Lista 2 - Quântica B (2013)

1. Quantization of the electromagnetic field
Consider the mode expansion of the vector potential (in the Schrödinger representation)

$$\mathbf{A}(\mathbf{r}) = \sqrt{\frac{\hbar}{2\epsilon_0 V}} \sum_{\mathbf{k}} \sum_{\lambda=\pm} \sqrt{\frac{1}{\omega_k}} a_{\mathbf{k},\lambda} e^{i\mathbf{k}\cdot\mathbf{r}} \hat{e}_{\mathbf{k},\lambda} + \text{h.c.},$$
$$\dot{\mathbf{A}}(\mathbf{r}) = -i\sqrt{\frac{\hbar}{2\epsilon_0 V}} \sum_{\mathbf{k}} \sum_{\lambda=\pm} \sqrt{\omega_k} a_{\mathbf{k},\lambda} e^{i\mathbf{k}\cdot\mathbf{r}} \hat{e}_{\mathbf{k},\lambda} + \text{h.c.},$$

where $a^\dagger_{\mathbf{k},\lambda}$ ($a_{\mathbf{k},\lambda}$) are creation (annihilation) operators of photons the wavevector and polarization of which are respectively $\mathbf{k} = k\left(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta\right)$ and λ, $\omega_k = ck$ is their angular frequency, and $\hat{e}_{\mathbf{k},\pm}$ are the polarization vectors

$$\hat{e}_{\mathbf{k},1} = \left(\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta\right),$$
$$\hat{e}_{\mathbf{k},2} = \left(-\sin\phi, \cos\phi, 0\right),$$
$$\hat{e}_{\mathbf{k},\pm} = \frac{1}{\sqrt{2}}\left(\mp\hat{e}_{\mathbf{k},1} - i\hat{e}_{\mathbf{k},2}\right).$$

(a) Show that $\nabla\cdot\mathbf{A} = 0$. What is the physical interpretation of this result?
(b) Show that the angular momentum

$$\mathbf{L} = \frac{1}{\mu_0 c^2}\int \mathrm{d}^3 r \mathbf{r}\times\left(\mathbf{E}\times\mathbf{B}\right) = \mathbf{L}^{(o)} + \mathbf{L}^{(s)}, \text{ with}$$
$$\mathbf{L}^{(o)} = \frac{1}{\mu_0 c^2}\int \mathrm{d}^3 r \sum_{i=1}^{3} E_i\left(\vec{\ell}A_i\right), \text{ with } \vec{\ell}\psi = \mathbf{r}\times\nabla\psi,$$
$$\mathbf{L}^{(s)} = \frac{1}{\mu_0 c^2}\int \mathrm{d}^3 r \mathbf{E}\times\mathbf{A}.$$

Hint: It is convenient to use techniques of tensor calculus, in particular the Levi-Civita antisymmetrical tensor ε_{ijk}: $\varepsilon_{ijk} = 0$ if $i = j$, or $i = k$, or , $j = k$; $\varepsilon_{ijk} = 1$ if (ijk) equals (123) or any cyclic permutation of these indices, and $\varepsilon_{ijk} = -1$ otherwise. In addition, use the "contract epsilon identy" $\sum_{k=1}^{3}\varepsilon_{ijk}\varepsilon_{klm} = \delta_{i,l}\delta_{j,m} - \delta_{i,m}\delta_{j,l}$. Then show that

$$\left[\mathbf{r}\times\left(\mathbf{E}\times\mathbf{B}\right)\right]_i = \sum_{j,k,l} E_l\left(\varepsilon_{ijk}x_j\frac{\partial}{\partial x_k}A_l\right) - \sum_{j,k,l}\frac{\partial}{\partial x_l}\left(\varepsilon_{ijk}x_j E_l A_k\right) + \sum_{j,k}\varepsilon_{ijk}E_j A_k.$$

Recall that $\nabla\cdot\mathbf{E} = 0$, $\left(\mathbf{a}\times\mathbf{b}\right)_i = \sum_{j,k}\varepsilon_{ijk}a_j b_k$ and $\left(\nabla\times\mathbf{b}\right)_i = \sum_{j,k}\varepsilon_{ijk}\frac{\partial}{\partial x_j}b_k$. Finally, use the boundary conditions that the fields vanish when $r\to\infty$.
(c) Show that

$$\mathbf{L}^{(s)} = \frac{\epsilon_0 i}{\hbar}\int \mathrm{d}^3 r \mathbf{E}\cdot S\cdot\mathbf{A},$$

with S being 3×3 matrices satisfing angular momentum commutation relations and having eigenvalues 0, $\pm\hbar$.
(d) Show and give the physical interpretation of the result

$$\mathbf{L}^{(s)} = \sum_{\mathbf{k}}\hbar\left(a^\dagger_{\mathbf{k},+}a_{\mathbf{k},+} - a^\dagger_{\mathbf{k},-}a_{\mathbf{k},-}\right)\hat{k}.$$

(e) Write $\mathbf{A}$, $\mathbf{E}$ and $\mathbf{B}$ in the Heisenberg representation. (Consider the free-field Hamiltonian $H = \sum_{\mathbf{k},\lambda}\hbar\omega_{k,\lambda}a^\dagger_{\mathbf{k},\lambda}a_{\mathbf{k},\lambda}$, and ignore the zero-point energy.) Compute the commutation relations $[A_i(\mathbf{r},t), A_j(\mathbf{r}',t')]$, $[E_i(\mathbf{r},t), E_j(\mathbf{r}',t')]$, $[A_i(\mathbf{r},t), E_j(\mathbf{r}',t')]$, and $[E_i(\mathbf{r},t), B_j(\mathbf{r}',t')]$? Give a physical consequence of latter one.

(f) Do $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ commute with the *total* photon number operator

$$N(t) = \sum_{\mathbf{k},\lambda} a^\dagger_{\mathbf{k},\lambda} a_{\mathbf{k},\lambda}?$$

Interpret or give a physical consequence of your result.

(g) Consider a coherent state of photons with momentum $\mathbf{p} = \hbar\mathbf{k}$ and helicity λ given by

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^\dagger_{\mathbf{k},\lambda}} |0\rangle \, ,$$

where $|0\rangle$ is the vacuum state and α is a scalar. Compute the time evolution of $\Delta X = \sqrt{\langle X^2\rangle - \langle X\rangle^2}$ and $\Delta P = \sqrt{\langle P^2\rangle - \langle P\rangle^2}$ where $X = \sqrt{\frac{\hbar}{2\omega_k}}\left(a^\dagger_{\mathbf{k},\lambda} + a_{\mathbf{k},\lambda}\right)$ and $P = i\sqrt{\frac{\hbar\omega_k}{2}}\left(a^\dagger_{\mathbf{k},\lambda} - a_{\mathbf{k},\lambda}\right)$ are position and momentum operators of the associated harmonic oscillator, respectively.

(h) Show that the Schrödinger equation $i\hbar\frac{\partial}{\partial t}|\alpha(t)\rangle = H|\alpha(t)\rangle$ has a solution $|\alpha(t)\rangle = |\beta\rangle$, where $\beta = \alpha e^{-i\omega_k t}$. (Ignore the zero-point energy.) Now compute $\langle\alpha(t)|\mathbf{A}|\alpha(t)\rangle$. (Discuss your result relating it with classical electromagnetic waves such as laser.)

2. Interaction between matter and radiation: emission and absorption

(a) Consider a structureless free quantum particle in the infinity space. Show that this particle cannot spontaneously emit a single photon. Physically, why this is the case? *Hint*: Use that the initial and final states of the free particle have well define momenta and that the dispersion relation for the particle is quadratic while for the photon it is linear.

(b) Consider the spontaneous decay of the Hydrogen atom (fixed in space) in state $|2,1,1\rangle$. Compute the amplitude of the decay using plane waves for photons, and explain the angular dependence of the amplitude for each helicity ±1 of the final-state photon in terms of the angular momentum conservation. Show that the rate is the same as the decay rate of the $|2,1,0\rangle$ state.

(c) (Optional) Compare the previous decay rate with the case of a free Hydrogen atom, *i.e.*, for the case of a finite-mass proton. Without doing any calculation, in which case do you expect the transition rate to be larger? Justify.

(d) How can the 2s state decay to the 1s state? There is no need in computing it, but discuss in detail. Discuss about the electric and magnetic dipolar transitions. Discuss about the decay route 2s →2p→1s. (Recall that due to Lamb shift splitting, 2s and 2p are not degenerate.) (Optional) Compute this amplitude transition (see *Advanced Quantum Mechanics*, J. J. Sakurai, problem 2.6).

3. Consider the Jaynes- Cummings Hamiltonian given by

$$H = \hbar\omega a^\dagger a + \frac{1}{2}\hbar\omega_0\sigma^z + \frac{1}{2}\hbar\Omega\left[a^\dagger\left(\sigma^x - i\sigma^y\right) + a\left(\sigma^x + i\sigma^y\right)\right].$$

The creation and anihillation operators $a^\dagger$ and a act on the radiation field while the Pauli matrices $\sigma^{x,y,z}$ act on the matter. ω, ω_0 and Ω are constants (frequencies).

(a) Give a detailed physical interpretation of each term in the Hamiltonian.

(b) Compute all the Eigenenergies and Eigenvectors of H. (They are called dressed states of the matter.)

