Lista 2 - Quântica B (2013)

1. Quantization of the electromagnetic field
Consider the mode expansion of the vector potential (in the Schrödinger representation)

$$\mathbf{A}(\mathbf{r}) = \sqrt{\frac{\hbar}{2\epsilon_0 V}} \sum_{\mathbf{k}} \sum_{\lambda=\pm} \sqrt{\frac{1}{\omega_k}} a_{\mathbf{k},\lambda} e^{i\mathbf{k}\cdot\mathbf{r}} \hat{e}_{\mathbf{k},\lambda} + \text{h.c.},$$

$$\dot{\mathbf{A}}(\mathbf{r}) = -i\sqrt{\frac{\hbar}{2\epsilon_0 V}} \sum_{\mathbf{k}} \sum_{\lambda=\pm} \sqrt{\omega_k} a_{\mathbf{k},\lambda} e^{i\mathbf{k}\cdot\mathbf{r}} \hat{e}_{\mathbf{k},\lambda} + \text{h.c.},$$

where $a^\dagger_{\mathbf{k},\lambda}$ ($a_{\mathbf{k},\lambda}$) are creation (annihilation) operators of photons the wavevector and polarization of which are respectively $\mathbf{k} = k\left(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta\right)$ and λ, $\omega_k = ck$ is their angular frequency, and $\hat{e}_{\mathbf{k},\pm}$ are the polarization vectors

$$\begin{aligned}
\hat{e}_{\mathbf{k},1} &= \left(\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta\right), \\
\hat{e}_{\mathbf{k},2} &= \left(-\sin\phi, \cos\phi, 0\right), \\
\hat{e}_{\mathbf{k},\pm} &= \frac{1}{\sqrt{2}}\left(\hat{e}_{\mathbf{k},1} \pm i\hat{e}_{\mathbf{k},1}\right).
\end{aligned}$$

(a) Show that $\nabla\cdot\mathbf{A} = 0$. What is the physical interpretation of this result?
(b) Show that the angular momentum

$$\begin{aligned}
\mathbf{L} &= \frac{1}{\mu_0 c^2}\int \mathrm{d}^3 r\mathbf{r}\times(\mathbf{E}\times\mathbf{B}) = \mathbf{L}^{(o)} + \mathbf{L}^{(s)}, \text{ with} \\
\mathbf{L}^{(o)} &= \frac{1}{\mu_0 c^2}\int \mathrm{d}^3 r \sum_{i=1}^{3} E_i\left(\vec{\ell}A_i\right), \text{ with } \vec{\ell}\psi = -i\mathbf{r}\times\nabla\psi, \\
\mathbf{L}^{(s)} &= \frac{1}{\mu_0 c^2}\int \mathrm{d}^3 r \mathbf{E}\times\mathbf{A}.
\end{aligned}$$

Hint: It is convenient to use techniques of tensor calculus, in particular the Levi-Civita antisymmetrical tensor ε_{ijk}: $\varepsilon_{ijk} = 0$ if $i = j$, or $i = k$, or , $j = k$; $\varepsilon_{ijk} = 1$ if (ijk) equals (123) or any cyclic permutation of these indices, and $\varepsilon_{ijk} = -1$ otherwise. In addition, use the "contract epsilon identy" $\sum_{k=1}^{3}\varepsilon_{ijk}\varepsilon_{klm} = \delta_{i,l}\delta_{j,m} - \delta_{i,m}\delta_{j,l}$. Then show that

$$\left[\mathbf{r}\times(\mathbf{E}\times\mathbf{B})\right]_i = \sum_{j,k,l} E_l\left(\varepsilon_{ijk}x_j\frac{\partial}{\partial x_k}A_l\right) - \sum_{j,k,l}\frac{\partial}{\partial x_l}\left(\varepsilon_{ijk}x_j E_l A_k\right) + \sum_{j,k}\varepsilon_{ijk}E_j A_k.$$

