Soliton theories with ∞ Symmetry (APDiff) and stability (BPS)

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- Example: BPS Skyrme model
  - integrability, symmetries, exact solutions
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- **the baby Skyrme model** Piette, Schroers, Zakrzewski (95)
  - field variable $\vec{\phi} : \mathcal{M}_{2,1} \ni x \to \vec{\phi}(x) \in S^2$
    $\vec{\phi} = (\phi_1, \phi_2, \phi_3)$ and constraint $\vec{\phi}^2 = 1$
  - Lagrangian

$$L_{baby} = \frac{\nu^2}{2}(\partial_\mu \vec{\phi})^2 - \frac{\lambda^2}{4}(\partial_\alpha \vec{\phi} \times \partial_\beta \vec{\phi})^2 - \mu^2 V(\vec{n} \cdot \vec{\phi})$$

  - topology

    static finite energy solutions: $\vec{\phi} \to \vec{\phi}_\infty$
    isolated point-like vacuum: $V(\vec{n} \cdot \vec{\phi}_\infty) = 0$
    $\vec{\phi} : \mathbb{R}^2 \cup \{\infty\} \cong \mathbb{S}^2 \to \mathbb{S}^2 \Rightarrow \deg[\vec{\phi}] = Q \in \pi_2(\mathbb{S}^2)$
  - non-integrable, non-BPS, V-sensitive
  - application

    toy model for the Skyrme model
    condensed matter Adam, Naya,S. Guillen, Speight, Vazquez (in progr.)

- **the baby Skyrme model as sum of two BPS models**

$$L_{baby} = \underbrace{\frac{\nu^2}{2}(\partial_\mu \vec{\phi})^2}_{L_{O(3)} = L_2} - \underbrace{\frac{\lambda^2}{4}(\partial_\alpha \vec{\phi} \times \partial_\beta \vec{\phi})^2 - \mu^2 V(\vec{n} \cdot \vec{\phi})}_{L_{BPS} = L_4 + L_0}$$

- $L_{O(3)}$ - $O(3)$ σ-model
  - static sector integrable
  - (anti)-holomorphic exact solutions

- L_{BPS} - BPS baby Skyrme model Gisiger, Paranjape (97); Adam, Romanczukiewicz, Sanchez-Guillen, Wereszczynski (10); Speight (10)
  - no quadratic term
  - topological in nature

$$L_4 = -8\pi^2 \mathbb{B}_\alpha \mathbb{B}^\alpha, \quad \underbrace{\mathbb{B}_\alpha = (8\pi)^{-1} \epsilon_{\alpha\beta\gamma} \vec{\phi} \cdot (\partial^\beta \vec{\phi} \times \partial^\gamma \vec{\phi})}_{\text{topological current}}$$

- **the BPS baby Skyrme model**

$$L = -\frac{\lambda^2}{4}(\partial_\mu \vec{\phi} \times \partial_\nu \vec{\phi})^2 - \mu^2 \; V(\phi^3)$$

- symmetries
  - ∞ many target space symmetries: subgroup of SDiff($\mathbb{S}^2$)
    ∞ many conservation laws $\Rightarrow$ generalized integrability
    Ferreira, Alvarez, Sanchez-Guillen (98)

$$J_\mu = \frac{\delta G}{\delta \bar{u}} \mathcal{K}_\mu - \frac{\delta G}{\delta u} \bar{\mathcal{K}}_\mu, \quad \mathcal{K}^\mu = \frac{K^\mu}{(1+|u|^2)^2}, \quad K^\mu = (u_\nu \bar{u}^\nu)\bar{u}^\mu - \bar{u}_\nu^2 u^\mu$$

where $G = G(u\bar{u})$ and we use the stereographic projection

$$\vec{\phi} = \frac{1}{1+|u|^2}\left(u+\bar{u}, -i(u-\bar{u}), |u|^2 - 1\right)$$

Adam, Sanchez-Guillen, Wereszczynski (10)