(c) Consider now that the system is prepared in the state $|\psi_0\rangle = \sum_n C_n |n\rangle_{\text{radiation}} \otimes |0\rangle_{\text{matter}}$, with $C_1 = C_2$ and all others $C_i = 0$. Compute the probability of finding the two-level system in the excited state as a function of time.

4. Interaction between matter and radiation: scattering

We are interested in the scattering process in which the initial and final states are

$$|I\rangle = |i\rangle \otimes |n_{\mathbf{k},\lambda}, 0_{\mathbf{k}',\lambda'}\rangle \, , \text{ and } \; |F\rangle = |f\rangle \otimes \left|(n-1)_{\mathbf{k},\lambda}, 1_{\mathbf{k}',\lambda'}\right\rangle ,$$

i.e., in the beginning, there are n photons of momentum $\hbar\mathbf{k}$ and polarization λ while, in the end, there is one less photon in such state which was scattered into a photon of momentum $\hbar\mathbf{k}'$ and polarization λ'. Such process involves two photons and have contribution in second order of perturbation theory from the term $\frac{e}{m}\sum_i \mathbf{p}_i\cdot\mathbf{A}(\mathbf{r}_i)$ (where $\mathbf{p}_i$ are the momentum of the i-th electron in the system), and contribution in first order in perturbation theory from the diamagnetic term $V = \frac{e^2}{2m}\sum_i \mathbf{A}(\mathbf{r}_i)\cdot\mathbf{A}(\mathbf{r}_i)$. Here, consider only the effects of this latter term.

(a) Rewrite V in terms of the density operator $\rho(\mathbf{r})$.

(b) Compute the matrix element $\langle I|V|F\rangle$.

(c) Compute the differential cross section and show that

$$\frac{\mathrm{d}\sigma_{I\to F}}{\mathrm{d}\Omega} = r_0^2 \frac{\omega}{\omega'} \left|\hat{\epsilon}_{\mathbf{k},\lambda} \cdot \hat{\epsilon}_{\mathbf{k}',\lambda'}\right|^2 \left|\langle f \left|\tilde{\rho}(\mathbf{k}-\mathbf{k}')\right| i\rangle\right|^2,$$

where $r_0 = \frac{e^2}{4\pi\epsilon_0 mc^2}$ is the classical radius of the electron, and $\tilde{\rho}(\mathbf{k})$ is the Fourier transform of $\rho\left(\mathbf{r}\right)$.

(d) Consider the simplest case of the scattering by a single free electron in which $|i\rangle = |\hbar\mathbf{q}_i\rangle$ and $|f\rangle = |\hbar\mathbf{q}_f\rangle$ and compute the corresponding differential cross section (dubbed the Thomson cross section). Explain why this process is allowed.

ANSWER:
1.
(a) The spatial dependence appears only in the exponential:

$$\nabla\cdot\left(\hat{e}e^{\pm i\mathbf{k}\cdot\mathbf{r}}\right)=\pm i\hat{e}\cdot\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{r}}.$$

Moreover $\hat{e}_{\mathbf{k},1}\cdot\mathbf{k}=\hat{e}_{\mathbf{k},2}\cdot\mathbf{k}=0$. Since the circular polarized vectors are combinations of $\hat{e}_{\mathbf{k},1}$ and $\hat{e}_{\mathbf{k},2}$, then $\nabla\cdot\mathbf{A}=0$. Thus, this vector potential satisfy the transverse gauge. The physical interpretation is that the propagation direction $\mathbf{k}$ is perpendicular to the polarization, i.e., the light is a transverse wave.
(b) Lets use the identity

$$[\mathbf{r}\times(\mathbf{E}\times\mathbf{B})]_i=\sum_{j,k,l}E_l\left(\varepsilon_{ijk}x_j\frac{\partial}{\partial x_k}A_l\right)-\sum_{j,k,l}\frac{\partial}{\partial x_l}\left(\varepsilon_{ijk}x_jE_lA_k\right)+\sum_{j,k}\varepsilon_{ijk}E_jA_k. \tag{1}$$

Then, integrating over space,

$$\int d^3r\,[\mathbf{r}\times(\mathbf{E}\times\mathbf{B})]_i=\int d^3r\sum_{j,k,l}E_l\left(\varepsilon_{ijk}x_j\frac{\partial}{\partial x_k}A_l\right)-\int d^2r\sum_{j,k,l}\left(\varepsilon_{ijk}x_jE_lA_k\right)\Big|_{x_l=-\infty}^{x_l=+\infty}+\int d^3r\sum_{j,k}\varepsilon_{ijk}E_jA_k.$$

Notice that $\left(\varepsilon_{ijk}x_jE_lA_k\right)\big|_{x_i=-\infty}^{x_i=+\infty}=0$, since $x_i\neq x_j$ (guaranteed by ε_{ijk}) and that the fields vanish at $r\to\infty$. Finally, noticing that $\left[E_l\left(\mathbf{r}\times\nabla\right)A_l\right]_i=\sum_{j,k}\left[E_l\left(\varepsilon_{ijk}x_j\partial_k\right)A_l\right]_i$, we arrive at the final result

$$\int d^3r\,[\mathbf{r}\times(\mathbf{E}\times\mathbf{B})]=\int d^3r\sum_{i=1}^{3}E_i\left(\vec{\ell}A_i\right)+\int d^3r\mathbf{E}\times\mathbf{A}.$$

We now switch to the prove of Eq. (1):

$$\begin{aligned}
[\mathbf{r}\times(\mathbf{E}\times\mathbf{B})]_i&=\sum_{j,k}\varepsilon_{ijk}x_j\left(\mathbf{E}\times\mathbf{B}\right)_k=\sum_{j,k}\varepsilon_{ijk}x_j\sum_{l,m}\varepsilon_{klm}E_lB_m=\sum_{j,k}\varepsilon_{ijk}x_j\sum_{l,m}\varepsilon_{klm}E_l\sum_{n,o}\varepsilon_{mno}\frac{\partial}{\partial x_n}A_o\\
&=\sum_{j,k}\varepsilon_{ijk}x_j\sum_{l,n,o}E_l\frac{\partial}{\partial x_n}A_o\sum_m\varepsilon_{klm}\varepsilon_{mno}=\sum_{j,k}\varepsilon_{ijk}x_j\sum_{l,n,o}E_l\frac{\partial}{\partial x_n}A_o\left(\delta_{k,n}\delta_{l,o}-\delta_{k,o}\delta_{l,n}\right)\\
&=\sum_{j,k,l}\varepsilon_{ijk}x_jE_l\frac{\partial}{\partial x_k}A_l-\sum_{j,k,l}\varepsilon_{ijk}x_jE_l\frac{\partial}{\partial x_l}A_k\\
&=\sum_{j,k,l}\varepsilon_{ijk}x_jE_l\frac{\partial}{\partial x_k}A_l-\sum_{j,k,l}\varepsilon_{ijk}x_j\frac{\partial}{\partial x_l}\left(E_lA_k\right)+\sum_{j,k,l}\varepsilon_{ijk}x_j\left(\frac{\partial E_l}{\partial x_l}\right)A_k\\
&=\sum_{j,k,l}\varepsilon_{ijk}x_jE_l\frac{\partial}{\partial x_k}A_l-\sum_{j,k,l}\varepsilon_{ijk}x_j\frac{\partial}{\partial x_l}\left(E_lA_k\right)+\sum_{j,k}\varepsilon_{ijk}x_jA_k\nabla\cdot\mathbf{E}\\
&=\sum_{j,k,l}\varepsilon_{ijk}x_jE_l\frac{\partial}{\partial x_k}A_l-\sum_{j,k,l}\varepsilon_{ijk}\frac{\partial}{\partial x_l}\left(x_jE_lA_k\right)+\sum_{j,k,l}\varepsilon_{ijk}E_lA_k\left(\frac{\partial x_j}{\partial x_l}\right)\\
&=\sum_{j,k,l}\varepsilon_{ijk}x_jE_l\frac{\partial}{\partial x_k}A_l-\sum_{j,k,l}\varepsilon_{ijk}\frac{\partial}{\partial x_l}\left(x_jE_lA_k\right)+\sum_{j,k}\varepsilon_{ijk}E_jA_k,
\end{aligned}$$

where we used that $\nabla\cdot\mathbf{E}=0$ (there are no charges).
(c)

$$\begin{aligned}
\mathbf{L}_i^{(s)}&=\frac{1}{\mu_0c^2}\int d^3r\sum_{j,k}\varepsilon_{ijk}E_jA_k=\frac{\epsilon_0 i}{\hbar}\int d^3r\sum_{j,k}E_j\frac{\hbar\varepsilon_{ijk}}{i}A_k\\
&=\frac{\epsilon_0 i}{\hbar}\int d^3r\sum_{j,k}E_jS_{j,k}^{(i)}A_k=\frac{\epsilon_0 i}{\hbar}\int d^3r\mathbf{E}\cdot S^{(i)}\cdot\mathbf{A},
\end{aligned}$$

where

$$S^{(1)}=i\hbar\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix},\quad S^{(2)}=i\hbar\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix},\quad S^{(3)}=i\hbar\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix}.$$

It is easy to verify that the Eigenvalues of $S^{(i)}$ are $+\hbar$, 0, and $-\hbar$. Moreover, a simple inspection show that

$$\left[S^{(j)}, S^{(k)}\right] = i\hbar\varepsilon_{jkl}S^{(l)}.$$

Thus, S are spin-1 matrices.

