1ª Prova de Quântica B - SFI5707

#USP: Nome:

11 de abril de 2013

1. (*2,5 pontos*) Um átomo de H é colocado entre as placas de um capacitor onde um pulso de voltagem é aplicado de forma a produzir um campo elétrico homogêneo com dependência temporal $\mathbf{E} = \mathbf{E}_0 e^{-t/\tau}\theta(t)$. Sendo que o átomo está inicialmente no seu estado fundamental, pede-se:

(a) A probabilidade, em mais baixa ordem de teoria de perturbação não-nula, de encontrar o átomo em um dos estados 2p após decorrido um tempo muito longo.

(b) E para o estado 2s? (Para esse item, não é necessário fazer o cálculo, apenas indique qual a menor ordem de teoria de perturbação não-nula e descreva fisicamente o processo de transição.)

2. (*4,0 pontos*) Considere um partícula de spin-1/2 em um campo magnético $\mathbf{B}(t) = B_0\left(\hat{z} + \alpha e^{-(t/\tau)^2}\hat{x}\right)$. Sabe-se que em $t = -\infty$ a partícula encontra-se no estado $|\uparrow\rangle$. Lembrando que o momento magnético da partícula é $\boldsymbol{\mu} = \gamma\mathbf{S}$, pergunta-se:

(a) (*1,5 ponto*) Em primeira ordem de teoria de perturbação, qual a probabilidade de encontrar a partícula no estado $|\downarrow\rangle$ no tempo $t = +\infty$?

(b) (*1,0 ponto*) Quais condições B_0, α, τ e γ devem satisfazer para que este resultado seja válido?

(c) (*1,0 ponto*) Em primeira ordem na aproximação adiabática, qual a probabilidade de encontrar a partícula no estado $|\downarrow\rangle$ no tempo $t = +\infty$? (Aqui, apenas monte a integral a ser calculada.)

(d) (*0,5 ponto*) Quais condições B_0, α, τ e γ devem satisfazer para que a aproximação adiabática seja válida?

3. (*2,5 pontos*) Considere um oscilador harmônico unidimensional cujo Hamiltoniano não perturbado é $H_0 = \hbar\omega\left(a^\dagger a + 1/2\right)$.

(a) (*0,5 ponto*) Considere uma perturbação do tipo $H_1 = g_1 x e^{-(t/\tau)^2}$, com g_1 e τ constantes, e x sendo o operador posição. Qual o significado físico de H_1?

(b) (*0,5 ponto*) Considere que, em $t = -\infty$, o oscilador está no n-ésimo auto-estado de H_0: $|n\rangle$. Em que ordem de teoria de perturbação a probabilidade de achar o sistema no auto-estado de H_0 $|n+m\rangle$ (considere $m > 0$) é diferente de zero?

(c) (*0,8 ponto*) Em menor ordem de teoria de perturbação não nula, calcule a probabilidade de achar o oscilador no estado $|n+m\rangle$ no instante $t = +\infty$.

(d) (*0,4 ponto*) Consider agora uma perturbação do tipo $H_2 = g_2 x^2 e^{-(t/\tau)^2}$, com g_2 constante. Dê uma interpretação física para H_2.

(e) (*0,3 ponto*) Em quais ordens de teoria de perturbação, a probabilidade de transição do estado $|n\rangle$ para o estado $|n+m\rangle$ é não-nula?

4. (*2,5 pontos*) Considere um átomo de H fixo na origem de um certo sistema de coordenadas. Uma partícula infinitamente pesada de carga Ze (que deve ser tratada classicamente) é então lançada em direção ao átomo na trajetória $\mathbf{R}(t) = vt\hat{x} + d\hat{y}$.

(a) (*0,4 ponto*) Escreva a Hamiltoniano $H_1(t)$, que corresponde à interação Coulombiana entre a partícula e o elétron do átomo, em termos de Z, e, $\mathbf{R}$, e $\mathbf{r}$ (o operador posição do elétron).