  - static energy: ∞ many base space symmetries SDiff($\mathbb{S}^2$) $\Rightarrow$ symmetries of incompressible fluid Leznov, Piette, Zakrzewski (97)

- BPS bound Innocentis, Ward (01); Adam, Sanchez-Guillen, Wereszczynski (10); Speight (10)
  - energy functional

$$E = \frac{1}{2}\int d^2x \left(\lambda^2(\partial_1\vec{\phi}\times\partial_2\vec{\phi})^2 + 2\mu^2 V\right) = \frac{1}{2}\int d^2x(\lambda^2 q^2 + 2\mu^2 V)$$

where we use $\partial_1\vec{\phi}\times\partial_2\vec{\phi} = q\vec{\phi}$
$q \equiv \vec{\phi}\cdot(\partial_1\vec{\phi}\times\partial_2\vec{\phi})$ topological charge density

$$E = \frac{1}{2}\int d^2x \left(\lambda q \pm \mu\sqrt{2V}\right)^2 \mp \lambda\mu\int d^2xq\sqrt{2V} \geq \mp\lambda\mu\int d^2xq\sqrt{2V}$$

  - equality if BPS equation holds

$$\lambda q \pm \mu\sqrt{2V} = 0$$

  - bound is topological

$$d^2xq = \vec{\phi}^*(\Omega_{S^2}) \Rightarrow \int d^2xq\sqrt{2V} = 4\pi\mathrm{deg}[\vec{\phi}]\langle\sqrt{2V}\rangle_{\mathbb{S}^2}$$

area of $\mathbb{S}^2\times$ average of $\sqrt{2V(\phi_3)}$ on $\mathbb{S}^2\times$ times $\vec{\phi}$ covers $\mathbb{S}^2$
while $\vec{x}$ covers $\mathbb{R}^2$ once

  - energy bound

$$E \geq 4\pi\lambda\mu|\mathrm{deg}[\vec{\phi}]\langle\sqrt{2V}\rangle_{\mathbb{S}^2}|$$

- BPS $\Rightarrow$ full e.o.m.
- examples of exact solutions $V = 4\left(\frac{1-\phi^3}{2}\right)^s$

ansatz $u = \pm\sqrt{\frac{g}{1-g}}e^{in\varphi}$ and $g(r=0)=1,\ g(r=R)=g'(r=R)=0$

- $s \in (0,2)$ compactons

$$g(r) = \begin{cases} \left(1-\frac{r^2}{R^2}\right)^{\frac{2}{2-s}} & 0 \le r \le R = 2\sqrt{\frac{n}{\mu|2-s|}} \\ 0 & r \ge R \end{cases}$$

- $s = 2$

$$g(r) = e^{-\frac{\mu r^2}{2n}}$$

- $s > 2$

$$g(r) = \left(\frac{R^2}{R^2+r^2}\right)^{\frac{2}{s-2}}$$

- new perspective on the baby Skyrme model
  - improved topological bound (f. eg. $s = 1$)

$$E_{baby} = E_{O(3)} + E_{BPS} \ge 4\pi|Q|\left(\nu^2 + \frac{4}{3}\lambda\mu^2\right)$$

  - good approximation at strong coupling $\nu \to \infty$
  - exact solutions for the full model - the holomorphic potential and generalizations
  - higher charge baby skyrmions vs. compactons Karliner, Hen (08)