(d) From the definition

$$\begin{aligned}\mathbf{L}^{(s)} &= \frac{1}{\mu_0 c^2}\int \mathrm{d}^3r\mathbf{E}\times\mathbf{A} = -\epsilon_0\int \mathrm{d}^3r\dot{\mathbf{A}}\times\mathbf{A}\\ &= i\epsilon_0\left(\frac{\hbar}{2\epsilon_0 V}\right)\int \mathrm{d}^3r\sum_{\mathbf{k},\mathbf{k}'}\sum_{\lambda,\lambda'}\sqrt{\frac{\omega_{k'}}{\omega_k}}\left(a_{\mathbf{k},\lambda}e^{i\mathbf{k}\cdot\mathbf{r}}\hat{e}_{\mathbf{k},\lambda} - a^\dagger_{\mathbf{k},\lambda}e^{-i\mathbf{k}\cdot\mathbf{r}}\hat{e}^*_{\mathbf{k},\lambda}\right)\times\left(a_{\mathbf{k}',\lambda'}e^{i\mathbf{k}'\cdot\mathbf{r}}\hat{e}_{\mathbf{k}',\lambda'} + \text{h.c.}\right).\end{aligned}$$

We then have four terms. Integrating over space yields

$$\frac{1}{V}\int \mathrm{d}^3re^{i\left(\pm\mathbf{k}\pm\mathbf{k}'\right)\cdot\mathbf{r}} = \delta_{\pm\mathbf{k},\mp\mathbf{k}'},$$

and thus,

$$\mathbf{L}^{(s)} = i\left(\frac{\hbar}{2}\right)\sum_{\mathbf{k}}\sum_{\lambda,\lambda'}\left(a_{\mathbf{k},\lambda}a_{-\mathbf{k},\lambda'}\hat{e}_{\mathbf{k},\lambda}\times\hat{e}_{-\mathbf{k},\lambda'} + a_{\mathbf{k},\lambda}a^\dagger_{\mathbf{k},\lambda'}\hat{e}_{\mathbf{k},\lambda}\times\hat{e}^*_{\mathbf{k},\lambda'} - \dots\right).$$

The sum over the terms $a_{\mathbf{k},\lambda}a_{-\mathbf{k},\lambda'}\hat{e}_{\mathbf{k},\lambda}\times\hat{e}_{-\mathbf{k},\lambda'}$ and $a^\dagger_{\mathbf{k},\lambda}a^\dagger_{-\mathbf{k},\lambda'}\hat{e}^*_{\mathbf{k},\lambda}\times\hat{e}^*_{-\mathbf{k},\lambda'}$ vanish because they are odd functions of $\mathbf{k}$. This is easily shown by summing over the pair of vectors $\mathbf{q}$ and $-\mathbf{q}$:

$$a_{\mathbf{q},\lambda}a_{-\mathbf{q},\lambda'}\left(\hat{e}_{\mathbf{q},\lambda}\times\hat{e}_{-\mathbf{q},\lambda'}\right) + a_{-\mathbf{q},\lambda}a_{\mathbf{q},\lambda'}\left(\hat{e}_{-\mathbf{q},\lambda}\times\hat{e}_{\mathbf{q},\lambda'}\right) = 0,$$

since $[a_{\mathbf{q},\lambda}, a_{-\mathbf{q},\lambda'}] = 0$. The spin angular momentum then reduces to

$$\mathbf{L}^{(s)} = i\left(\frac{\hbar}{2}\right)\sum_{\mathbf{k}}\sum_{\lambda,\lambda'}\left(a_{\mathbf{k},\lambda}a^\dagger_{\mathbf{k},\lambda'}\hat{e}_{\mathbf{k},\lambda}\times\hat{e}^*_{\mathbf{k},\lambda'} - a^\dagger_{\mathbf{k},\lambda}a_{\mathbf{k},\lambda'}\hat{e}^*_{\mathbf{k},\lambda}\times\hat{e}_{\mathbf{k},\lambda'}\right).$$

We now sum over the polarizations. Using that

$$\begin{aligned}\hat{e}^*_{\mathbf{k},\pm}\times\hat{e}_{\mathbf{k},\pm} &= \frac{1}{2}\left(\pm i\hat{e}_{\mathbf{k},1}\times\hat{e}_{\mathbf{k},2} \mp i\hat{e}_{\mathbf{k},2}\times\hat{e}_{\mathbf{k},1}\right) = \pm i\hat{k},\\ \hat{e}^*_{\mathbf{k},+}\times\hat{e}_{\mathbf{k},-} &= \hat{e}^*_{\mathbf{k},-}\times\hat{e}_{\mathbf{k},+} = 0,\end{aligned}$$

since $\hat{e}_{\mathbf{k},1}\times\hat{e}_{\mathbf{k},2} = \hat{k}$, we arrive at the final result

$$\begin{aligned}\mathbf{L}^{(s)} &= i\left(\frac{\hbar}{2}\right)\sum_{\mathbf{k}}\left(a_{\mathbf{k},+}a^\dagger_{\mathbf{k},+}\left(-i\hat{k}\right) + a_{\mathbf{k},-}a^\dagger_{\mathbf{k},-}\left(i\hat{k}\right) - a^\dagger_{\mathbf{k},+}a_{\mathbf{k},+}\left(i\hat{k}\right) - a^\dagger_{\mathbf{k},-}a_{\mathbf{k},-}\left(-i\hat{k}\right)\right)\\ &= \frac{\hbar}{2}\sum_{\mathbf{k}}\left(a_{\mathbf{k},+}a^\dagger_{\mathbf{k},+}\hat{k} - a_{\mathbf{k},-}a^\dagger_{\mathbf{k},-}\hat{k} + a^\dagger_{\mathbf{k},+}a_{\mathbf{k},+}\hat{k} - a^\dagger_{\mathbf{k},-}a_{\mathbf{k},-}\hat{k}\right)\\ &= \sum_{\mathbf{k}}\left(a^\dagger_{\mathbf{k},+}a_{\mathbf{k},+} - a^\dagger_{\mathbf{k},-}a_{\mathbf{k},-}\right)\hbar\hat{k},\end{aligned}$$

where we used that $[a_{\mathbf{k},\lambda}, a^\dagger_{\mathbf{k},\lambda}] = 1$. This result means that the spin angular momentum of the photon is parallel to the direction of propagation $\hat{k}$. Moreorver, the possible values are only $\pm\hbar$. If the $L^{(s)}_k = +\hbar$, then the helicity is positive (+1), i.e., the classical picture of the photon is that it has clockwise circular polarization. In the same manner, If the $L^{(s)}_k = -\hbar$, then the helicity is negative. Finally, notice the component $L^{(s)}_k = 0$ is absent. This a feature of massless particles travelling at light speed.

(e) In the Heisenberg representation

$$\mathbf{A}(\mathbf{r}, t) = e^{i\frac{H}{\hbar}t}\mathbf{A}(\mathbf{r})e^{-i\frac{H}{\hbar}t},$$

where $H = \sum \hbar\omega_k a^\dagger_{\mathbf{k},\lambda} a_{\mathbf{k},\lambda}$. We then need to compute

$$\begin{aligned} e^{i\omega t a^\dagger a} a e^{-i\omega t a^\dagger a} &= a + i\omega t \left[a^\dagger a, a\right] + \frac{1}{2!}\left(i\omega t\right)^2 \left[a^\dagger a, \left[a^\dagger a, a\right]\right] + \dots \\ &= a - i\omega t a + \frac{1}{2!}\left(-i\omega t\right)^2 a + \dots = e^{-i\omega t} a. \end{aligned}$$

Thus,

$$\mathbf{A}(\mathbf{r},t) = \sqrt{\frac{\hbar}{2\epsilon_0 V}} \sum_{\mathbf{k}} \sum_{\lambda=\pm} \sqrt{\frac{1}{\omega_k}} \left(a_{\mathbf{k},\lambda} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \hat{e}_{\mathbf{k},\lambda} + a^\dagger_{\mathbf{k},\lambda} e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \hat{e}^*_{\mathbf{k},\lambda}\right).$$

As a consequence,

$$\begin{aligned} \mathbf{E}(\mathbf{r},t) &= -\frac{\partial \mathbf{A}}{\partial t} = i\sqrt{\frac{\hbar}{2\epsilon_0 V}} \sum_{\mathbf{k}} \sum_{\lambda=\pm} \sqrt{\omega_k} \left(a_{\mathbf{k},\lambda} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \hat{e}_{\mathbf{k},\lambda} - a^\dagger_{\mathbf{k},\lambda} e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \hat{e}^*_{\mathbf{k},\lambda}\right), \\ \mathbf{B}(\mathbf{r},t) &= \nabla\times\mathbf{A} = i\sqrt{\frac{\hbar}{2\epsilon_0 V}} \sum_{\mathbf{k}} \sum_{\lambda=\pm} \sqrt{\frac{1}{\omega_k}} \left(a_{\mathbf{k},\lambda} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \mathbf{k}\times\hat{e}_{\mathbf{k},\lambda} - a^\dagger_{\mathbf{k},\lambda} e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \mathbf{k}\times\hat{e}^*_{\mathbf{k},\lambda}\right). \end{aligned}$$