(b) (*1,2 ponto*) Considere H_1 como uma perturbação dependente do tempo. Mostre que para $d \gg a_0$, a probabilidade de transição eletrônica é

$$P_{f\leftarrow i} \approx \frac{Z^2 e^4}{(4\pi\epsilon_0\hbar)^2}\left|\int_{-\infty}^{\infty} \mathrm{d}t e^{i\omega_{fi}t}\frac{vt\langle f|x|i\rangle + d\langle f|y|i\rangle}{\left((vt)^2 + d^2\right)^{3/2}}\right|^2.$$

Interprete cada um desses elementos de matriz.

(c) (*0,4 ponto*) Para que valores de t a integral acima é dominada? Qual a interpretação física desse intervalo?

(d) (*0,5 ponto*) Chamando o intervalo do item anterior de τ, calcule $P_{f\leftarrow i}$ no limite em que $|\omega_{fi}|\tau \gg 1$. Qual o significado físico desse limite?

DADOS:

$$Y_{00} = \sqrt{\frac{1}{4\pi}};\quad Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta;\quad Y_{1\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta e^{\pm i\varphi};$$

$$R_{10} = 2\left(\frac{1}{a_0}\right)^{\frac{3}{2}} e^{-\frac{r}{a_0}};\quad R_{20} = \left(\frac{1}{2a_0}\right)^{\frac{3}{2}}\left(2-\frac{r}{a_0}\right)e^{-\frac{r}{2a_0}};\quad R_{21} = \left(\frac{1}{2a_0}\right)^{\frac{3}{2}}\left(\frac{r}{a_0\sqrt{3}}\right)e^{-\frac{r}{2a_0}};$$

$$S^x\left|\uparrow\downarrow\right\rangle = \frac{\hbar}{2}\left|\downarrow\uparrow\right\rangle;\quad a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + i\frac{p}{m\omega}\right);\quad \left[a, a^\dagger\right] = 1;$$

$$\int_{-\infty}^{\infty}\mathrm{d}t e^{-\left(\frac{t}{\tau}\right)^2+2i\omega t} = \sqrt{\pi}\tau e^{-(\omega\tau)^2};$$

$$\int_0^\infty \mathrm{d}x\frac{x\sin ax}{\left(x^2+\beta^2\right)^{3/2}} = aK_0\left(a\beta\right);\quad \int_0^\infty \mathrm{d}x\frac{\cos ax}{\left(x^2+\beta^2\right)^{3/2}} = -\frac{1}{\beta}\frac{\partial}{\partial\beta}K_0\left(a\beta\right);$$

A função de Bessel $K_0(z) \approx \ln\left(\frac{2}{z}\right)$ no limite $z \ll 1$.

A função de Bessel $K_0(z) \approx \sqrt{\frac{2}{\pi z}}e^{-z}$ no limite $z \gg 1$.

GABARITO:
1.
(a) In first order of perturbation theory,

$$P_{2p\leftarrow 1s} = \frac{1}{\hbar^2}\left|\int_{-\infty}^{\infty} \mathrm{d}t \left\langle 2p\left|V(t)\right|1s\right\rangle e^{-i\omega t}\right|^2,$$

where $\hbar\omega = E_{2p} - E_{1s}$ is the energy difference between the final and initial states (which is 13.6 eV times 3/4). As we show below, this is finite. Thus, 1st order of perturbation theory is the lowest order in which the transition probability is different from zero.

The perturbation is

$$V(t) = -eE_0 z e^{-t/\tau}\theta(t),$$

where we chose the electric field to be in the z-direction: $\mathbf{E} = E_0 e^{-t/\tau}\theta(t)\hat{z}$.

