basis

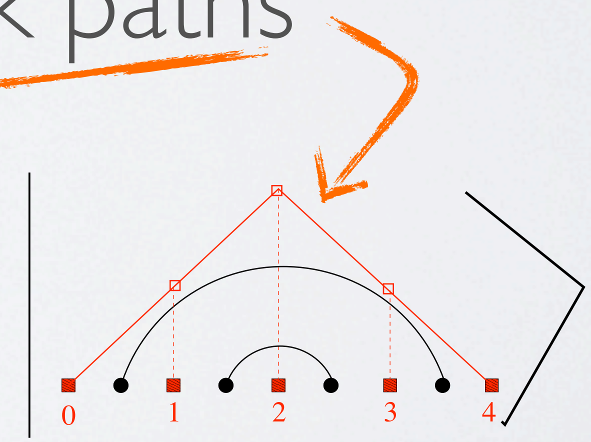
Ground state L=4 sites

$|\Psi> = c_1$ 

$+ \quad c_2$ 

One-to-one correspondence - Dyck paths

$|\Psi> = c_1$ 

$+ \quad c_2$ 

$c_1, c_2 \geq 0 \quad Real,$ Like for a stationary state

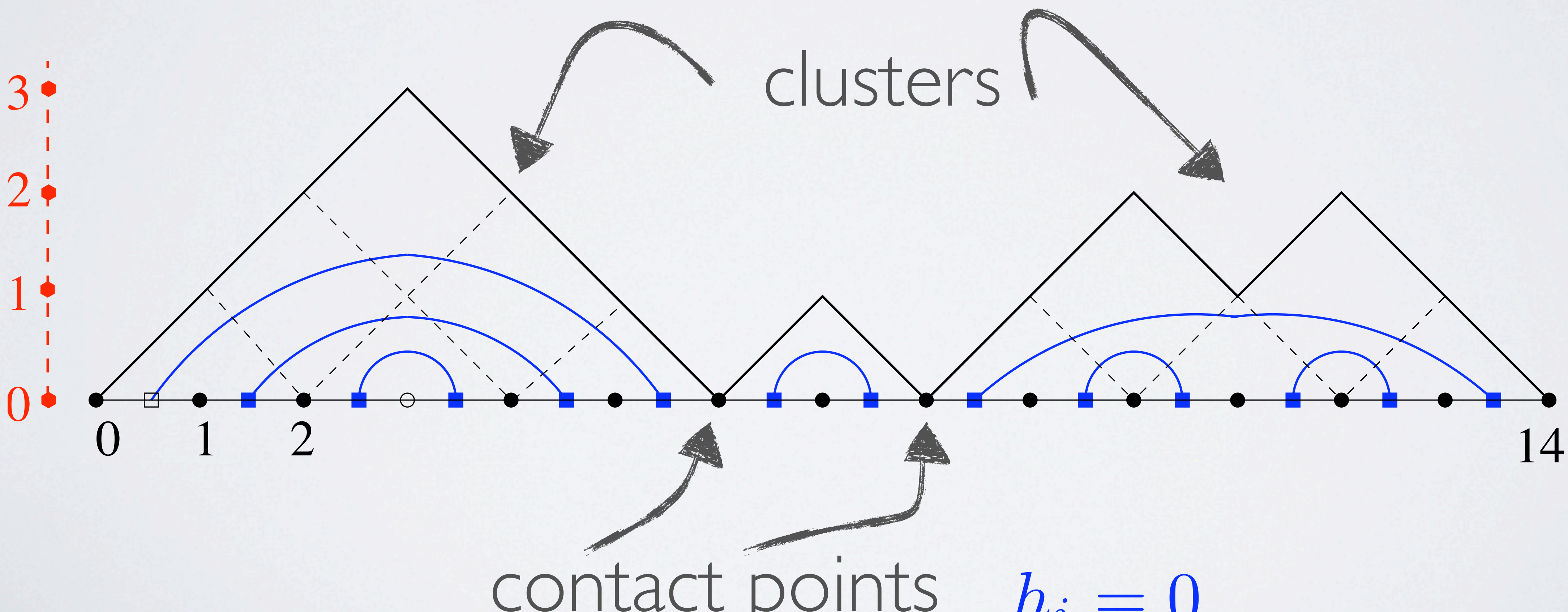
To simplify: models defined on the same configuration space

Dyck Path: restrict solid-on-solid (RSOS) conf. in L+1 sites

configurations: $(h_1, h_2, \ldots, h_l)$

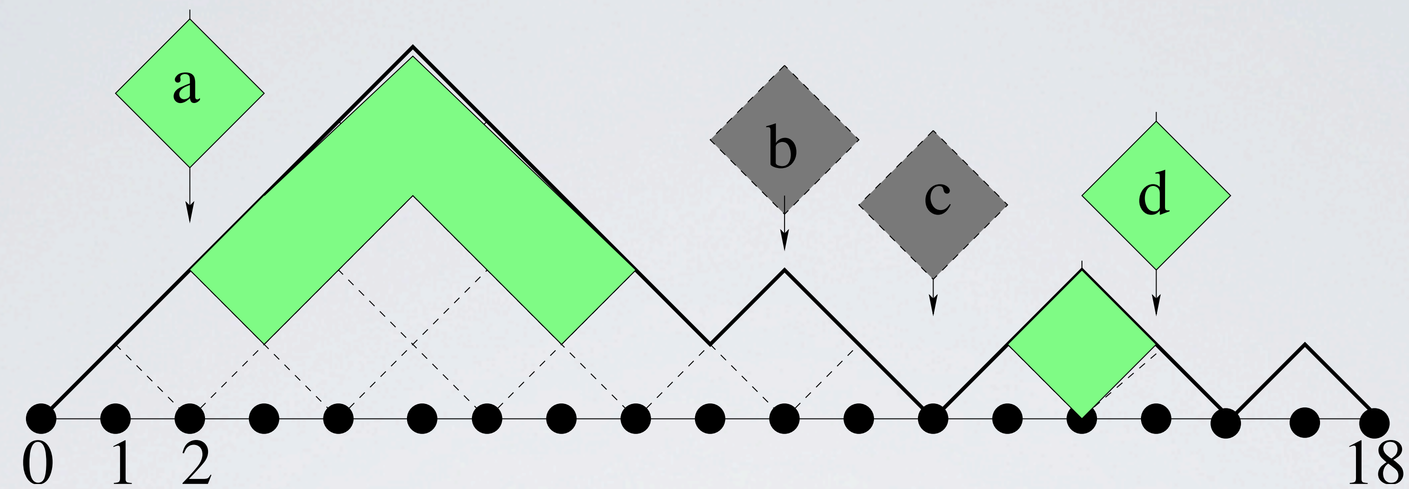
$$h_{i+1} - h_i = \pm 1, \quad h_0 = h_L = 0 \quad (i = 0, 1, \ldots, L-1)$$