- baby skyrmion - vortex duality Adam, Sanchez-Guillen, Wereszczynski, Zakrzewski (13)
  - BPS baby Skyrme (in complex field)
$$L_{BPS\ baby} = -\lambda^2 \frac{K_\mu u^\mu}{(1+|u|^2)^4} - \mu^2 \left(\frac{|u|^2}{1+|u|^2} \right)^s$$
  - BPS vortex model with a Higgs type potential
$$L_{BPS\ vortex} = -\lambda^2 K_\mu u^\mu - \mu^2 \left(1 - |u|^2\right)^s$$
    - the same SDiff symetries
    - integrable and solvable with exact BPS solutions - vortices
      $V = 0 \ \Leftrightarrow \ |u| = 1$ i.e., $\mathcal{M}_{vac} \ \cong \mathbb{S}^1$
      $u_\infty \equiv \lim_{\vec{x}\to\infty} : \mathbb{S}^1 \to \mathbb{S}^1 \quad \Rightarrow$ top. charge $Q_v \in \pi_1(\mathbb{S}^1)$
  - duality map
$$u_b = \Sigma_b e^{i\Phi_b}, \qquad u_v = \Sigma_v e^{i\Phi_v}$$
$$\Sigma_v^2 = \frac{1}{1+\Sigma_b}, \qquad \Phi_v = \Phi_b$$
    - the models are equivalent: baby skyrmions described by vortices
    - survives after $U(1)$ gauging: abelian Higgs model with SDiff symmetry
    - also in higher dimensions $\Rightarrow$ skyrmions - monopoles duality
      Adam, Sanchez-Guillen, Wereszczynski, Zakrzewski (in progress)

- **the gauged BPS baby Skyrme model**

$$L = -\frac{\lambda^2}{4}(D_\alpha \vec{\phi} \times D_\beta \vec{\phi})^2 - \mu^2 V(\vec{n} \cdot \vec{\phi}) - \frac{1}{4g^2} F_{\alpha\beta}^2$$

Adam, Naya, Sanchez-Guillen, Wereszczynski (12)

covariant derivative $D_\alpha \vec{\phi} \equiv \partial_\alpha \vec{\phi} + A_\alpha \vec{n} \times \vec{\phi}$
natural as $V(\phi^3)$ has unbroken $U(1)$ Gladikowski, Piette, Schroers (96)

- symmetries
  - ∞ many target space symmetries: subgroup of SDiff($\mathbb{S}^2$)
    ∞ many conservation laws $\Rightarrow$ generalized integrability

$$J_\mu = \frac{\delta G}{\delta \bar{u}} \mathcal{K}_\mu - \frac{\delta G}{\delta u} \bar{\mathcal{K}}_\mu, \quad \mathcal{K}^\mu = \frac{K^\mu}{(1+|u|^2)^2}$$

$$K^\mu = (D_\nu u D^\nu \bar{u}) D^\mu \bar{u} - (D_\nu \bar{u})^2 D^\mu u$$

    where $G = G(u\bar{u})$
  - static energy: ∞ many base space symmetries SDiff($\mathbb{R}^2$)

- BPS bound - energy functional Adam, Naya, Sanchez-Guillen, Wereszczynski (12); Stepien (12)

$$E = \frac{1}{2}\int d^2x \left(\lambda^2(D_1\vec{\phi}\times D_2\vec{\phi})^2 + 2\mu^2 V(\vec{n}\cdot\vec{\phi}) + \frac{1}{g^2}B^2\right)$$

$$= \frac{1}{2}\int d^2x(\lambda^2 Q^2 + 2\mu^2 V + (1/g^2)B^2), \qquad Q \equiv \vec{\phi}\cdot(D_1\vec{\phi}\times D_2\vec{\phi}) = q + \epsilon_{ij}A_i\partial_j\phi_3$$

- consider

$$0 \leq \frac{1}{2}\int d^2x \left(\lambda^2(Q - w(\phi_3))^2 + \frac{1}{g^2}(B + b(\phi_3))^2\right)$$

$$\Downarrow$$

$$0 \leq \frac{1}{2}\int d^2x \Bigg(\lambda^2 Q^2 + \lambda^2 w^2 + \frac{1}{g^2}b^2 + \frac{1}{g^2}B^2 - 2\lambda^2 wq \underbrace{-2\lambda^2 w\epsilon_{ij}A_i\partial_j\phi_3 + \frac{2}{g^2}\epsilon_{ij}(\partial_i A_j)b}\Bigg)$$

$$b(\phi_3) = g^2\lambda^2 W(\phi_3) \equiv g^2\lambda^2\int_1^{\phi_3} d\phi w(\phi) \quad \Leftarrow \quad \text{total derivative}$$

$$\Downarrow$$

$$0 \leq \frac{1}{2}\int d^2x \left(\lambda^2 Q^2 + \underbrace{\lambda^2 W'^2 + g^2\lambda^4 W^2}_{2\mu^2 V} + \frac{1}{g^2}B^2 - \underbrace{2\lambda^2 W' q}_{\text{top.}}\right)$$