We now have to compute the integral. The time-dependent one is

$$\left|\int_0^{\infty} \mathrm{d}t e^{-(1/\tau + i\omega)t}\right|^2 = \frac{\tau^2}{1+(\omega\tau)^2}.$$

The spatial integral is the matrix element

$$\left|-eE_0\left\langle 2p\left|z\right|1s\right\rangle\right|^2 = (eE_0)^2\left|\int r^2\mathrm{d}r\mathrm{d}\Omega R_{21}^*(r)Y_{1m}^*(\theta,\varphi)\left(r\cos\theta\right)R_{10}(r)Y_{00}(\theta,\varphi)\right|^2,$$

where we used that $z = r\cos\theta$. The integral over the angle coordinates yields the selection rules. It is different from zero only if $m = 0$. Thus,

$$\int \mathrm{d}\Omega Y_{1m}^*(\theta,\varphi)\left(\cos\theta\right)Y_{00}(\theta,\varphi) = \int \mathrm{d}\Omega Y_{1m}^*(\theta,\varphi)\frac{1}{\sqrt{3}}Y_{10}(\theta,\varphi) = \frac{1}{\sqrt{3}}\delta_{m,0}.$$

The integral over r is

$$\int_0^{\infty} \mathrm{d}r R_{21}^*(r)\left(r^3\right)R_{10}(r) = \frac{1}{a_0^4\sqrt{6}}\int_0^{\infty}\mathrm{d}r r^4 e^{-\frac{3r}{2a_0}} = \frac{1}{a_0^4\sqrt{6}}\left(\frac{4!}{\left(\frac{3r}{2a_0}\right)^5}\right) = \frac{2^8}{\sqrt{6}3^4}a_0.$$

Putting everything together:

$$P_{2p\leftarrow 1s} = \sum_m \frac{1}{\hbar^2}\left(\frac{\tau^2}{1+(\omega\tau)^2}\right)\left(\frac{2^8}{\sqrt{6}3^4}eE_0a_0\right)^2\frac{\delta_{m,0}}{3} = \frac{1}{\hbar^2}\left(\frac{\tau^2}{1+(\omega\tau)^2}\right)(eE_0a_0)^2\frac{2^{15}}{3^{10}}.$$

Notice we are summing over all m in order to find the probability of transitioning into any 2p state. The same result would hold if we had chosen any other direction of the electric field.

(b) In 1st order of perturbation theory, there is no transition to the 2s state. The corresponding selection rule involves the integral

$$\int r^2\mathrm{d}r\mathrm{d}\Omega\left\langle 200\left|z\right|100\right\rangle \propto \int \mathrm{d}\Omega Y_{00}^*\cos\theta Y_{00} = 0.$$

The same would be true if we had chosen any other direction ($\hat{x}$ or $\hat{y}$).

The fundamental reason for no probability transition in first order is due to angular momentum conservation. A direct transition (i.e., involving only a single photon - even though we are treating the electric field classically) between two s-states is not possible because the photon carries angular momentum.

Therefore, a transition to the 2s state can only be possible in 2nd order of perturbation theory. This would involve two photons. The first (absorbed) photon, would transition the electron to the higher energy p-state, say, 3p, and the second (emitted) photon, would carry away the energy excess from this intermediate high-energy p-state allowing the electron to transition down to the 2s state. The corresponding integral would be

$$\frac{1}{\hbar^4}\left|\sum_{n>2}\sum_{m=-1}^{1}\int_{-\infty}^{\infty}\mathrm{d}t\int_{-\infty}^{t}\mathrm{d}t' e^{i\omega_{2s-n}t}e^{i\omega_{n-1s}t'}\left\langle 2,0,0\left|V(t)\right|n,1,m\right\rangle\left\langle n,1,m\left|V(t')\right|1,0,0\right\rangle\right|^2.$$

2.

(a) We recognize $H_0=-\gamma S^z B_0$ as the unperturbed Hamiltonian. The Eigenstates of which are $|\uparrow\downarrow\rangle$ with energies $E_{\uparrow\downarrow}=\mp\frac{1}{2}\hbar\gamma B_0=\mp\hbar\omega$. The perturbation is therefore $V=-\alpha\gamma S^x B_0 e^{-(t/\tau)^2}$. In first order of perturbation theory, the transition probability is

$$P=\frac{1}{\hbar^2}\left|\int_{-\infty}^{\infty}\mathrm{d}t e^{2i\omega t}\left\langle\downarrow\left|\alpha\gamma B_0 S^x\right|\uparrow\right\rangle e^{-(t/\tau)^2}\right|^2=\frac{1}{\hbar^2}\left(\alpha\frac{\hbar}{2}\gamma B_0\right)^2\left|\tau e^{-(\omega\tau)^2}\sqrt{\pi}\right|^2$$
$$=\pi\left(\alpha\omega\tau\right)^2 e^{-2(\omega\tau)^2}.$$