$$Z_1(L) = L!/(L/2)!(L/2+1)!$$

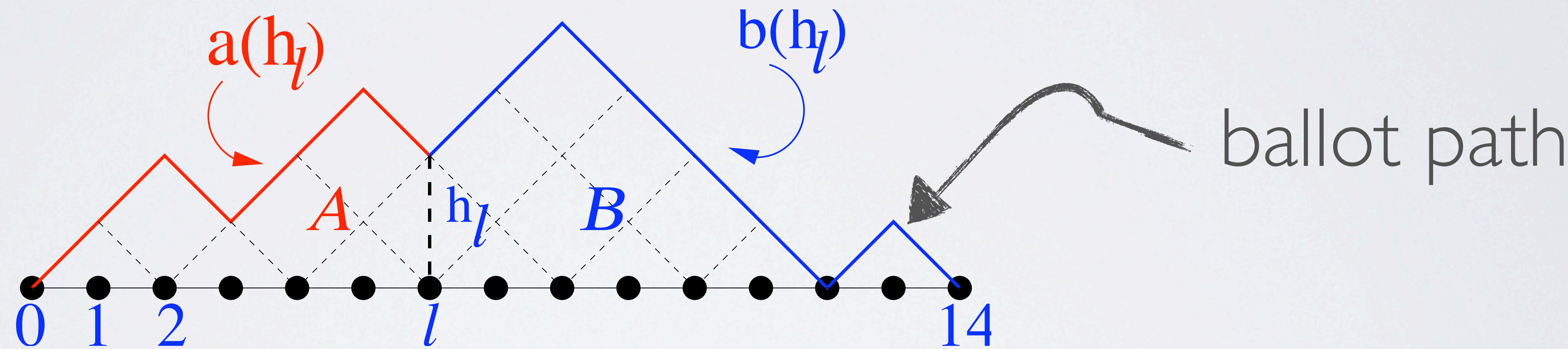


$h_i = 0$

Stochastic model: Dyck paths = config. of an interface



Bipartitions of $\mathcal{C}$ (L+1 sites): Part $\mathcal{A}$ (ℓ+1 sites) and part $\mathcal{B}$ (L-ℓ sites)



if $h_l = 0$ → no shared information

h_l large → large shared information

Constraints $h_0 = h_L = 0$ and RSOS rules models in which large probab. for large h_i → large values for the estimators

$P(a(h_l), b(h_l))$ Prob. of Dyck path composed by $a(h_l)$ $b(h_l)$

Marginals $$P(a(h_l)) = \sum_b P(a(h_l), b(h_l))$$

Prob. height h_l at site l

$$P_l(h, L) = \sum_a P(a(h_l)) = \sum_b P(b(h_l))$$

Estimators

- **Mutual information:**

$$I(l,L) = \sum_{h_l, a(h_l), b(h_l)} P(a(h_l), b(h_l)) \ln \frac{P(a(h_l), b(h_l))}{P(a(h_l))P(b(h_l))}$$

Standard estimator for shared information

- **Interdependency:**

$$H_h(l,L) = -\sum_h P_l(h,L) \ln P_l(h,L)$$

Shannon entropy for heights

New estimator: imitates the entanglement entropy using the Schmidt decomposition

> If the configurations have the same probability
> $$I(l,L) = H_h(l,L)$$

Renyi Interdependencies

$$R_n(l,L) = 1/(1-n)\ln\sum_h P_l(h,L)^n, \quad n = 2,3,\ldots$$

- Valence bond entanglement entropy:

$$h(l,L) = \sum_h h P_l(h,L)$$

Average height at separation site.

Used in the context of SU(2) spin 1/2 quantum chains (Chhajlany, Tomczak, Wojcic 2007, Jacobsen, Saleur 2008)

- Density of contact points:

$$D(l,L) = -\ln P_l(0,L)$$

If $\rho(l,L) = P_l(0,L)$ small → large clusters

← local operator

- Separation Shanon entropy:

$$S(l,L) = H(L) - H(l) - H(L-l)$$

$$H(M) = -\sum_k P_k \ln P_k$$

P_k is the probability of the configuration k

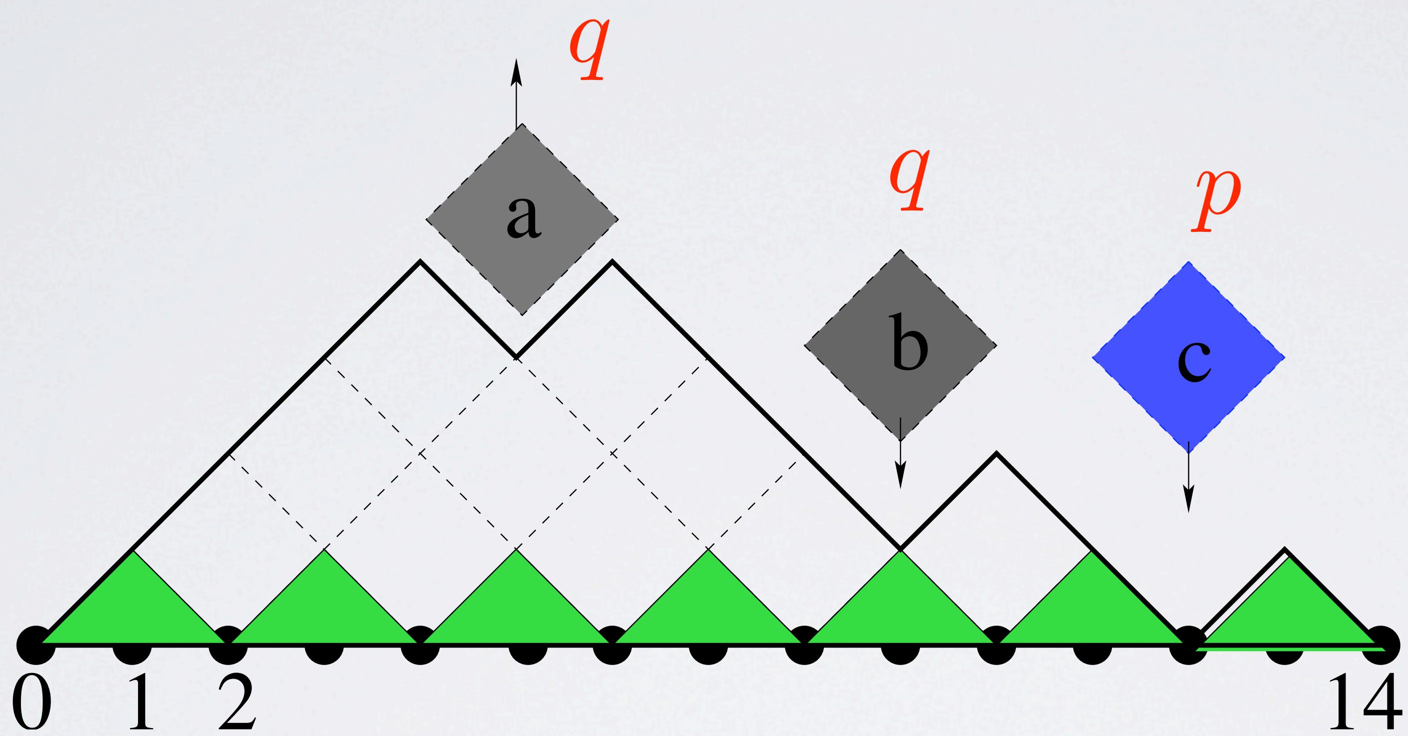
Measures the increase of disorder in $\mathcal{C}$ due to $\mathcal{A}$ and $\mathcal{B}$

If the configuration have the same probability
$D(l,L) = S(l,L)$

- All the estimators vanishes when $\mathcal{A}$ and $\mathcal{B}$ separated
- Universality at criticality
- If one compares two models, all estimators are larger (smaller)
- Some estimators can be computed using Monte Carlo simulations
- Applications: NEXT TALK

Stochastic models

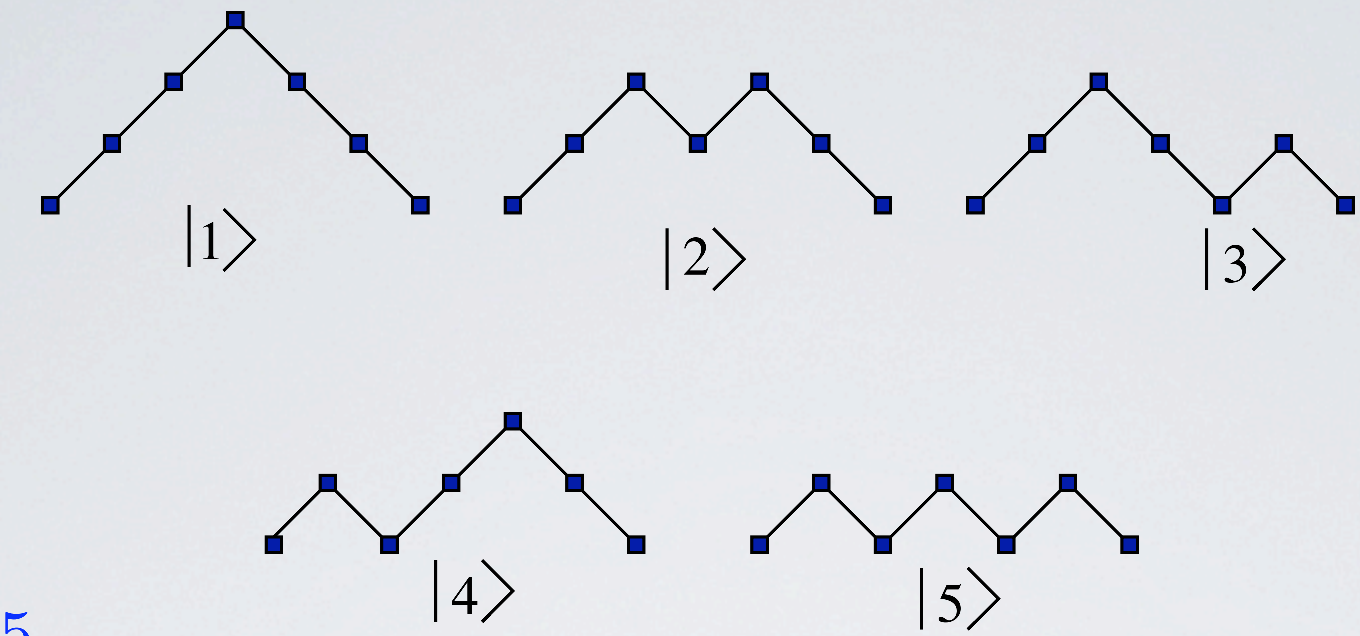
Model for polymer adsorption



$$q/p = K = u^{-1}$$

$u = 1$ is the Rouse Model (Rouse, 1953)

L=6



$$|P(t)>=\sum_{a=1}^{5} P_a(t)|a>, \quad P_a = \lim_{t\to\infty} P_a(t), \quad |0>=\sum_{a=1}^{5} P_a|a>$$