We now turn to the commutations relations:

$$\begin{aligned} [A_i, A_j] =& \frac{\hbar}{2\epsilon_0 V} \sum_{\mathbf{k},\mathbf{k}'} \sum_{\lambda,\lambda'} \sqrt{\frac{1}{\omega_k \omega_{k'}}} e^{i\left(\mathbf{k}\cdot\mathbf{r}-\mathbf{k}'\cdot\mathbf{r}'-\omega_k t+\omega_{k'} t'\right)} \left(\hat{e}_{\mathbf{k},\lambda}\right)_i \left(\hat{e}^*_{\mathbf{k}',\lambda'}\right)_j \left[a_{\mathbf{k},\lambda}, a^\dagger_{\mathbf{k}',\lambda'}\right] \\ &+ \frac{\hbar}{2\epsilon_0 V} \sum_{\mathbf{k},\mathbf{k}'} \sum_{\lambda,\lambda'} \sqrt{\frac{1}{\omega_k \omega_{k'}}} e^{-i\left(\mathbf{k}\cdot\mathbf{r}-\mathbf{k}'\cdot\mathbf{r}'-\omega_k t+\omega_{k'} t'\right)} \left(\hat{e}^*_{\mathbf{k},\lambda}\right)_i \left(\hat{e}_{\mathbf{k}',\lambda'}\right)_j \left[a^\dagger_{\mathbf{k},\lambda}, a_{\mathbf{k}',\lambda'}\right] \\ =& \frac{\hbar}{2\epsilon_0 V} \sum_{\mathbf{k},\lambda} \frac{1}{\omega_k} \left(e^{i\left(\mathbf{k}\cdot\left(\mathbf{r}-\mathbf{r}'\right)-\omega_k\left(t-t'\right)\right)} \left(\hat{e}_{\mathbf{k},\lambda}\right)_i \left(\hat{e}^*_{\mathbf{k},\lambda}\right)_j - e^{-i\left(\mathbf{k}\cdot\left(\mathbf{r}-\mathbf{r}'\right)-\omega_k\left(t-t'\right)\right)} \left(\hat{e}^*_{\mathbf{k},\lambda}\right)_i \left(\hat{e}_{\mathbf{k},\lambda}\right)_j\right). \end{aligned}$$

Summing over the polarizations,

$$\begin{aligned} \sum_\lambda \left(\hat{e}_{\mathbf{k},\lambda}\right)_i \left(\hat{e}^*_{\mathbf{k},\lambda}\right)_j &= \frac{1}{2}\left(\hat{e}_1 + i\hat{e}_2\right)_i \left(\hat{e}_1 - i\hat{e}_2\right)_j + \frac{1}{2}\left(\hat{e}_1 - i\hat{e}_2\right)_i \left(\hat{e}_1 + i\hat{e}_2\right)_j \\ &= \left(\hat{e}_1\right)_i \left(\hat{e}_1\right)_j + \left(\hat{e}_2\right)_i \left(\hat{e}_2\right)_j = \sum_\lambda \left(\hat{e}^*_{\mathbf{k},\lambda}\right)_i \left(\hat{e}_{\mathbf{k},\lambda}\right)_j \\ &= \delta_{i,j} - \frac{k_i k_j}{k^2}, \end{aligned}$$

where the last equality can be checked by inspection. Finally, we arrive at

$$\begin{aligned} [A_i, A_j] &= \frac{\hbar}{2\epsilon_0 V} \sum_{\mathbf{k}} \frac{1}{\omega_k} \left(e^{i\left(\mathbf{k}\cdot\left(\mathbf{r}-\mathbf{r}'\right)-\omega_k\left(t-t'\right)\right)} - e^{-i\left(\mathbf{k}\cdot\left(\mathbf{r}-\mathbf{r}'\right)-\omega_k\left(t-t'\right)\right)}\right) \left(\delta_{i,j} - \frac{k_i k_j}{k^2}\right) \\ &= \frac{i\hbar}{\epsilon_0 V} \sum_{\mathbf{k}} \frac{1}{\omega_k} \sin\left(\mathbf{k}\cdot\left(\mathbf{r}-\mathbf{r}'\right) - ck\left(t-t'\right)\right) \left(\delta_{i,j} - \frac{k_i k_j}{k^2}\right). \end{aligned}$$

We now use the continuum limit $\sum_{\mathbf{k}} \to (2\pi)^{-3} V \int \mathrm{d}^3 k$ and set, without loss of generality, that $\mathbf{r}-\mathbf{r}'$ points in the $\hat{k}_z$ direction. For $i \neq j$, it is easy to see that the integral vanishes since the integration over the ϕ angle (which runs from 0 to 2π) will involve either $\sin\phi\cos\phi$, or $\sin\phi$, or $\cos\phi$. For $i = j = z$, we have

$$\begin{aligned} &\int_0^\pi \mathrm{d}\theta \int_0^\infty \mathrm{d}k \left(k^2 \sin\theta \sin\left(k\left(\Delta r\cos\theta - c\Delta t\right)\right) \frac{1}{ck}\left(1-\cos^2\theta\right)\right) \\ &= \frac{4}{c\left(\Delta r\right)^3} \int_0^\infty \mathrm{d}k \sin\left(ck\Delta t\right)\left(k\Delta r\cos\left(k\Delta r\right) - \sin\left(k\Delta r\right)\right) = 0. \end{aligned}$$

The last result is obtained by regularizing the integrand: multiply it by $e^{-\alpha k}$ and after integration, take the limit $\alpha \to 0_+$. Analogously, for $i = j = x$, we have the integral

$$\begin{aligned}\int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\theta \int_0^{\infty} \mathrm{d}k \left(k^2 \sin\theta \sin\left(k\left(\Delta r \cos\theta - c\Delta t\right)\right) \frac{1}{ck}\left(1-\cos^2\phi\sin^2\theta\right)\right) \\ = \frac{\pi}{c}\int_0^{\pi}\mathrm{d}\theta\int_0^{\infty}\mathrm{d}k\left(k\sin\theta\sin\left(k\left(\Delta r\cos\theta - c\Delta t\right)\right)\left(2-\sin^2\theta\right)\right) \\ = -\frac{4\pi}{c\left(\Delta r\right)^3}\int_0^{\infty}\mathrm{d}k \sin\left(ck\Delta t\right)\left(k\Delta r\cos\left(k\Delta r\right) - \left(1-\left(k\Delta r\right)^2\right)\sin\left(k\Delta r\right)\right) = 0.\end{aligned}$$

Symmetry arguments ensures the same happens for $i = j = y$. Therefore, we conclude that $[A_i(\mathbf{r},t), A_j(\mathbf{r}',t')] = 0$.

The commutator between the electric fields is similar:

$$\begin{aligned}[E_i, E_j] =& \frac{\hbar}{2\epsilon_0 V}\sum_{\mathbf{k},\mathbf{k}'}\sum_{\lambda,\lambda'}\sqrt{\omega_k\omega_{k'}}e^{i\left(\mathbf{k}\cdot\mathbf{r}-\mathbf{k}'\cdot\mathbf{r}'-\omega_k t+\omega_{k'}t'\right)}\left(\hat{e}_{\mathbf{k},\lambda}\right)_i\left(\hat{e}^*_{\mathbf{k}',\lambda'}\right)_j\left[a_{\mathbf{k},\lambda}, a^\dagger_{\mathbf{k}',\lambda'}\right] \\ &+ \frac{\hbar}{2\epsilon_0 V}\sum_{\mathbf{k},\mathbf{k}'}\sum_{\lambda,\lambda'}\sqrt{\frac{1}{\omega_k\omega_{k'}}}e^{-i\left(\mathbf{k}\cdot\mathbf{r}-\mathbf{k}'\cdot\mathbf{r}'-\omega_k t+\omega_{k'}t'\right)}\left(\hat{e}^*_{\mathbf{k},\lambda}\right)_i\left(\hat{e}_{\mathbf{k}',\lambda'}\right)_j\left[a^\dagger_{\mathbf{k},\lambda}, a_{\mathbf{k}',\lambda'}\right] \\ =& \frac{i\hbar}{\epsilon_0 V}\sum_{\mathbf{k}}\omega_k\sin\left(\mathbf{k}\cdot\left(\mathbf{r}-\mathbf{r}'\right)-ck\left(t-t'\right)\right)\left(\delta_{i,j}-\frac{k_ik_j}{k^2}\right).\end{aligned}$$

The difference between the previous integrals is that ω_k now appears in the numerator, and thus contributes with k factor instead of k^{-1}. It turns out that, as before, all the integrals vanish. Thus,$[E_i(\mathbf{r},t), E_j(\mathbf{r}',t')] = 0$.