(b) In order to perform perturbation theory, we need the perturbation $V(t)$ to be small during all times compared to H_0. Thus $\alpha\ll 1$. Furthermore, in order to 1st order be valid, we need to compare with the correction coming from higher orders. Notice there is no contribution from 2nd order:

$$\delta P^{(2)}=\frac{1}{\hbar^4}\left|\sum_e\int_{-\infty}^{\infty}\mathrm{d}t\int_{-\infty}^{t}\mathrm{d}t' e^{\frac{i}{\hbar}(E_\downarrow-E_e)t}e^{\frac{i}{\hbar}(E_e-E_\uparrow)t'}\left\langle\downarrow\left|V(t)\right|e\right\rangle\left\langle e\left|V(t')\right|\uparrow\right\rangle\right|^2,$$

since $S^x|\uparrow\downarrow\rangle\propto|\downarrow\uparrow\rangle$. In third order of perturbation theory,

$$\delta P^{(3)}=\frac{1}{\hbar^6}\left|\sum_{e,e'}\int_{-\infty}^{\infty}\mathrm{d}t\int_{-\infty}^{t}\mathrm{d}t'\int_{-\infty}^{t'}\mathrm{d}t'' e^{\frac{i}{\hbar}(E_\downarrow-E_e)t}e^{\frac{i}{\hbar}(E_e-E_{e'})t'}e^{\frac{i}{\hbar}(E_{e'}-E_\uparrow)t''}\left\langle\downarrow\left|V(t)\right|e\right\rangle\left\langle e\left|V(t')\right|e'\right\rangle\left\langle e'\left|V(t'')\right|\uparrow\right\rangle\right|^2.$$

The only contribution comes from $e'=\downarrow$ and $e=\uparrow$. Thus,

$$\delta P^{(3)}=\frac{1}{\hbar^6}\left(\alpha\hbar\omega\right)^6\left|\int_{-\infty}^{\infty}\mathrm{d}t\int_{-\infty}^{t}\mathrm{d}t'\int_{-\infty}^{t'}\mathrm{d}t'' e^{2i\omega\left(t-t'+t''\right)}e^{-\left(t^2+t'^2+t''^2\right)/\tau^2}\right|^2.$$

It is not simple to compute exactly $\delta P^{(3)}$. However, we can perform a crude estimate by extending the integration limits of t' and t'' to infinity. In this way, each integral separates and we find

$$\delta P^{(3)}\lessapprox\pi^{\frac{3}{2}}\left(\alpha\omega\tau\right)^6 e^{-6(\omega\tau)^2}.$$

Finally, we conclude that, in order to 1st order of perturbation theory to be valid, we need that $\alpha\ll 1$, and that $\omega\tau\gg 1$, i.e., $\gamma B_0\tau\gg 1$.

(c) Let us compute the instantaneous Eigensates. Diagonalizing

$$H=-\boldsymbol{\mu}\cdot\mathbf{B}=-\hbar\omega\begin{pmatrix}1 & \alpha e^{-\left(\frac{t}{\tau}\right)^2}\\ \alpha e^{-\left(\frac{t}{\tau}\right)^2} & -1\end{pmatrix},$$

we find

$$|1\rangle=\frac{1}{N}\left[\alpha e^{-\left(\frac{t}{\tau}\right)^2}|\uparrow\rangle+\left(f_\alpha(t)-1\right)|\downarrow\rangle\right]\text{ and }|2\rangle=\frac{1}{N}\left[\left(f_\alpha(t)-1\right)e|\uparrow\rangle-\alpha e^{-\left(\frac{t}{\tau}\right)^2}|\downarrow\rangle\right],$$

with energies $\mp\hbar\omega f_\alpha(t)$, respectively, and $f_\alpha(t)=\sqrt{1+\alpha^2 e^{-2\left(\frac{t}{\tau}\right)^2}}$. The factor N is the normalization. The adiabatic Eigensates are (notice there is no Berry phase since the solid angle enclosed by $\mathbf{B}(t)$ is zero)

$$\left|\tilde{i}\right\rangle=e^{\frac{i}{\hbar}\int_{-\infty}^{t}E_i(t')\mathrm{d}t'}|i\rangle,\text{ with } i=1,2.$$