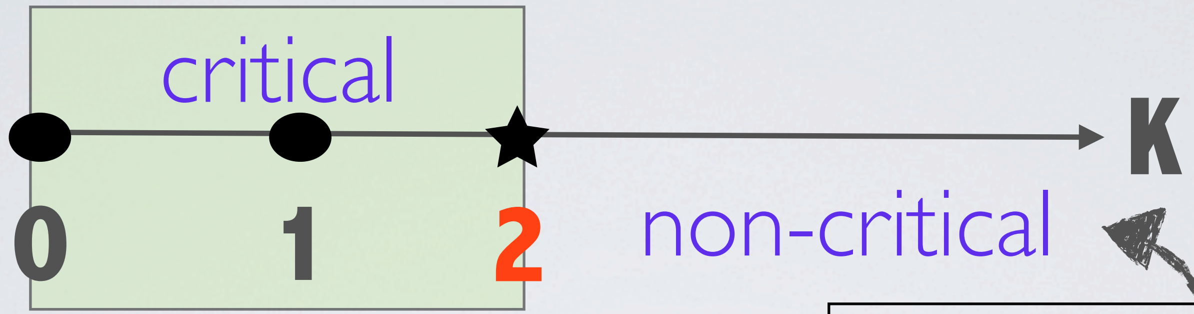
$$H = \left(\begin{array}{c|ccccc} & |1\rangle & |2\rangle & |3\rangle & |4\rangle & |5\rangle \\ \hline \langle 1| & 1 & -1 & 0 & 0 & 0 \\ \langle 2| & -1 & 3 & -\mathrm{u} & -\mathrm{u} & 0 \\ \langle 3| & 0 & -1 & 1+\mathrm{u} & 0 & -\mathrm{u} \\ \langle 4| & 0 & -1 & 0 & 1+\mathrm{u} & -\mathrm{u} \\ \langle 5| & 0 & 0 & -1 & -1 & 2\mathrm{u} \end{array}\right). \qquad u = \frac{1}{K}$$

$$|0\rangle = |1\rangle + |2\rangle + K(|3\rangle + |4\rangle) + K^2|5\rangle$$

$$|0\rangle = \sum_{\psi} K^{m(\psi)}|\psi\rangle$$

contact points

The phase diagram (Owczarek 2009)



Small clusters on the substrate

K=1 (Random Walker)

$$P_l(h,L) \sim \frac{4}{\sqrt{\pi}} \frac{z^2 e^{-z^2}}{\sqrt{\tilde{L}_{RW}}}, \quad z = h/\sqrt{\tilde{L}_{RW}}, \quad \tilde{L}_{RW}/2 = l(1-\frac{l}{L})$$

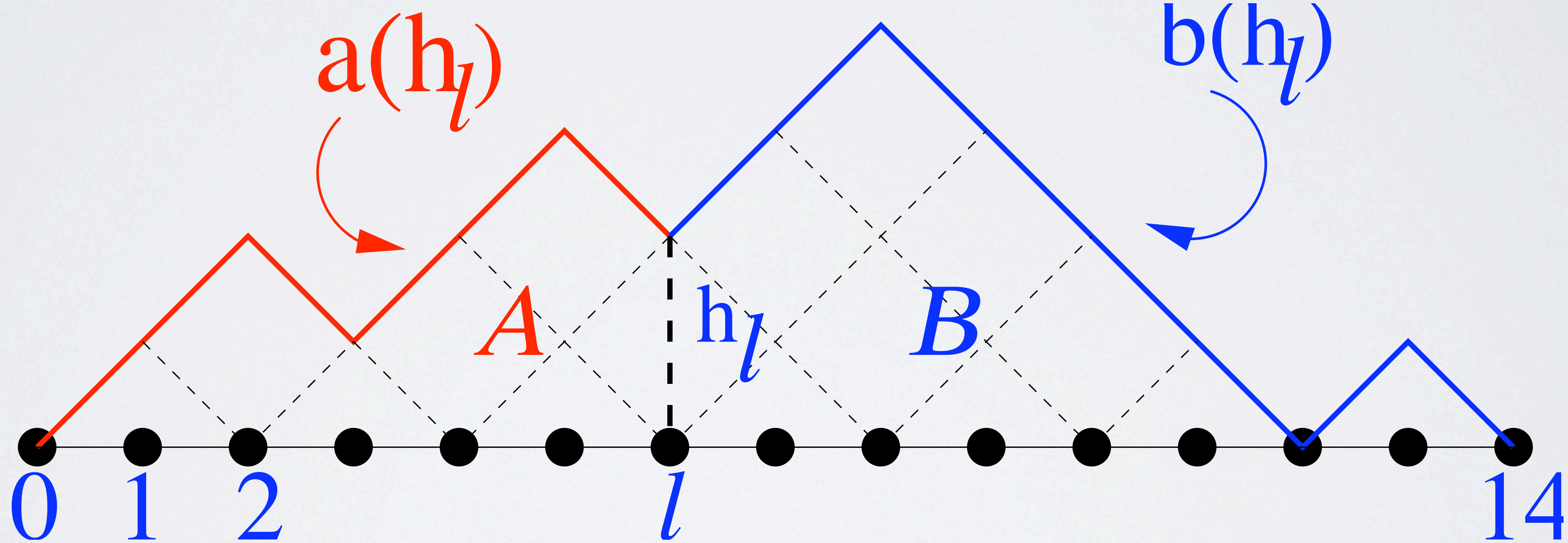
$$I(l,L) = H_h(l,L) \sim \frac{1}{2} \ln \tilde{L}_{RW} + C(K)$$