The commutator for the magnetic fields is

$$\begin{aligned}[B_i, B_j] =& \frac{\hbar}{2\epsilon_0 V}\sum_{\mathbf{k},\mathbf{k}'}\sum_{\lambda,\lambda'}\sqrt{\frac{1}{\omega_k\omega_{k'}}}e^{i\left(\mathbf{k}\cdot\mathbf{r}-\mathbf{k}'\cdot\mathbf{r}'-\omega_k t+\omega_{k'}t'\right)}\left(\mathbf{k}\times\hat{e}_{\mathbf{k},\lambda}\right)_i\left(\mathbf{k}'\times\hat{e}^*_{\mathbf{k}',\lambda'}\right)_j\left[a_{\mathbf{k},\lambda}, a^\dagger_{\mathbf{k}',\lambda'}\right] \\ &+ \frac{\hbar}{2\epsilon_0 V}\sum_{\mathbf{k},\mathbf{k}'}\sum_{\lambda,\lambda'}\sqrt{\frac{1}{\omega_k\omega_{k'}}}e^{-i\left(\mathbf{k}\cdot\mathbf{r}-\mathbf{k}'\cdot\mathbf{r}'-\omega_k t+\omega_{k'}t'\right)}\left(\mathbf{k}\times\hat{e}^*_{\mathbf{k},\lambda}\right)_i\left(\mathbf{k}'\times\hat{e}_{\mathbf{k}',\lambda'}\right)_j\left[a^\dagger_{\mathbf{k},\lambda}, a_{\mathbf{k}',\lambda'}\right] \\ =& \frac{-i\hbar}{2\epsilon_0 V}\sum_{\mathbf{k},\lambda}\frac{k^2}{\omega_k}\lambda\left(e^{i\left(\mathbf{k}\cdot\left(\mathbf{r}-\mathbf{r}'\right)-\omega_k\left(t-t'\right)\right)}\left(\hat{e}_{\mathbf{k},\lambda}\right)_i\left(\hat{e}^*_{\mathbf{k},\lambda}\right)_j - e^{-i\left(\mathbf{k}\cdot\left(\mathbf{r}-\mathbf{r}'\right)-\omega_k\left(t-t'\right)\right)}\left(\hat{e}^*_{\mathbf{k},\lambda}\right)_i\left(\hat{e}_{\mathbf{k},\lambda}\right)_j\right),\end{aligned}$$

since $\mathbf{k}\times\hat{e}_{\mathbf{k},\lambda} = -i\lambda k\hat{e}_{\mathbf{k},\lambda}$. The integrals involved are similar to those of the electric field, and hence, $[B_i(\mathbf{r},t), B_j(\mathbf{r}',t')] = 0$.

Let us now turn to the commutator

$$\begin{aligned}[A_i, E_j] =& \frac{-i\hbar}{2\epsilon_0 V}\sum_{\mathbf{k},\mathbf{k}'}\sum_{\lambda,\lambda'}\sqrt{\frac{\omega_{k'}}{\omega_k}}e^{i\left(\mathbf{k}\cdot\mathbf{r}-\mathbf{k}'\cdot\mathbf{r}'-\omega_k t+\omega_{k'}t'\right)}\left(\hat{e}_{\mathbf{k},\lambda}\right)_i\left(\hat{e}^*_{\mathbf{k}',\lambda'}\right)_j\left[a_{\mathbf{k},\lambda}, a^\dagger_{\mathbf{k}',\lambda'}\right] \\ &+ \frac{i\hbar}{2\epsilon_0 V}\sum_{\mathbf{k},\mathbf{k}'}\sum_{\lambda,\lambda'}\sqrt{\frac{\omega_{k'}}{\omega_k}}e^{-i\left(\mathbf{k}\cdot\mathbf{r}-\mathbf{k}'\cdot\mathbf{r}'-\omega_k t+\omega_{k'}t'\right)}\left(\hat{e}^*_{\mathbf{k},\lambda}\right)_i\left(\hat{e}_{\mathbf{k}',\lambda'}\right)_j\left[a^\dagger_{\mathbf{k},\lambda}, a_{\mathbf{k}',\lambda'}\right] \\ =& \frac{-i\hbar}{2\epsilon_0 V}\sum_{\mathbf{k},\lambda}\left(e^{i\left(\mathbf{k}\cdot\left(\mathbf{r}-\mathbf{r}'\right)-\omega_k\left(t-t'\right)\right)}\left(\hat{e}_{\mathbf{k},\lambda}\right)_i\left(\hat{e}^*_{\mathbf{k},\lambda}\right)_j + e^{-i\left(\mathbf{k}\cdot\left(\mathbf{r}-\mathbf{r}'\right)-\omega_k\left(t-t'\right)\right)}\left(\hat{e}^*_{\mathbf{k},\lambda}\right)_i\left(\hat{e}_{\mathbf{k},\lambda}\right)_j\right) \\ =& \frac{i\hbar}{\epsilon_0 V}\sum_{\mathbf{k}}\cos\left(\mathbf{k}\cdot\left(\mathbf{r}-\mathbf{r}'\right)-ck\left(t-t'\right)\right)\left(\delta_{i,j}-\frac{k_ik_j}{k^2}\right).\end{aligned}$$

At this point, it is already clear that not all commutators vanish, specially when $\mathbf{k}\cdot\left(\mathbf{r}-\mathbf{r}'\right) = ck\left(t-t'\right)$. Let us focus on the equal time commutator $(t = t')$. Then,

$$\begin{aligned}[A_i(\mathbf{r},t), E_j(\mathbf{r}',t)] &= \frac{i\hbar}{\epsilon_0}\left[\delta\left(\mathbf{r}-\mathbf{r}'\right)\delta_{ij} - \frac{\partial^2}{\partial x_i\partial x'_j}\sum_{\mathbf{k}}\frac{\cos\left(\mathbf{k}\cdot\left(\mathbf{r}-\mathbf{r}'\right)\right)}{Vk^2}\right] \\ &= \frac{i\hbar}{\epsilon_0}\left[\delta\left(\mathbf{r}-\mathbf{r}'\right)\delta_{ij} - \frac{\partial^2}{\partial x_i\partial x'_j}\left(\frac{1}{4\pi\left|\mathbf{r}-\mathbf{r}'\right|}\right)\right].\end{aligned}$$

The final commutator is

$$
\begin{aligned}
\left[E_i, B_j\right] =& \frac{\hbar}{2\epsilon_0 V} \sum_{\mathbf{k},\mathbf{k}'} \sum_{\lambda,\lambda'} \sqrt{\frac{\omega_{k'}}{\omega_k}} e^{i\left(\mathbf{k}\cdot\mathbf{r}-\mathbf{k}'\cdot\mathbf{r}'-\omega_k t+\omega_{k'} t'\right)} \left(\hat{e}_{\mathbf{k},\lambda}\right)_i \left(\mathbf{k}'\times\hat{e}^*_{\mathbf{k}',\lambda'}\right)_j \left[a_{\mathbf{k},\lambda}, a^\dagger_{\mathbf{k}',\lambda'}\right] \\
&+ \frac{\hbar}{2\epsilon_0 V} \sum_{\mathbf{k},\mathbf{k}'} \sum_{\lambda,\lambda'} \sqrt{\frac{\omega_{k'}}{\omega_k}} e^{-i\left(\mathbf{k}\cdot\mathbf{r}-\mathbf{k}'\cdot\mathbf{r}'-\omega_k t+\omega_{k'} t'\right)} \left(\hat{e}^*_{\mathbf{k},\lambda}\right)_i \left(\mathbf{k}'\times\hat{e}_{\mathbf{k}',\lambda'}\right)_j \left[a^\dagger_{\mathbf{k},\lambda}, a_{\mathbf{k}',\lambda'}\right] \\
=& \frac{\hbar}{2\epsilon_0 V} \sum_{\mathbf{k},\lambda} \sum_{l,m} \varepsilon_{jlm} \left(e^{i\left(\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')-\omega_k(t-t')\right)} \left(\hat{e}_{\mathbf{k},\lambda}\right)_i k_l \left(\hat{e}^*_{\mathbf{k},\lambda}\right)_m - e^{-i\left(\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')-\omega_k(t-t')\right)} \left(\hat{e}^*_{\mathbf{k},\lambda}\right)_i k_l \left(\hat{e}_{\mathbf{k},\lambda}\right)_m\right). \\
=& \frac{i\hbar}{\epsilon_0 V} \sum_{\mathbf{k}} \sum_{l,m} \varepsilon_{jlm} k_l \sin\left(\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}') - ck\,(t-t')\right) \left(\delta_{i,m} - \frac{k_i k_m}{k^2}\right).
\end{aligned}
$$

Notice that $\sum \varepsilon_{jlm} k_l k_m = (\mathbf{k}\times\mathbf{k})_j = 0$. Thus, for the equal time commutator

$$
\begin{aligned}
\left[E_i(\mathbf{r},t), B_j(\mathbf{r}',t)\right] &= \frac{i\hbar}{\epsilon_0 V} \sum_{\mathbf{k}} \sum_{l,m} \varepsilon_{jlm} k_l \sin\left(\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')\right) \delta_{im} = \frac{i\hbar}{\epsilon_0 V} \sum_{\mathbf{k},l} \varepsilon_{ijl} k_l \sin\left(\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')\right) \\
&= \frac{i\hbar}{\epsilon_0 V} \sum_{l} \varepsilon_{ijl} \frac{\partial}{\partial x'_l} \sum_{\mathbf{k}} k_l \cos\left(\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')\right) = \frac{i\hbar}{\epsilon_0} \sum_{l} \varepsilon_{ijl} \frac{\partial}{\partial x'_l} \left(\delta(\mathbf{r}-\mathbf{r}')\right).
\end{aligned}
$$

The consequence of this result is that it is not possible to simultaneously measure the electric and magnetic field with arbitrarily precision.