- the bound

$$E \geq \lambda^2 \int d^2x q W' = 4\pi\lambda^2 |\deg[\vec{\phi}] \langle W' \rangle_{\mathbb{S}^2}| = 2\pi\lambda^2 |\deg[\vec{\phi}] W(-1)|$$

- superpotential equation

$$\lambda^2 W'^2 + g^2 \lambda^4 W^2 = 2\mu^2 V(\phi_3)$$

  - equivalent to equation relating potential and superpotential in SUGRA Skenderis, Townsend (06); Trigiante, Van Riet, Vercnocke (12)
  - at vacuum $\phi_3 = 1$, $V(\phi_3 = 1) = 0 \;\Rightarrow\; W'(1) = 0, W(1) = 0$
  - we need globally existing W for $\phi_3 \in [-1, 1]$ - subtle problem
  - $V = 1 - \phi_3$ and $V = (1 - \phi_3)^2$ both W and solitons exist
  - for potentials with more than one vacuum, BPS solitons do not exist (but do exist in the ungauged case)

- the bound is saturated $\Rightarrow$ BPS equations

$$Q = W' \,, \quad B = -g^2 \lambda^2 W$$

  - the BPS equations $\Rightarrow$ full static e.o.m.
  - the BPS equations as gradient flow (axial ansatz) $\dot{q}^a = \frac{\partial \mathcal{W}(q)}{\partial q_a}$
    $q^a \leftrightarrow (\phi_3, B)$

- **the Skyrme model** Skyrme (61)
  - field variable $U : \mathcal{M}_{3,1} \ni x \to U(x) \in SU(2) \cong \mathbb{S}^3$
  - Lagrangian

$$L = \frac{f_\pi^2}{2} Tr(L_\mu L^\mu) - \frac{1}{32e^2} \underbrace{Tr[L_\mu, L_\nu]^2}_{\text{Skyrme term}} - \lambda^2 \pi^2 \underbrace{\mathbb{B}_\mu \mathbb{B}^\mu}_{\text{top. Skyrme term}} - \mu^2 \mathcal{V}(U, U^\dagger)$$

    - topological baryon current Witten (83)

$$\mathbb{B}^\mu = \frac{1}{24\pi^2} \epsilon^{\mu\nu\rho\sigma} Tr(L_\nu L_\rho L_\sigma), \quad L_\mu = U^\dagger \partial_\mu U$$

    - most general if: Lorentz inv., max. first time deriv. squared
    - standard Hamiltonian $\Rightarrow$ semiclassical quantization Adkins, Nappi, Witten (83)
  - topology
    - static finite energy solutions: $U \to U_\infty$
    - isolated point-like vacuum: $V(U_\infty) = 0$

$$U : \mathbb{R}^3 \cup \{\infty\} \cong \mathbb{S}^3 \to \mathbb{S}^3 \;\Rightarrow\; \deg[\vec{\phi}] = Q \in \pi_3(\mathbb{S}^3)$$

  - non-integrable, non-BPS, V-sensitive
    - complicated numerics Battye, Sutcliffe (97)...
    - $\mathcal{V} = 0$ - RMA/instanton holonomies Houghton, Manton, Sutcliffe (98)); Atiyah, Manton (89), (93)
    - $\mathcal{V} \neq 0$ - massless skyrmions in a curved space Atiyah, Sutcliffe (04); Dunajski (12)

- **perturbative vs. non-perturbative contribution**

$$L = \frac{f_\pi^2}{2} Tr(L_\mu L^\mu) - \frac{1}{32e^2} Tr[L_\mu, L_\nu]^2 - \lambda^2\pi^2 \mathbb{B}_\mu \mathbb{B}^\mu - \mu^2 \mathcal{V}(U, U^\dagger)$$