Notice

$$\theta=\theta_1=\frac{1}{\hbar}\int_{-\infty}^{t}E_1(t')\mathrm{d}t'=-\omega\int_{-\infty}^{t}f_\alpha(t')\mathrm{d}t'=-\theta_2.$$

The adiabatic wave function evolve in time as

$$\left|\tilde{\psi}(t)\right\rangle=\tilde{c}_1(t)\left|\tilde{1}\right\rangle+\tilde{c}_2(t)\left|\tilde{2}\right\rangle,$$

with the initial condition that $\left|\tilde{\psi}(-\infty)\right\rangle = |\psi(-\infty)\rangle = |\uparrow\rangle$. Thus, $\tilde{c}_1(-\infty) = c_1(-\infty) = 1$ and $\tilde{c}_2(-\infty) = c_2(-\infty) = 0$ (since $|1(\pm\infty)\rangle = |\uparrow\rangle$ and $|2(\pm\infty)\rangle = |\downarrow\rangle$).

The transition probability is

$$P = |\langle\downarrow|\psi(\infty)\rangle|^2 = \left|\left\langle\downarrow|\tilde{\psi}(\infty)\right\rangle\right|^2 = |\tilde{c}_2(\infty)|^2 .$$

In 1st order of adiabatic approximation, the coefficients evolve as

$$\tilde{c}_i(t) = c_i(-\infty) + \sum_{j\neq i}\int_{-\infty}^{t} \mathrm{d}t' \frac{\left\langle \tilde{i}\left|\dot{H}\right|\tilde{j}\right\rangle}{E_i - E_j} c_j(-\infty).$$

Therefore,

$$\begin{aligned}
\tilde{c}_2(\infty) &= \int_{-\infty}^{\infty} \mathrm{d}t \frac{\left\langle \tilde{2}\left|\dot{H}\right|\tilde{1}\right\rangle}{E_2 - E_1} = \int_{-\infty}^{\infty} \mathrm{d}t \frac{\left\langle 2\left|\frac{4\alpha\omega}{\tau^2} t e^{-\left(\frac{t}{\tau}\right)^2} S^x\right|1\right\rangle}{2\hbar\omega f_\alpha(t)} e^{2i\theta(t)} \\
&= \frac{\alpha}{\tau^2}\int_{-\infty}^{\infty} \mathrm{d}t \frac{t e^{-\left(\frac{t}{\tau}\right)^2}}{f_\alpha(t) N^2}\left((f_\alpha(t)-1)^2 - \alpha^2 e^{-2\left(\frac{t}{\tau}\right)^2}\right) e^{2i\theta(t)} \\
&= \frac{\alpha}{\tau^2}\int_{-\infty}^{\infty} \mathrm{d}t \frac{t e^{-\left(\frac{t}{\tau}\right)^2 + 2i\theta(t)}}{1 + \alpha^2 e^{-2\left(\frac{t}{\tau}\right)^2}}.
\end{aligned}$$

(d) The adiabatic approximation is justified whenever the angular frequency that the instantaneous basis rotate is much smaller than the Bohr frequency. For our problem, the angular frequency can be estimated by the maximum value of

$$\alpha = \langle 2|\frac{\mathrm{d}}{\mathrm{d}t}|1\rangle .$$

The Bohr frequency is $\omega = \left(E_\downarrow - E_\uparrow\right)/\hbar$. The adiabatic approximation is valid when $\alpha_{\max}/\omega_{\min} \ll 1$, i.e., the basis rotate much slower than the system resonance frequencies. It is easy to compute $\alpha_{\max}$, though it is combersome. Consider α small and that its maximum value happens when $t \approx \tau$, we find that

$$\alpha_{\max} \propto \frac{\alpha}{\tau}.$$

The mininum energy gap happens for $t \to \pm\infty$. Thus, $\omega_{\min} = 2\gamma B_0$. Finally, conclude that $\gamma B_0 \tau \gg \alpha$ in order to the adiabatic approximation be valid. Notice that the restrictions in item (b) are more restrictives than this one.