(f) The electric and magnetic field do not commutate with the number operator. In order to show this, consider a single mode. Then, we need the commutators

$$
\begin{aligned}
\left[a_{\mathbf{k}',\lambda'}, a^\dagger_{\mathbf{k},\lambda} a_{\mathbf{k},\lambda}\right] &= \delta_{\mathbf{k},\mathbf{k}'} \delta_{\lambda,\lambda'} a_{\mathbf{k},\lambda}, \\
\left[a^\dagger_{\mathbf{k}',\lambda'}, a^\dagger_{\mathbf{k},\lambda} a_{\mathbf{k},\lambda}\right] &= -\delta_{\mathbf{k},\mathbf{k}'} \delta_{\lambda,\lambda'} a^\dagger_{\mathbf{k},\lambda},
\end{aligned}
$$

in order to compute

$$
\begin{aligned}
\left[\mathbf{E}, N\right] &= i\sqrt{\frac{1}{2\epsilon_0 V}} \sum_{\mathbf{k}} \sum_{\lambda=\pm} (\hbar\omega_k)^{\frac{3}{2}} \left(a_{\mathbf{k},\lambda} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \hat{e}_{\mathbf{k},\lambda} + a^\dagger_{\mathbf{k},\lambda} e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \hat{e}^*_{\mathbf{k},\lambda}\right) \neq 0, \\
\left[\mathbf{B}, N\right] &= i\sqrt{\frac{\hbar}{2\epsilon_0 V}} \sum_{\mathbf{k}} \sum_{\lambda=\pm} (\hbar\omega_k)^{\frac{1}{2}} \left(a_{\mathbf{k},\lambda} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \mathbf{k}\times\hat{e}_{\mathbf{k},\lambda} + a^\dagger_{\mathbf{k},\lambda} e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \mathbf{k}\times\hat{e}^*_{\mathbf{k},\lambda}\right) \neq 0.
\end{aligned}
$$

Thus, the number of photons is not conserved when the electromagnetic field is coupled to matter.

(g) Using the result of item (h), we have that

$$
\begin{aligned}
\langle X\rangle =& \sqrt{\frac{\hbar}{2\omega_k}} \left\langle \alpha(t) \left| a^\dagger_{\mathbf{k},\lambda} + a_{\mathbf{k},\lambda} \right| \alpha(t) \right\rangle. \\
=& \sqrt{\frac{\hbar}{2\omega_k}} \left(\left\langle 0 \left| e^{-\frac{1}{2}|\beta|^2} e^{\beta^* a_{\mathbf{k},\lambda}} a^\dagger_{\mathbf{k},\lambda} e^{-\frac{1}{2}|\beta|^2} e^{\beta a^\dagger_{\mathbf{k},\lambda}} \right| 0 \right\rangle + \left\langle 0 \left| e^{-\frac{1}{2}|\beta|^2} e^{\beta^* a_{\mathbf{k},\lambda}} a_{\mathbf{k},\lambda} e^{-\frac{1}{2}|\beta|^2} e^{\beta a^\dagger_{\mathbf{k},\lambda}} \right| 0 \right\rangle \right) \\
=& \sqrt{\frac{\hbar}{2\omega_k}} e^{-|\beta|^2} \left(\left\langle 0 \left| e^{\beta^* a_{\mathbf{k},\lambda}} a^\dagger_{\mathbf{k},\lambda} e^{\beta a^\dagger_{\mathbf{k},\lambda}} \right| 0 \right\rangle + \left\langle 0 \left| e^{\beta^* a_{\mathbf{k},\lambda}} a_{\mathbf{k},\lambda} e^{\beta a^\dagger_{\mathbf{k},\lambda}} \right| 0 \right\rangle \right) \\
=& \sqrt{\frac{\hbar}{2\omega_k}} e^{-|\beta|^2} \left(\left\langle 0 \left| \left[e^{\beta^* a_{\mathbf{k},\lambda}}, a^\dagger_{\mathbf{k},\lambda}\right] e^{\beta a^\dagger_{\mathbf{k},\lambda}} \right| 0 \right\rangle + \left\langle 0 \left| e^{\beta^* a_{\mathbf{k},\lambda}} \left[a_{\mathbf{k},\lambda}, e^{\beta a^\dagger_{\mathbf{k},\lambda}}\right] \right| 0 \right\rangle \right).
\end{aligned}
$$

We then need to compute the commutators:

$$
\begin{aligned}
e^{-\alpha a^\dagger} a e^{\alpha a^\dagger} =& a + \left[-\alpha a^\dagger, a\right] + \frac{1}{2!} \left[-\alpha a^\dagger, \left[-\alpha a^\dagger, a\right]\right] + \ldots \\
=& a + \alpha + \frac{1}{2!} \left[-\alpha a^\dagger, \alpha\right] + \cdots = a + \alpha,
\end{aligned}
$$

which yields

$$ae^{\alpha a^\dagger} = e^{\alpha a^\dagger} a + e^{-\alpha a^\dagger}\alpha, \qquad \Rightarrow \qquad \left[a, e^{\alpha a^\dagger}\right] = \alpha e^{-\alpha a^\dagger}.$$

Thus,

$$\begin{aligned}\langle X\rangle &= \sqrt{\frac{\hbar}{2\omega_k}} e^{-|\beta|^2}\left(\beta^* \left\langle 0 \left| e^{-\beta^* a_{\mathbf{k},\lambda}} e^{\beta a^\dagger_{\mathbf{k},\lambda}}\right| 0\right\rangle + \beta \left\langle 0 \left| e^{\beta^* a_{\mathbf{k},\lambda}} e^{\beta a^\dagger_{\mathbf{k},\lambda}}\right| 0\right\rangle\right)\\ &= \sqrt{\frac{\hbar}{2\omega_k}}\left(\beta^* + \beta\right).\end{aligned}$$

In the same manner,

$$\langle P\rangle = i\sqrt{\frac{\hbar\omega_k}{2}}\left(\beta^* - \beta\right).$$

Now, we turn to the quadratric mean values (where we drop out the subindeces)

$$\begin{aligned}\left\langle X^2\right\rangle &= \frac{\hbar}{2\omega_k}\left\langle a^\dagger a^\dagger + a a^\dagger + a^\dagger a + aa\right\rangle = \frac{\hbar}{2\omega_k}\left\langle a^\dagger a^\dagger + 1 + 2a^\dagger a + aa\right\rangle\\ &= \frac{\hbar}{2\omega_k} e^{-|\beta|^2}\left\langle 0 \left| \left[e^{-\beta^* a}, a^\dagger a^\dagger\right] e^{\beta a^\dagger} + 1 + 2\left[e^{-\beta^* a}, a^\dagger\right]\left[a, e^{\beta a^\dagger}\right] + e^{-\beta^* a}\left[aa, e^{\beta a^\dagger}\right]\right| 0\right\rangle.\end{aligned}$$

We need the commutators

$$\begin{aligned}\left[aa, e^{\alpha a^\dagger}\right] &= \alpha a e^{-\alpha a^\dagger} + \alpha e^{-\alpha a^\dagger} a = \alpha^2 e^{-\alpha a^\dagger} + 2\alpha e^{-\alpha a^\dagger} a,\\ \left[e^{\alpha^* a}, a^\dagger a^\dagger\right] &= \left[aa, e^{\alpha a^\dagger}\right]^\dagger = \alpha^{*2} e^{-\alpha^* a} + 2\alpha^* a^\dagger e^{-\alpha^* a}.\end{aligned}$$

Therefore,

$$\begin{aligned}\left\langle X^2\right\rangle =& \frac{\hbar}{2\omega_k}\left(\beta^{*2} + 1 + 2\beta\beta^* + \beta^2\right) = \frac{\hbar}{2\omega_k}\left(1 + \left(\beta + \beta^*\right)^2\right) = \frac{\hbar}{2\omega_k}\left(1 + \left(\beta + \beta^*\right)^2\right),\\ \left\langle P^2\right\rangle =& -\frac{\hbar\omega_k}{2}\left(\beta^{*2} - 1 - 2\beta\beta^* + \beta^2\right) = \frac{\hbar\omega_k}{2}\left(1 - \left(\beta - \beta^*\right)^2\right).\end{aligned}$$

Finally,

$$\Delta X = \sqrt{\frac{\hbar}{2\omega_k}}, \text{ and } \Delta P = \sqrt{\frac{\hbar\omega_k}{2}},$$

which do not depend on time, which is the main feature of coherent states. Moreover, $\Delta X \Delta P = \frac{1}{2}\hbar$, which saturates the Heisenberg uncertainty principle.