Recall that $\nabla\cdot\mathbf{E} = 0$, $(\mathbf{a}\times\mathbf{b})_i = \sum_{j,k}\varepsilon_{ijk}a_j b_k$ and $(\nabla\times\mathbf{b})_i = \sum_{j,k}\varepsilon_{ijk}\frac{\partial}{\partial x_j}b_k$. Finally, use the boundary conditions that the fields vanish when $r\to\infty$.
(c) Show that

$$\mathbf{L}^{(s)} = \frac{\epsilon_0 i}{\hbar}\int \mathrm{d}^3 r\mathbf{E}\cdot S\cdot\mathbf{A},$$

with S being 3×3 matrices satisfing angular momentum commutation relations and having eigenvalues $0, \pm\hbar$.
(d) Show and give the physical interpretation of the result

$$\mathbf{L}^{(s)} = \sum_{\mathbf{k}}\hbar\left(a^\dagger_{\mathbf{k},+}a_{\mathbf{k},+} - a^\dagger_{\mathbf{k},-}a_{\mathbf{k},-}\right)\hat{k}.$$

(e) Write $\mathbf{A}$, $\mathbf{E}$ and $\mathbf{B}$ in the Heisenberg representation. (Consider the free-field Hamiltonian $H = \sum_{\mathbf{k},\lambda}\hbar\omega_{k,\lambda}a^\dagger_{\mathbf{k},\lambda}a_{\mathbf{k},\lambda}$, and ignore the zero-point energy.) Compute the commutation relations $[A_i(\mathbf{r},t), A_j(\mathbf{r}',t')]$, $[E_i(\mathbf{r},t), E_j(\mathbf{r}',t')]$, $[A_i(\mathbf{r},t), E_j(\mathbf{r}',t')]$, and $[E_i(\mathbf{r},t), B_j(\mathbf{r}',t')]$? Give a physical consequence of latter one.

(f) Do $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ commute with the *total* photon number operator

$$N(\mathbf{r},t) = \sum_{\mathbf{k},\lambda} a^{\dagger}_{\mathbf{k},\lambda} a_{\mathbf{k},\lambda} ?$$

Interpret or give a physical consequenc of your result.

(g) Consider a coherent state of photons with momentum $\mathbf{p} = \hbar\mathbf{k}$ and helicity λ given by

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^{\dagger}_{\mathbf{k},\lambda}} |0\rangle ,$$

where $|0\rangle$ is the vacuum state and α is a scalar. Compute the time evolution of $\Delta X = \sqrt{\langle X^2\rangle - \langle X\rangle^2}$ and $\Delta P = \sqrt{\langle P^2\rangle - \langle P\rangle^2}$ where $X = \sqrt{\frac{\hbar}{2\omega_k}} \left(a^{\dagger}_{\mathbf{k},\lambda} + a_{\mathbf{k},\lambda}\right)$ and $P = i\sqrt{\frac{\hbar\omega_k}{2}} \left(a^{\dagger}_{\mathbf{k},\lambda} - a_{\mathbf{k},\lambda}\right)$ are position and momentum operators of the associated harmonic oscillator, respectively.

(h) Show that the Schrödinger equation $i\hbar\frac{\partial}{\partial t}|\alpha(t)\rangle = H|\alpha(t)\rangle$ has a solution $|\alpha(t)\rangle = |\beta\rangle$, where $\beta = \alpha e^{-i\omega_k t}$. (Ignore the zero-point energy.) Now compute $\langle\alpha(t)|\mathbf{A}|\alpha(t)\rangle$. (Discuss your result relating it with classical electromagnetic waves such as laser.)

2. Interaction between matter and radiation: emission and absorption

(a) Consider a structureless free quantum particle in the infinity space. Show that this particle cannot spontaneously emit a single photon. Physically, why this is the case? *Hint*: Use that the initial and final states of the free particle have well define momenta and that the dispersion relation for the particle is quadratic while for the photon it is linear.

(b) Consider the decay of the Hydrogen atom (fixed in space) in state $|2,1,1\rangle$. Compute the amplitude of the decay using plane waves for photons, and explain the angular dependence of the amplitude for each helicity ± 1 of the final-state photon in terms of the angular momentum conservation. Show that the rate is the same as the decay rate of the $|2,1,0\rangle$ state.