$$R_n(l,L) \sim 1/2 \ln \tilde{L}_{RW} + C_n(K)$$

$$h(l,L) \sim \frac{4}{\sqrt{2\pi}} \tilde{L}_{RW}^{1/2}$$

$$D(l,L) = S(l,L) \sim \frac{3}{2} \ln \tilde{L}_{RW} + C_S(K)$$

$0 < K < 2$
Same universal behavior


a(h_l)
b(h_l)
A
h_l
B
0 1 2
l
14

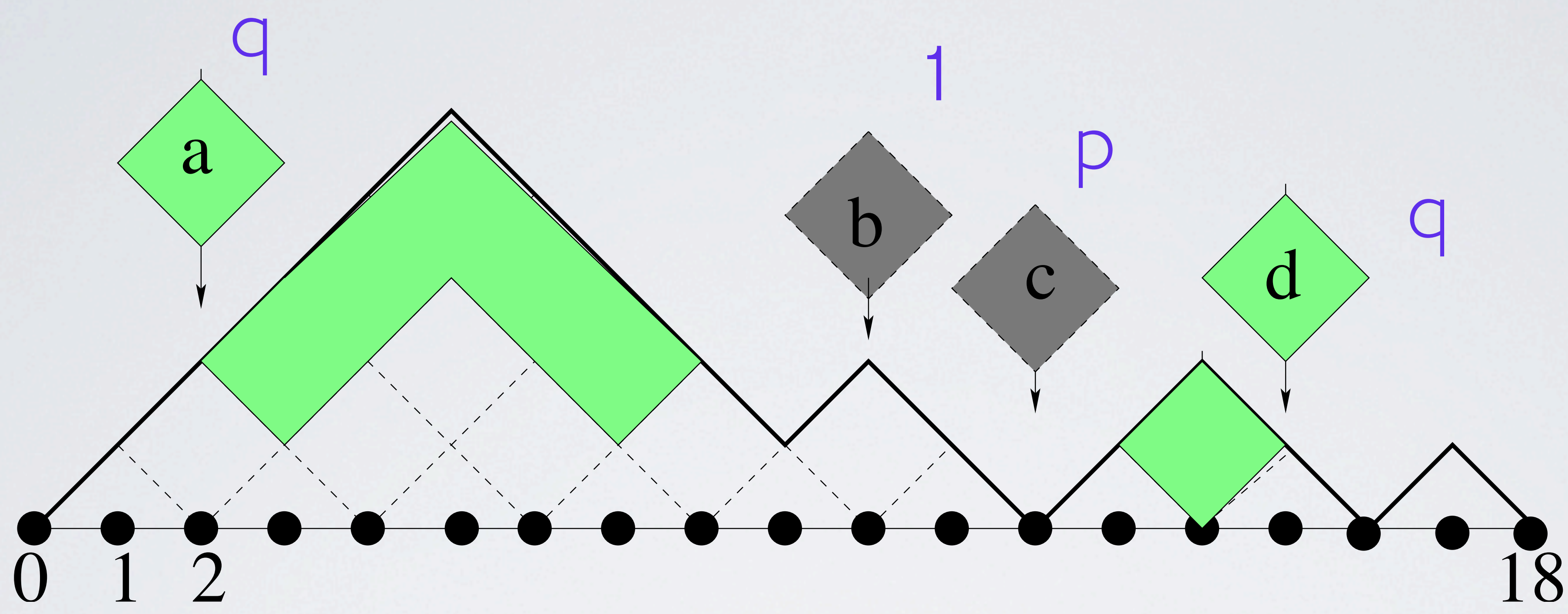
K =2 (critical) The shared information is distinct and smaller (Random Walker in the plane):
Separation entropy:

$$S(l,L) \sim 1/2 \ln(l(1-l/L)) + C_S \qquad \gamma_s = 1/2$$

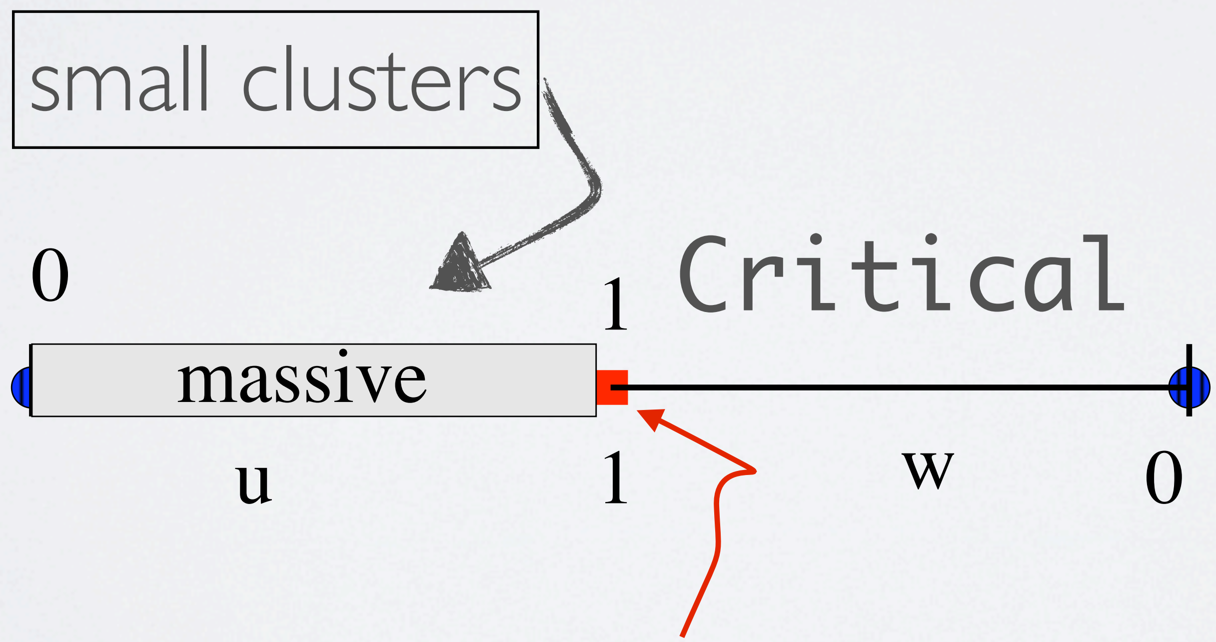
$$h(l,L) \sim \frac{2}{\sqrt{2\pi}} \tilde{L}_{RW}^{1/2} \qquad \text{smaller than in } 0 < k < 2 \quad \text{region}$$

K >2 (non critical) The shared information is finite

Raise and peel model (de Gier, Nienhuis,Pearce,Rittenberg, 2003; FCA, Rittenberg, 2007 [review])