(h) Let us show this result by inspection. The left-hand side of the equation is

$$\begin{aligned}i\hbar\frac{\partial}{\partial t}\left|\beta\right\rangle &= i\hbar\frac{\partial}{\partial t}\left(e^{-\frac{1}{2}|\beta|^2} e^{\beta a^\dagger}\right)\left|0\right\rangle = i\hbar e^{-\frac{1}{2}|\beta|^2}\frac{\partial}{\partial t}\left(e^{\beta a^\dagger}\right)\left|0\right\rangle\\ &= i\hbar e^{-\frac{1}{2}|\beta|^2}\dot{\beta} a^\dagger e^{\beta a^\dagger}\left|0\right\rangle = \hbar\omega_k e^{-\frac{1}{2}|\beta|^2}\beta a^\dagger e^{\beta a^\dagger}\left|0\right\rangle.\end{aligned}$$

The right-hand side is

$$\begin{aligned}H\left|\beta\right\rangle =& \sum_{\mathbf{k}',\lambda'}\hbar\omega_{k'} a^\dagger_{\mathbf{k}',\lambda'} a_{\mathbf{k}',\lambda'} e^{-\frac{1}{2}|\beta|^2} e^{\beta a^\dagger_{\mathbf{k},\lambda}}\left|0\right\rangle = \hbar\omega_k a^\dagger_{\mathbf{k},\lambda} a_{\mathbf{k},\lambda} e^{-\frac{1}{2}|\beta|^2} e^{\beta a^\dagger_{\mathbf{k},\lambda}}\left|0\right\rangle\\ =& \hbar\omega_k e^{-\frac{1}{2}|\beta|^2} a^\dagger_{\mathbf{k},\lambda}\left[a_{\mathbf{k},\lambda}, e^{\beta a^\dagger_{\mathbf{k},\lambda}}\right]\left|0\right\rangle = \hbar\omega_k e^{-\frac{1}{2}|\beta|^2} a^\dagger_{\mathbf{k},\lambda}\beta e^{\beta a^\dagger_{\mathbf{k},\lambda}}\left|0\right\rangle.\end{aligned}$$

Notice that this equals the left-hand side. This ends the proof that

$$\alpha(t) = \beta = \alpha e^{-i\omega_k t}.$$

Let us now compute

$$\langle\beta|\mathbf{A}|\beta\rangle = \sqrt{\frac{\hbar}{2\epsilon_0 V}}\sum_{\mathbf{q}}\sum_{\sigma=\pm}\sqrt{\frac{1}{\omega_q}}\left(\langle a_{\mathbf{q},\sigma}\rangle e^{i\mathbf{q}\cdot\mathbf{r}}\hat{e}_{\mathbf{q},\sigma} + \langle a^\dagger_{\mathbf{q},\sigma}\rangle e^{-i\mathbf{q}\cdot\mathbf{r}}\hat{e}^*_{\mathbf{q},\sigma}\right)$$
$$= \sqrt{\frac{\hbar}{2\epsilon_0 V\omega_k}}\left(\beta e^{i\mathbf{k}\cdot\mathbf{r}}\hat{e}_{\mathbf{k},\lambda} + \beta^* e^{-i\mathbf{k}\cdot\mathbf{r}}\hat{e}^*_{\mathbf{k},\lambda}\right)$$
$$= \sqrt{\frac{\hbar}{2\epsilon_0 V\omega_k}}\left(\alpha e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}\hat{e}_{\mathbf{k},\lambda} + \alpha^* e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}\hat{e}^*_{\mathbf{k},\lambda}\right).$$

This is an important result. Notice it is simply the $\mathbf{k}$-wave solution of the plane equation $\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\mathbf{A} = 0$. Thus, the state $|\beta\rangle$ is a coherent electromagnetic wave, just like a laser. (Some people like to say that laser is a Bose-Einstein condensate of photons.)

Let explore some insightful cases as $\mathbf{p} = p\hat{z}$, $\lambda = +1$, and $\alpha \in \Re$. Thus, $\hat{e}_{\mathbf{k},\pm} = -\frac{1}{\sqrt{2}}(1, i, 0)$, and

$$\langle\mathbf{A}\rangle = -\sqrt{\frac{\hbar}{\epsilon_0 V\omega_k}}\alpha\left(\cos\left(k\left(z - ct\right)\right)\hat{x} - \sin\left(k\left(z - ct\right)\right)\hat{y}\right),$$

which is a circular polarized vector which rotates in time clockwise in the xy-plane (the z-direction coming out of the plane).

2.

(a) The reason why the spontaneous emission is not possible is due to the impossibility of conserving both energy and linear momentum in the process. Linear momentum requires

$$\mathbf{k}_i = \mathbf{k}_f + \mathbf{q},$$

where $\mathbf{k}_{i,f}$ are the initial and final momentum of the particle, respectively, and $\mathbf{q}$ is the momentum of the emitted photon. Energy conservation requires

$$\frac{\hbar^2 k_i^2}{2m} = \frac{\hbar^2 k_f^2}{2m} + \hbar c q.$$

In the reference frame of the particle in its initial state, $k_i = 0$. Then, without loss of generality, it is clear that energy conservation can not be fulfilled unless $k = k_f = 0$, which means that no photon was emitted.

All this arguments can be made precise considering the transition amplitude

$$\sum_{\lambda}\langle\mathbf{k}_f; n_{\mathbf{q},\lambda}|\mathbf{k}_i; 0, (t)\rangle = 0,$$

where $||\mathbf{k}_i; 0, t\rangle = e^{-iHt/\hbar}\,||\mathbf{k}_i; 0\rangle$. In order to accomplish this, we just need to study the matrix element

$$\langle\mathbf{k}_f; n_{\mathbf{q},\lambda}|H|\,\mathbf{k}_i; 0\rangle,$$

where the only interesting term in the Hamiltonian is the interaction $H_{\text{int}} = -\frac{1}{m}q\mathbf{A}\cdot\mathbf{p} + \frac{1}{2m}q^2A^2$. The second term conserves the number of photons and thus can be neglected. The first term changes the number of photons by one and thus can allow the emission:

$$\langle\mathbf{k}_f; n_{\mathbf{q},\lambda}|H|\,\mathbf{k}_i; 0\rangle \propto \hat{e}^*_{\mathbf{q},\lambda}\cdot\left\langle\mathbf{k}_f\left|e^{-i\mathbf{q}\cdot\mathbf{r}}\mathbf{p}\right|\mathbf{k}_i\right\rangle \propto \int d^3r e^{-i\mathbf{k}_f\cdot\mathbf{r}}e^{-i\mathbf{q}\cdot\mathbf{r}}e^{i\mathbf{k}_i\cdot\mathbf{r}} \propto \delta\left(\mathbf{k}_i - \mathbf{k}_f - \mathbf{q}\right),$$

which is the momentum conservation. To complete the story, we need to consider the Dyson series where the energy conservation comes from (Fermi's golden rule). The energy conservation will happen in all orders of perturbation theory (recall the classes on light scattering) because the whole Hamiltonian is time-independent. Thus, we also have to satisfy energy conservation. As argued in the beginning, both conservation laws can not be satisfied simultaneously. Thus, a single photon emission can not happen.

Notice that the total number of photons does not commute with the Hamiltonian (due to the vector field term). Can you think which process can happen in order to not conserve the number of photons?

(b) As seen in class, in first order of perturbation theory the transition rate is given by

$$\begin{aligned} W_{FI} =& \frac{2\pi}{\hbar} \left|\langle f \left| H_{\text{int,static}} \right| i\rangle\right|^2 \rho|_{\hbar\omega=E_i-E_f} \\ =& \frac{2\pi}{\hbar}\frac{e^2}{m^2}\frac{\hbar\left(n_{\mathbf{q},\lambda}+1\right)}{2\omega_q\epsilon_0 V}\left|\hat{e}^*_{\mathbf{q},\lambda}\cdot\left\langle 1,0,0\left|e^{i\mathbf{q}\cdot\mathbf{r}}\mathbf{p}\right|2,1,1\right\rangle\right|^2\frac{V}{(2\pi)^3}\frac{\omega_q^2\mathrm{d}\Omega}{\hbar c^3} \\ =& \left|\hat{e}^*_{\mathbf{q},\lambda}\cdot\left\langle 1,0,0\left|e^{i\mathbf{q}\cdot\mathbf{r}}\mathbf{p}\right|2,1,1\right\rangle\right|^2\frac{e^2\omega_{fi}}{2\left(2\pi\right)^2 m^2\epsilon_0\hbar c^3}\mathrm{d}\Omega, \end{aligned}$$

where, for spontaneous emission, $n_{\mathbf{q},\lambda}=0$. In the electric dipole approximation $e^{i\mathbf{q}\cdot\mathbf{r}}\approx 1$. Moreover, we use that $\mathbf{p}=im\left[H_0,\mathbf{r}\right]/\hbar$. Thus,

$$W_{FI}=\left(\frac{e^2}{4\pi\epsilon_0\hbar c}\right)\frac{\omega_{fi}^3}{2\pi c^2}\left|\hat{e}^*_{\mathbf{q},\lambda}\cdot\langle 1,0,0\left|\mathbf{r}\right|2,1,1\rangle\right|^2\mathrm{d}\Omega=\alpha\frac{\omega_{fi}^3}{2\pi c^2}\left|\hat{e}^*_{\mathbf{q},\lambda}\cdot\langle 1,0,0\left|\mathbf{r}\right|2,1,1\rangle\right|^2\mathrm{d}\Omega,$$

where $\alpha\approx 1/137$ is the fine structure constant. We now have to compute the matrix element

$$\begin{aligned} \langle 1,0,0\left|\mathbf{r}\right|2,1,1\rangle &= \int \mathrm{d}^3 r R^*_{10}\left(r\right)Y^*_{00}\left(\theta,\phi\right)r\left(\sin\theta\sin\phi,\sin\theta\cos\phi,\cos\theta\right)R_{21}\left(r\right)Y_{11}\left(\theta,\phi\right) \\ &= \int \mathrm{d}^3 r\sqrt{\frac{1}{\pi a_0^3}}e^{-\frac{r}{a_0}}r\left(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta\right)\frac{r}{8a_0}\sqrt{\frac{1}{\pi a_0^3}}e^{-\frac{r}{2a_0}}\sin\theta e^{i\phi} \\ &= \frac{2^7}{3^5}a_0\left(1,i,0\right). \end{aligned}$$