3.

(a) This corresponds a constant force (in space), for instance, a constant (in space) electric field if the particle is charged.

(b) Let us rewrite the perturbation in terms of the raising and lowering operators:

$$H_1 = g_1\sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right) e^{-(t/\tau)^2} = \hbar\omega_1\left(a + a^\dagger\right) e^{-(t/\tau)^2},$$

where m is the particle mass. From the Dyson series,

$$\langle j(t)|k(t)\rangle = \delta_{j,k} + \frac{1}{i\hbar}\int_{-\infty}^{t} \mathrm{d}t_1 \langle j|H_{1,I}(t_1)|k\rangle + \left(\frac{1}{i\hbar}\right)^2 \int_{-\infty}^{t} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2 \langle j|H_{1,I}(t_1)H_{1,I}(t_2)|k\rangle + \ldots,$$

where $H_{1,I} = e^{iH_0t/\hbar}H_1e^{-iH_0t/\hbar}$ is the perturbation in the interaction picture, we notice that the states $|j\rangle$ and $|k\rangle$ can only be "connected" if there are enough a and $a^\dagger$ in between. Thus, if $j > k$, the first contribution comes from the $(j-k)$-th order in perturbation theory. The transition amplitude reads (set $l = j - k$)

$$\left(\frac{1}{i\hbar}\right)^l \sum_{\psi_1,\ldots,\psi_{l-1}} \int_{-\infty}^{t} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2 \ldots \int_{-\infty}^{t_{l-1}} \mathrm{d}t_l e^{i\omega_{j1}t_1 + i\omega_{12}t_2 + \cdots + \omega_{l-1l}t_l} \langle j|H_1(t_1)|\psi_1\rangle\langle\psi_1|H_1(t_2)|\psi_2\rangle\ldots\langle\psi_{k-1}|H_1(t_l)|k\rangle .$$

The only contribution term of these sums are when $|\psi_{k-1}\rangle = |k+1\rangle$, $|\psi_{k-2}\rangle = |k+2\rangle$, and so on until $|\psi_1\rangle = |j-1\rangle$. Each these states are connected via $a^\dagger$ coming from H_1. Then, it simplifies to

$$\frac{\omega_1^l}{i^l}\int_{-\infty}^{t} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2 \ldots \int_{-\infty}^{t_{l-1}} \mathrm{d}t_l e^{i\omega(t_1+t_2+\cdots+t_l)} \langle j|\, a^\dagger\, |j-1\rangle \langle j-1|\, a^\dagger\, |j-2\rangle \ldots \langle k+1|\, a^\dagger\, |k\rangle\, e^{-\frac{1}{\tau^2}\left(t_1^2+t_2^2+\cdots+t_l^2\right)}.$$

Notice that in $(l-1)$-th order, there will be no way to connect $|k\rangle$ to $|j\rangle$. Thus, the least order in perturbation theory connecting states $|n\rangle$ and $|m+n\rangle$ is $|m|$.

(c) We now need to compute the matrix element obtained in (b). Using that $a^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle$, it simplifies to

$$\frac{\omega_1^m}{i^m}\sqrt{(m+n)\ldots(n+2)(n+1)}\int_{-\infty}^{t} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2 \ldots \int_{-\infty}^{t_{m-1}} \mathrm{d}t_m e^{i\omega(t_1+t_2+\cdots+t_m)} e^{-\frac{1}{\tau^2}\left(t_1^2+t_2^2+\cdots+t_m^2\right)}.$$

Notice also that the integrals are all symmetric by exchanging the labels. Thus,

$$\int_{-\infty}^{t} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2 \ldots \int_{-\infty}^{t_{m-1}} \mathrm{d}t_m e^{i\omega(t_1+t_2+\cdots+t_m)} e^{-\frac{1}{\tau^2}\left(t_1^2+t_2^2+\cdots+t_m^2\right)} = \frac{1}{m!}\left[\int_{-\infty}^{t} \mathrm{d}t_1 e^{-t_1^2/\tau^2 + i\omega t_1}\right]^m.$$