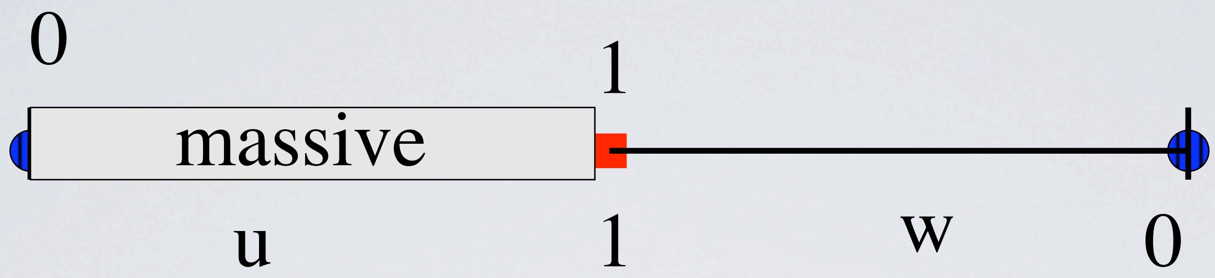


$$u = p/q$$

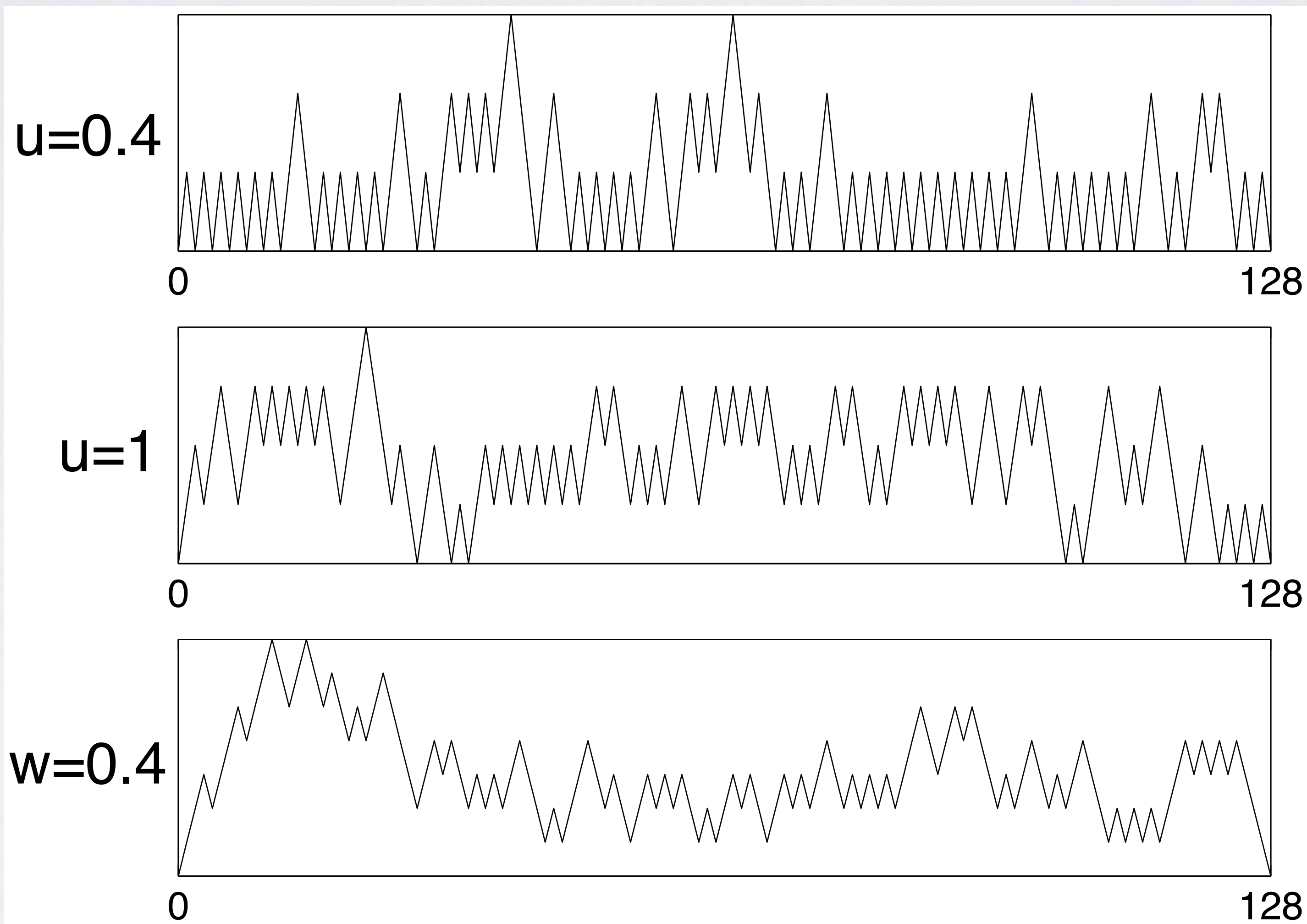


$$w = 1/u$$

Conformal invariant

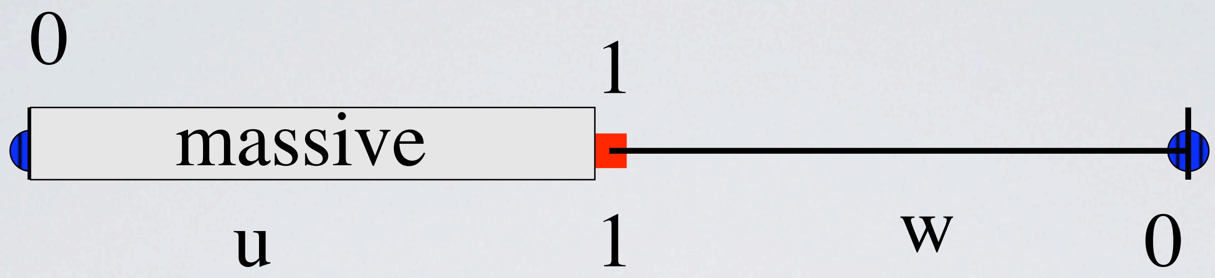


Typical configurations (L=128)



Small clusters

Large clusters



★ At $u = 1$ we have a conformally invariant stochastic model - central charge $c = 0$

H spin 1/2 XXZ quantum chain with $U_q(sl(2))$ symmetry with

$\Delta = q + 1/q = 1/2, \quad q = \exp(i\pi/3)$

The link patterns correspond to $U_q(sl(2))$ singlets (non-local basis) ($c = 1$ in the spin basis)

★ $u > 1$ Almost a single cluster.