$\Downarrow$ kinetic, $\Downarrow$ two-body: perturbative surface contribution

$\Downarrow$ collective: non-perturbative volume contribution

- non-perturbative part (strong fields) described by the BPS Skyrme model
  - topological in nature
  - residual pion d.o.f. encoded in *U* and the potential
  - provides a BPS sector (solvable) of the full theory

- perturbative part
  - near vacuum behaviour
  - (far) interactions
  - spoils the BPS (solvability) property

- **the BPS Skyrme model**

$$L = -\lambda^2 \pi^2 \mathbb{B}_\mu \mathbb{B}^\mu - \mu^2 \mathcal{V}(U, U^\dagger)$$

Adam, Sanchez-Guillen, Wereszczynski (10)

useful decomposition $U(x) = e^{i\xi(x)\vec{n}(x)\cdot\vec{\tau}}$

$$L = \frac{\lambda^2 \sin^4 \xi}{(1+|u|^2)^4} \left(\epsilon^{\mu\nu\rho\sigma} \xi_\nu u_\rho \bar{u}_\sigma\right)^2 - \mu^2 \mathcal{V}(\xi)$$

- symmetries
  - ∞ many target space symmetries: subgroup of SDiff($\mathbb{S}^3$)
    • derivative term is SDiff($\mathbb{S}^3$) inv - square of the pull back of the target space volume form on $\mathbb{S}^3$

    $$dV = -i \frac{\sin^2 \xi}{(1+|u|^2)^2} d\xi du d\bar{u}$$

    • potential inv under SDiff($\mathbb{S}^3$): $\xi \to \xi, \quad u \to \tilde{u}(u, \bar{u}, \xi)$
    ∞ many conservation laws $\Rightarrow$ generalized integrability
  - static energy: ∞ many base space symmetries SDiff($\mathbb{R}^3$) $\Rightarrow$ symmetries of incompressible fluid

- BPS bound

$$E = \int d^3x \left(\frac{\lambda^2 \sin^4 \xi}{(1+|u|^2)^4} (\epsilon^{mnl} i \xi_m u_n \bar{u}_l)^2 + \mu^2 \mathcal{V} \right)$$

$$= \int d^3x \left(\frac{\lambda \sin^2 \xi}{(1+|u|^2)^2} \epsilon^{mnl} i \xi_m u_n \bar{u}_l \pm \mu \sqrt{\mathcal{V}} \right)^2 \mp \int d^3x \frac{2\mu\lambda \sin^2 \xi \sqrt{\mathcal{V}}}{(1+|u|^2)^2} \epsilon^{mnl} i \xi_m u_n \bar{u}_l$$

$$\geq \pm (2\lambda\mu\pi^2) \left[\frac{-i}{\pi^2} \int d^3x \frac{\sin^2 \xi \sqrt{\mathcal{V}}}{(1+|u|^2)^2} \epsilon^{mnl} \xi_m u_n \bar{u}_l \right] \equiv 2\lambda\mu\pi^2 \langle \sqrt{\mathcal{V}} \rangle_{S^3} |B|$$

  - the bound is saturated $\Rightarrow$ BPS equation
$$\frac{\lambda \sin^2 \xi}{(1+|u|^2)^2} \epsilon^{mnl} i \xi_m u_n \bar{u}_l = \mp \mu \sqrt{\mathcal{V}}$$
  - BPS $\Rightarrow$ full e.o.m.
  - the BPS equation has full SDIff($\tilde{S}^3$) symmetry - $ds^2 = d\xi^2 + \frac{\sin^2 \xi}{\sqrt{\mathcal{V}}} \frac{du d\bar{u}}{(1+|u|^2)^2}$

- example: the Skyrme potential $\mathcal{V} = 1 - \cos \xi$
  - ∞ many exact solutions - core-type compactons
$$u = \tan \frac{\theta}{2} e^{-i\phi}, \quad \xi = \begin{cases} 2 \arccos \frac{r}{R} & r \leq R \equiv \sqrt[3]{\frac{2\sqrt{2}|n|\lambda}{\mu}} \\ 0 & r \geq R \end{cases}$$
  - $E \sim |B|, \ R \sim |B|^{1/3}$
  - symmetry transformations $\Rightarrow$ new solutions with discrete sym. Adam, Fosco, Queiruga, Sanchez-Guillen, Wereszczynski (12)

application: baryons as (near) BPS objects

how to make skyrmions near BPS?