Finally, at $t\to\infty$, the probability transition is

$$P = \left|\frac{\omega_1^m}{i^m}\sqrt{\frac{(m+n)!}{n!}}\frac{1}{m!}\left(\sqrt{\pi}\tau e^{-\frac{1}{4}(\omega\tau)^2}\right)^m\right|^2 = \left(\pi\frac{(m+n)!}{m!n!}\right)^m (\omega_1\tau)^{2m}\, e^{-\frac{m}{2}(\omega\tau)^2}.$$

(d) This perturbation means that the strength of the confining potential changes in time. For instance, one can think about atoms optically trapped and the intensity of the lasers is being changed.

(e) Following the same reasoning of item (b), we rewrite

$$H_2 = g_2\left(\frac{\hbar}{2m\omega}\right)\left(a^2 + 1 + 2a^\dagger a + a^{\dagger 2}\right) e^{-\left(\frac{t}{\tau}\right)^2}.$$

Hence, in the Dyson's series, the perturbation will only connect intermediate states that differs by of none or two energy quanta. Thus, states $|n\rangle$ and $|m+n\rangle$ can only be connected if m is even in any order of perturbation theory. Furthermore, the least order that contributes is $\frac{m}{2}$.

4.

(a) This is the usual Coulomb interaction

$$H_1(t) = -\frac{Ze^2}{4\pi\epsilon_0}\frac{1}{|\mathbf{r}-\mathbf{R}(t)|},$$

with $\mathbf{r}$ being the operator (which is constant in the Schrödinger representation) and $\mathbf{R}$ being the classical trajectory of the heavy particle.

(b) In the limit $R \gg a_0$, the heavy particle is faraway from the electron which is bounded around the origin from distances of order a_0. Thus, we have that $R(t) \gg \langle r\rangle$ for all times. We then approximate

$$\frac{1}{|\mathbf{r}-\mathbf{R}(t)|} = \frac{1}{R\sqrt{1-2\frac{\mathbf{r}\cdot\mathbf{R}}{R^2}+\frac{r^2}{R^2}}} \approx \frac{1}{R}\left(1+\frac{\mathbf{r}\cdot\mathbf{R}}{R^2}\right),$$

where term $\mathcal{O}(r/R)^2$ were neglected. Thus, in 1st order of perturbation theory

$$P_{f\leftarrow i} = \left(\frac{Ze^2}{4\pi\epsilon_0\hbar}\right)^2 \left|\int \mathrm{d}t e^{i\omega_{fi}t}\frac{\langle f\,|\mathbf{r}|\,i\rangle\cdot\mathbf{R}}{R^3}\right|^2.$$

Notice the 1st term $\propto R^{-1}$ only shifts the total energy and does not induces transitions. Using the trajectory $\mathbf{R}(t) = vt\hat{x} + d\hat{y}$, we arrive at the desired result

$$P_{f\leftarrow i} = \left(\frac{Ze^2}{4\pi\epsilon_0\hbar}\right)^2 \left|\int_{-\infty}^{\infty} \mathrm{d}t e^{i\omega_{fi}t}\frac{\langle f\,|x|\,i\rangle\, vt + \langle f\,|y|\,i\rangle\, d}{\left((vt)^2+d^2\right)^{3/2}}\right|^2.$$

Finally, notice that the matrix elements $\langle f\,|x,y|\,i\rangle$ are the usual electric dipole ones, as expected from this level of approximation.

(c) Notice that for long times, the integrand vanishes as $\sim t^{-2}$. The main contribution comes from the time interval in which $vt \approx d$. Thus, $\tau = d/v$. This corresponds to the time needed by the particle to go from the nucleous to the point of its maximum approach $d\hat{y}$. For matters of effect, we can say that τ is the characteristic time of the perturbation, since for $t \gg \tau$, the perturbing effects are much smaller. We can thus conclude that the main contribution comes when the heavy particle at distances of order d around the yz plane (the plane perpendicular to its trajectory containing the atom).