The polariation vectors are $\hat{e}^*_{\mathbf{q},\pm}=\frac{1}{\sqrt{2}}\left(\mp\cos\theta\cos\phi-i\sin\phi,\mp\cos\theta\sin\phi+i\cos\phi,\pm\sin\theta\right)$, and hence,

$$\begin{aligned} W_{FI,\pm} &= \alpha\frac{\omega_{fi}^3}{2\pi c^2}\left(\frac{2^7}{3^5}a_0\right)^2\left|\frac{1}{\sqrt{2}}\left(\mp\cos\theta\cos\phi-i\sin\phi\mp i\cos\theta\sin\phi-\cos\phi\right)\right|^2\mathrm{d}\Omega \\ &= \alpha\frac{\omega_{fi}^3}{2\pi c^2}\left(\frac{2^7}{3^5}a_0\right)^2\left|\frac{1}{\sqrt{2}}\left(1\pm\cos\theta\right)e^{i\phi}\right|^2\mathrm{d}\Omega=\alpha\frac{\omega_{fi}^3}{4\pi c^2}\left(\frac{2^7}{3^5}a_0\right)^2\left(1\pm\cos\theta\right)^2\mathrm{d}\Omega. \end{aligned}$$

We now interpret this result with respect to angular momentum conservation. Consider for instance the case $\lambda=+1$, which means that the emitted photon has angular momentum $\hbar\hat{q}$. For $\theta=0$, this means that the total alngular momentum of the final state is $+\hbar$. Since the electron in the final state has no angular momentum, this means that the maximum chance of the emitted photon with helicity $+1$ is at this the z-direction where the momentum is classically conserved, as expected. On the other hand for $\lambda=-1$, the probability amplitude that the emitted photon is in the positive z-direction is null, since this direction maximally violates the classical picture of angular momentum conservation. Conversely, the photon with negative helicity has higher probability of being emitted in the negative z-direction ($\theta=\pi$), since in this case, the angular momentum it carries is $+\hbar$.

Finally, in order to compute the total transition rate, we integrate over all possible outcomes $\mathrm{d}\Omega$ and sum over all helicities:

$$\begin{aligned} \alpha\frac{\omega_{fi}^3}{4\pi c^2}\left(\frac{2^7}{3^5}a_0\right)^2\int\left(1\pm\cos\theta\right)^2\mathrm{d}\Omega &= \alpha\frac{\omega_{fi}^3}{4\pi c^2}\left(\frac{2^7}{3^5}a_0\right)^2\times\frac{16\pi}{3}=\alpha\frac{\omega_{fi}^3}{c^2}\frac{2^{16}}{3^{11}}a_0^2, \\ &= \left(\frac{2^7}{3^8}\right)\alpha^5\frac{mc^2}{\hbar}, \end{aligned}$$

where $a_0=\frac{\hbar}{mc\alpha}$ and $\hbar\omega_{fi}=\frac{3}{4}\left(\frac{mc^2\alpha^2}{2}\right)$. Notice it independs on λ and equals half of the transition rate between states $|2,1,0\rangle$ and $|1,0,0\rangle$, as would be expected from the isotropy of the Coulomb potential. Finally, summing over the helicities, we arrive at the desired result

$$W_{FI}=\sum_\lambda W_{FI,\lambda}=\left(\frac{2}{3}\right)^2\alpha^5 mc^2\hbar^2\approx\left(1.6\,\mathrm{ns}\right)^{-1}.$$

(c) We expect that transition rate for the finite-mass Hydrogen atom to be smaller since it approaches to the case of a structureless particle and both energy and momentum have to be conserved simultaneously.

Performing the calculations, we find that

$$W_{FI,\text{finite-mass}} = W_{FI,\text{infinite-mass}} \times \frac{\mu}{m}\left(\frac{2}{1+x+\sqrt{1+x}}\right),$$

where $\mu = mM/(M+m)$ is the reduced mass, M is the proton mass, and $x = \frac{3}{4}\frac{\mu}{M}\alpha^2 \approx 3\,10^{-8}$.

(d) As the photon carries angular momentum of $\hbar$, there can not be a single-photon electric-dipole transition between two atomic states having the same angular momentum. Thus, it seems the transition between states 2s→1s is highly suppressed. There is a route in which this transition can take place by emitting a single photon which is via a magnetic dipole transition $2s_{1/2}\to 1s_{1/2}$ in which the electron spin flips. It turns out this transition rate is very small as we will argue later. Therefore, the main decay process must be a 2-photon emission process.

We then look back to the interaction Hamiltonian in order to understand the possible 2-photon decay routes. There are two terms: one is $V = -\frac{e}{m}\mathbf{p}\cdot\mathbf{A}$ and the other is $V = \frac{e^2}{2m^2}A^2$. The latter one does not contribute in the electric dipole approximation ($e^{i\mathbf{k}\cdot\mathbf{r}} \approx 1$) because it will vanish identically: $\langle 1s|2s\rangle = 0$. We need to consider the next term and thus, the overall transition rate will pick up a factor of $(\mathbf{k}\cdot\mathbf{r})^2$. The other possibility (of same order of magnitude) is to consider the former term in second-order of perturbation theory. Then, we will have matrix elements of type

$$\sum_l \frac{\langle 100; 2\gamma\,|V|\,l; 1\gamma\rangle\,\langle l; 1\gamma\,|V|\,200; 0\gamma\rangle}{E_{200}-E_l},$$

where $|l\rangle$ are atomic intermediate states and $|n\gamma\rangle$ depicts states with n photons. The order of magnitude can be easily worked out (dropping out all numerical factors). As can be seen from the previous item, the electric-dipole transition matrix elements

$$\langle f\,|V|\,i\rangle \sim \frac{e}{m}\sqrt{\frac{\hbar}{\epsilon_0\omega V}}\frac{m}{\hbar}E_0 a_0 \sim \sqrt{\frac{\alpha\hbar c E_0}{V}}a_0,$$

and the phase space of a single emitted photon (proportional to the density of states) is $\sim VE_0^2(\hbar c)^{-3}$. For a 2-photon process, we will have to integrate over the 2-photon phase space, and thus we pick this factor squared. In addition, notice we have only one delta function $\delta(E_{2s}-E_{1s}-\hbar\omega_1-\hbar\omega_2)$, thus, there will be an additional factor of energy, which will be of order of the transition energy E_0. Then, in second order we have that the transition rate will be of order

$$\sim \frac{1}{\hbar}\left(\frac{\alpha\hbar c E_0}{V}a_0^2 \times \frac{1}{E_0}\right)^2 \times \left(\frac{VE_0^2}{(\hbar c)^3}\right)^2 E_0 \sim \alpha^2 a_0^4 E_0^5 \times \frac{1}{\hbar^5 c^4} \sim \alpha^2\left(\frac{\hbar}{mc\alpha}\right)^4 \left(mc^2\alpha^2\right)^5 \frac{1}{\hbar^5 c^4} \sim \alpha^8\frac{mc^2}{\hbar},$$

which is $\alpha^3 \approx 4\,10^{-7}$ smaller than the single-photon transition 2p→1s. A detailed calculation shows that the correct transition rate is $\approx 8.229\,\text{s}^{-1}$, which is one order of magnitude smaller than our naive calculation. Therefore, the transition rates is 8 orders of magnitude smaller for the 2s state, meaning it is a metastable state. This could be used to produce entangled photon pairs.

The alternative route decay 2s→2p→1s seems to be an important one because the transition rate 2p→1s is very high (of order 10^8 bigger than the current rate). However, the splitting between the 2s and 2p (due to Lamb shift) is very small. This makes the transition rate indeed even smaller than the previous 2-photon decay considered. We can estimate the order of magnitude of the 2s→2p electric dipole transition. As before,

$$\sim \frac{1}{\hbar}\left(\frac{\alpha\hbar c\Delta E}{V}a_0^2\right)\times\left(\frac{V\Delta E^2}{(\hbar c)^3}\right) \sim \frac{\alpha a_0^2 \Delta E^3}{\hbar^3 c^2},$$

where ΔE is of order of the splitting between the 2s and 2p states, which is of order of $1\,\text{GH}_\text{z}$ (as seen in class) or $h\nu \approx 4\,10^{-6}\,\text{eV}$. Thus,

$$W_{2s\to 2p} \sim W_{2p\to 1s} \times \left(\frac{\Delta E}{E_0}\right)^3 \sim 2.5\,10^{-20} W_{2p\to 1s}.$$

This is extreme smaller than the previous 2-photon decays. Thus, although the route decay 2s→2p→1s is favored by the fast decay of the last process, it is hindered by the very first step due to the almost degeneracy between states 2s and 2p.

3. Will be typed soon...

(*a*)
(*b*)
(*c*)
(*d*)
4.
(*a*)
(*b*)
(*c*)

5.
(*a*)
(*b*)
(*c*)
(*d*)
(*e*)
(*f*)

6.
(*a*)
(*b*)
(*c*)