Results

$0 < u < 1$

(non critical) density of clusters finite ⟶ shared information finite

$u = 1$

$$E(l, L) \sim \gamma_E \ln \tilde{L}_C + C_E, \quad \tilde{L}_C = L \sin(\pi l / L)/\pi$$

$$E(l, L) \sim \gamma_E \ln l + C_E, \quad 1 << l << L$$

Similar as $S_{vN}(l, L)$ to the quantum case

Results

$$E(l,L) \sim \gamma_E \ln \tilde{L}_C + C_E, \quad \tilde{L}_C = L\sin(\pi l/L)/\pi$$

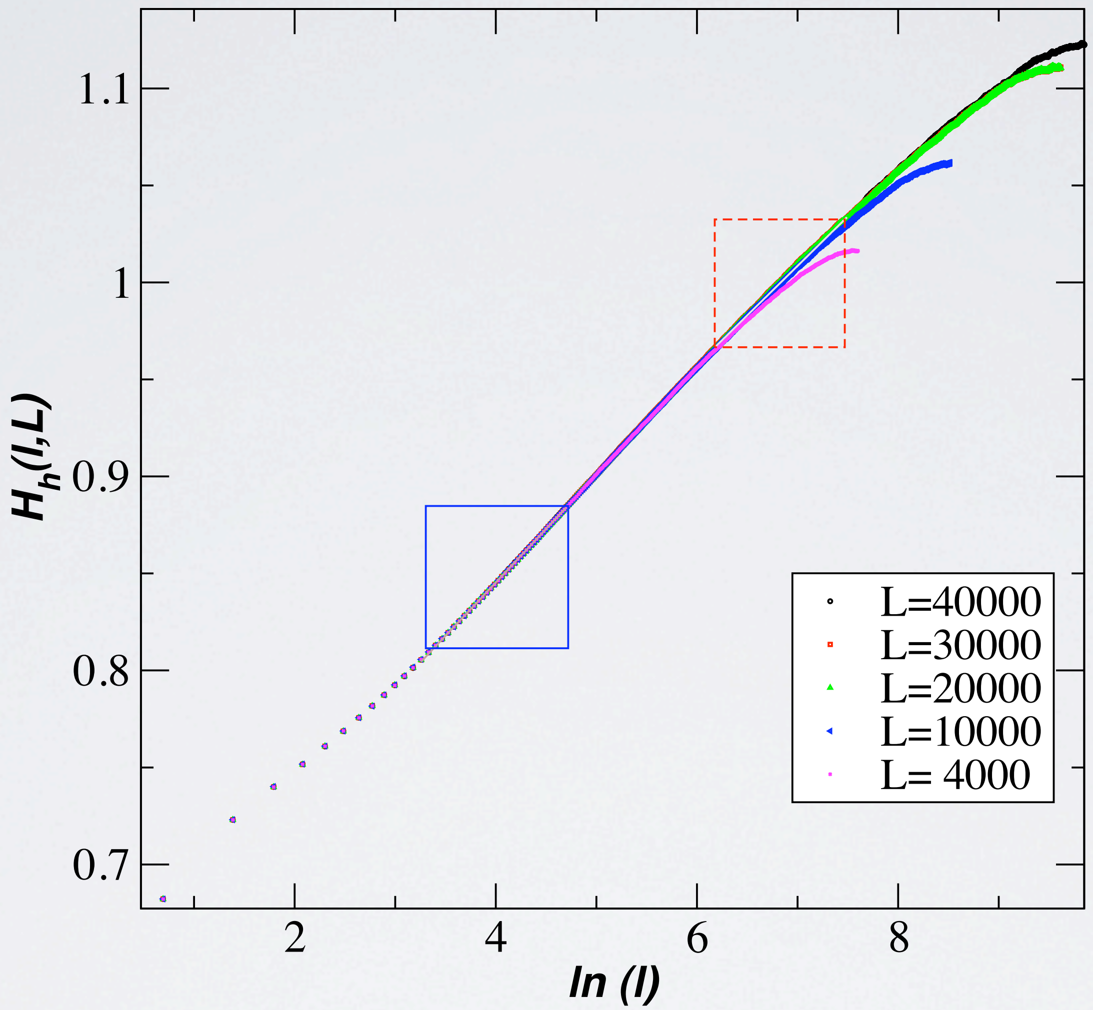
- Mutual information and separation Shannon entropy (not precise L up to 26)

  $\gamma_I = 0.07 \quad C_I = 0.65 \quad \gamma_S = 0.4 \quad C_S = 0.7$

- The other measures (precise)

$P_l(h,l)$ Evaluated by MC up to L =60000

★ Interdependency $H_h(l,L) = -\sum_n P_l(h,L)\ln P_l(h,L)$



$$H_h(l,L) \sim 0.050\ln l$$

★ Valence bond entanglement entropy

$$h(l,L) = \sum_h h P_l(h,L)$$

$$\gamma_h = 0.277 \quad C_h = 0.75$$

Related to a periodic model (Jacobsen and Saleur, 2008)

$$\gamma_h = \sqrt{3}/2\pi \approx 0.275$$

Second moment of $P_l(h,l)$

$$\kappa_2(l,L) \sim \beta_2 \ln \tilde{L}_C + b_2 \quad \beta_2 = 0.19, \quad b_2 = 0.25$$

$$\beta_2 = (2\pi\sqrt{3} - 9)/\pi^2 \approx 0.190767$$

half of the value of a related periodic model! (correct behavior)

$\kappa_2(l,L)$ also a possible estimator

★ **Density of contact points** $D(l,L) = -\ln P_l(o,L)$

$$\rho(l,L) \sim \frac{\alpha}{\tilde{L}_C^{1/3}}, \quad \alpha = -\frac{\sqrt{3}\Gamma(-\frac{1}{6})}{6\pi^{5/6}}$$ (FCA, Pyatov and Rittenberg, 2007