(d) We now have to compute

$$\int_{-\infty}^{\infty} \mathrm{d}t e^{i\omega_{fi}t}\frac{x_{fi}vt + y_{fi}d}{\left((vt)^2+d^2\right)^{3/2}} = x_{fi}v\int_{-\infty}^{\infty}\mathrm{d}t\frac{te^{i\omega t}}{\left((vt)^2+d^2\right)^{3/2}} + y_{fi}d\int_{-\infty}^{\infty}\mathrm{d}t\frac{e^{i\omega t}}{\left((vt)^2+d^2\right)^{3/2}}$$

$$= 2i\frac{x_{fi}}{v^2}\int_0^{\infty}\mathrm{d}t\frac{t\sin\omega t}{(t^2+\tau^2)^{3/2}} + 2\frac{y_{fi}d}{v^3}\int_{-\infty}^{\infty}\mathrm{d}t\frac{\cos\omega t}{(t^2+\tau^2)^{3/2}}$$

$$= 2i\frac{x_{fi}}{v^2}\omega K_0(\omega\tau) - 2\frac{y_{fi}d}{v^3}\frac{1}{\tau}\frac{\partial}{\partial\tau}K_0(\omega\tau) \to 2i\frac{x_{fi}}{v^2}\omega\sqrt{\frac{2}{\pi\omega\tau}}e^{-\omega\tau} + 2\frac{y_{fi}d}{v^3}\frac{\omega}{\tau}\sqrt{\frac{2}{\pi\omega\tau}}e^{-\omega\tau}.$$

Finally,

$$P_{f\leftarrow i} \approx \left(\frac{Ze^2}{4\pi\epsilon_0\hbar}\right)^2\left|2i\frac{x_{fi}}{v^2}+2\frac{y_{fi}}{v^2}\right|^2\left(\frac{2\omega_{fi}}{\pi\tau}\right)e^{-2\omega_{fi}\tau} = \frac{Z^2e^4}{2\pi^3\epsilon_0^2\hbar^2v^2d^2}\left|ix_{fi}+y_{fi}\right|^2\omega_{fi}\tau e^{-2\omega_{fi}\tau}.$$

Notice that $x - iy = r_-$ is exactly the operator promoting a variation of angular momentum in the $\hat{z}$-direction of $\Delta m = -\hbar$.

The meaning of the limit $\omega_{fi}\tau \ll 1$ $\omega_{fi}\tau \gg 1$ is that of a slow heavy particle. τ is the characteristic time of the heavy particle passing around the atom and perturbing it. ω_{fi} is the Bohr frequency the inverse of which is the characteristic time of the electron resonating between states $|f\rangle$ and $|i\rangle$. Thus, if $\tau \gg \omega_{fi}^{-1}$, this means that the particle passed slowly around the atom. In this limit, the atom "sees" the particle's electric field changing adiabatically during the interval time τ.

If one wishes to study the fast particle limit $\omega\tau \ll 1$ (which is not appropriate in the current formulation), then one would find

$$= 2i\frac{x_{fi}}{v^2}\omega K_0(\omega\tau) - 2\frac{y_{fi}d}{v^3}\frac{1}{\tau}\frac{\partial}{\partial\tau}K_0(\omega\tau) \to 2i\frac{x_{fi}}{v^2}\omega\ln\left(\frac{2}{\omega\tau}\right) + 2\frac{y_{fi}d}{v^3}\frac{1}{\tau^2}.$$

Then,

$$P_{f\leftarrow i} \approx \left(\frac{Ze^2}{4\pi\epsilon_0\hbar}\right)^2\left|2i\frac{x_{fi}}{v^2}\omega_{fi}\ln\left(\frac{2}{\omega_{fi}\tau}\right)+2\frac{y_{fi}}{v^2\tau}\right|^2 = \frac{Z^2e^4}{4\pi^2\epsilon_0^2\hbar^2v^2d^2}\left|ix_{fi}\omega_{fi}\tau\ln\left(\frac{2}{\omega_{fi}\tau}\right)+y_{fi}\right|^2.$$