- extended model: $YM_5(SU(2))$ Sutcliffe (10)
  - ∞ many interacting vector mesons
  - if all included $\Rightarrow$ BPS (self-dual) configurations
  - no potential

- truncated model: the BPS Skyrme model Adam, Sanchez-Guillen, Wereszczynski (10)
  - other terms spoil the BPS property
  - potential term crucial

• solitons in the BPS Skyrme model as nuclei

- $E \sim |B|$
  - zero binding energy
  - $E_{BPS} \leq E_{experiment}$
  - other (perturbative) terms spoil the BPS property $\Rightarrow$ should be "small"
- compactons
  - $R \sim |B|^{1/3}$
  - finite range of interactions
  - long range interactions $\Rightarrow$ perturbative terms
- symmetries of incompressible fluid
  - solitonic realization of a liquid drop model of nuclei
  - broken by perturbative terms
- quenched type limit

- quantized $B = 1$ sector (fit to m_p, m_Δ)

radius	BPS Skyrme	massive Skyrme	experiment
compacton	0.897	-	-
$r_{e,0}$	0.635	0.68	0.72
$r_{e,1}$	0.669	1.04	0.88
$r_{m,0}$	0.710	0.95	0.81

ratios	BPS Skyrme	massive Skyrme	experiment
$r_{e,1}/r_{e,0}$	1.054	1.529	1.222
$r_{m,0}/r_{e,0}$	1.118	1.397	1.125
$r_{e,1}/r_{m,0}$	0.943	1.095	1.086

  - $r < r_{experiment} \Rightarrow$ no pion cloud outside of baryons
- $B > 1$ sector and the Coulomb interaction Adam, Naya, Sanchez-Guillen, Wereszczynski (in progress)

- new perspective on the Skyrme model
  - distinguishing between perturbative vs non-perturbative terms
    - non-perturbative i.e., collective: $\mathbb{B}_\mu^2$ and potential
      - solvable, SDiff BPS part
      - liquid drop model
    - perturbative
      - near vacuum
      - long range of interactions and pion cloud
      - SDiff symmetry breaking
  - role of the potential
    - new approximate methods for the Skyrme + potential (?)
    - strongly coupled limit (?)
      - restoration of SDiff symmetry
      - melting of the Skyrme crystal (?)
  - new starting point for the effective action (??)
  - new large N_c limit (??)
    - SDiff are symmetries of YM at $N_c \to \infty$

• why interesting?

- new soliton models
  - solvable, integrable
    - ∞ many exact, topologically nontrivial solutions
  - BPS bound - saturated
  - rich mathematical structure
    - fake supersymmetry
    - gradient flow
- new perspective on the standard soliton models

- SDiff symmetry very important
  - $SU(N_c) \to$SDiff (Σ) as $N_c \to \infty$ Goldstonde, Hoppe
    - new large N_c limit (?)
  - symmetry of membranes
  - tensionless strings
  - higher dimensional integrability (?)

- **Concluding Conjecture**

  Typical integrable models in (2+0), (1+1) are certain deformations of conformal theories (Zamolodchikov).

  - Virasoro algebra naturally included although APD and conformal groups are very different. Babelon, Ferreira
  - ∞-dim algebra (group) for any dimensions (Volume Preserving Diff). Ferreira, Razumov

$$\Downarrow$$

APDiff one alternative to standard 2d Integrability in higher dimensions?
A symmetry of large N_c (or beyond) effective action?