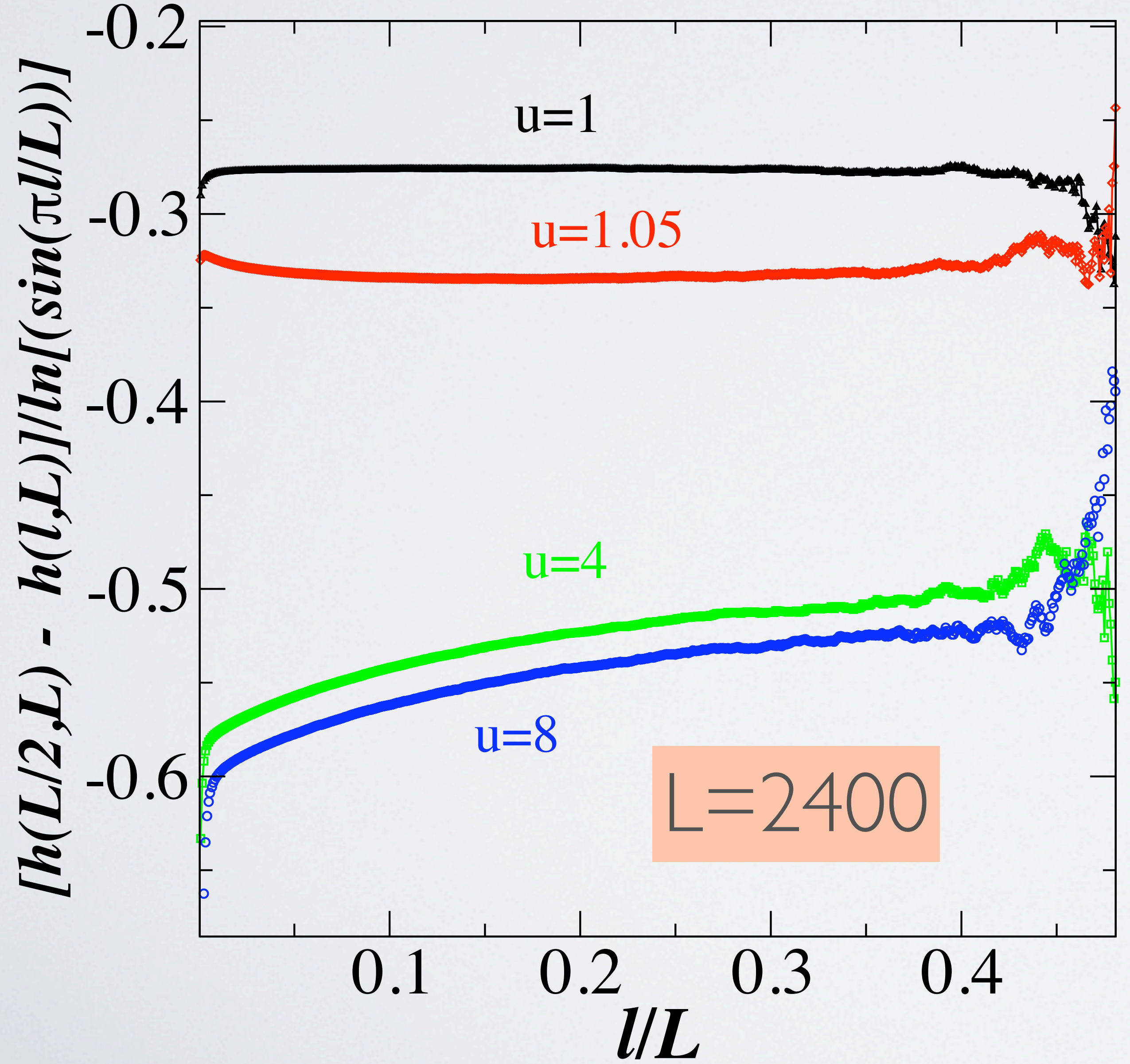
Local operator CFT

$$D(l,L) = -\ln \rho(l,L) \sim \frac{1}{3} \ln \tilde{L}_C + 0.28349$$

$u > 1$ Dynamical critical exponent $z < 1$

$$E(l, L) \sim \gamma_E \ln \tilde{L}_u + C_E$$

γ_E changes continuously with u



finite-size scaling $\tilde{L}_u(l, L)$ depends on u

Summary of results RPM

	u=1 γ_E	u=1 C_E	u=4 γ_E	u=4 C_E
Mutual Information	0.07	0.65	-	-
Interdependency	0.050	0.67	0.09	0.91
Rényi $(n = 2)$	0.05	0.39	0.06	0.09
Valence Bond Ent.	0.277	0.75	0.63	1.37
Dens. Contact Points	0.333	0.284	0.73	0.71
Separation Entropy	0.4	0.7	-	-

As u increases the shared information increases

Conclusions

- Estimators introduced for the shared information of subsystems in stationary states of one-dimensional Markov processes show universal behavior
- Some of the estimators can be evaluated by Monte Carlo simulation (distinct from the quantum case)
- Estimators with properties similar to the counterparts in the quantum case

a) Vanish if the subsystems are separated

b) If ξ finite $\longrightarrow$ are finite

c) If we have logarithmic behavior : $E(l,L) \sim \gamma_E \ln \tilde{L}_E(l,L) + C_E$

γ_E and $\tilde{L}_E$ are universal and C_E non universal

d)If we have power-law behavior: $E(l,L) \sim \gamma_E \tilde{L}^{\delta_E} + D_E$

γ_E , δ_E and $\tilde{L}_E$ are universal and D_E non universal

★ Other results:

- Stochastic process with a source at one end (asymmetric FSS functions)
- Even-odd sites finite-size corrections (Affleck et al 2009, Calabrese et al 2010, Song et al 2010, Cabrese and Cardy 2010)
- Other models (different configuration spaces)- Restricted Motzkin Paths - strongly constrained systems, larger values for γ_E , ASEP, etc.

Thank you

The End

CALCULATIONS OF SHARED INFORMATION IN CRITICAL AND NON-CRITICAL STOCHASTIC STATIONARY SYSTEMS

Francisco C. Alcaraz

Universidade de São Paulo - IFSC - São Carlos - BRAZIL

FCA, Rittenberg V and Sierra G - Phys. Rev. E **80**, 030102(R) (2009)
FCA and Rittenberg V - JSTAT P03024 (2010)

- Application 1: Model for polymer adsorption

- Aplication 2: The raise and peel model

- Conclusions