(c) (Optional) Compare the previous decay rate with the case of a free Hydrogen atom, *i.e.*, for the case of a finite-mass proton. Without doing any calculation, in which case do you expect the transition rate to be larger? Justify.

(d) How can the 2s state decay to the 1s state? There is no need in computing it, but discuss in detail. Discuss about the electric and magnetic dipolar transitions. Discuss about the decay route 2s →2p→1s. (Recall that due to Lamb shift splitting, 2s and 2p are not degenerate.) (Optional) Compute this amplitude transition (see *Advanced Quantum Mechanics*, J. J. Sakurai, problem 2.6).

3. Consider the Jaynes- Cummings Hamiltonian given by

$$H = \hbar\omega a^{\dagger}a + \frac{1}{2}\hbar\omega_0\sigma^z + \frac{1}{2}\hbar\Omega\left[a^{\dagger}\left(\sigma^x - i\sigma^y\right) + a\left(\sigma^x + i\sigma^y\right)\right].$$

The creation and anihillation operators $a^{\dagger}$ and a act on the radiation field while the Pauli matrices $\sigma^{x,y,z}$ act on the matter. ω, ω_0 and Ω are constants (frequencies).

(a) Give a detailed physical interpretation of each term in the Hamiltonian.

(b) Compute all the Eigenenergies and Eigenvectors of H. (They are called dressed states of the matter.)

(c) Consider now that the system is prepared in the state $|\psi_0\rangle = \sum_n C_n |n\rangle_{\text{radiation}} \otimes |0\rangle_{\text{matter}}$, with $C_1 = C_2$ and all others $C_i = 0$. Compute the probability of finding the two-level system in the excited state as a function of time.

4. Interaction between matter and radiation: scattering

We are interested in the scattering process in which the initial and final states are

$$|I\rangle = |i\rangle \otimes |n_{\mathbf{k},\lambda}, 0_{\mathbf{k}',\lambda'}\rangle , \text{ and } |F\rangle = |f\rangle \otimes \left|(n-1)_{\mathbf{k},\lambda}, 1_{\mathbf{k}',\lambda'}\right\rangle ,$$

i.e., in the beginning, there are n photons of momentum $\hbar\mathbf{k}$ and polarization λ while, in the end, there is one less photon in such state which was scattered into a photon of momentum $\hbar\mathbf{k}'$ and polarization λ'. Such process involves two photons and have contribution in second order of perturbation theory from the term $\frac{e}{m}\sum_i \mathbf{p}_i \cdot \mathbf{A}(\mathbf{r}_i)$ (where $\mathbf{p}_i$ are the momentum of the i-th electron in the system), and contribution in first order in perturbation theory from the diamagnetic term $V = \frac{e^2}{2m}\sum_i \mathbf{A}(\mathbf{r}_i)\cdot\mathbf{A}(\mathbf{r}_i)$. Here, consider only the effects of this latter term.

(a) Rewrite V in terms of the density operator $\rho(\mathbf{r})$.

(b) Compute the matrix element $\langle I|V|F\rangle$.

(c) Compute the differential cross section and show that

$$\frac{\mathrm{d}\sigma_{I\to F}}{\mathrm{d}\Omega} = r_0^2 \frac{\omega}{\omega'} \left|\hat{\epsilon}_{\mathbf{k},\lambda} \cdot \hat{\epsilon}_{\mathbf{k}',\lambda'}\right|^2 \left|\langle f \left|\tilde{\rho}(\mathbf{k} - \mathbf{k}')\right| i\rangle\right|^2 ,$$

where $r_0 = \frac{e^2}{4\pi\epsilon_0 mc^2}$ is the classical radius of the electron, and $\tilde{\rho}(\mathbf{k})$ is the Fourier transform of $\rho\left(\mathbf{r}\right)$.

(d) Consider the simplest case of the scattering by a single free electron in which $|i\rangle = |\hbar\mathbf{q}_i\rangle$ and $|f\rangle = |\hbar\mathbf{q}_f\rangle$ and compute the corresponding differential cross section (dubbed the Thomson cross section). Explain why this process is allowed.