REVIEW OF CLASSICAL E-M.

$$\nabla\cdot\vec{E} = \frac{1}{\epsilon_0}\rho \qquad \nabla\cdot\vec{B} = 0$$

$$\nabla\times\vec{B} = \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E} + \frac{1}{\epsilon_0 c^2}\vec{J} \qquad \nabla\times\vec{E} = -\frac{\partial}{\partial t}\vec{B}$$

FROM THESE TWO

$\vec{B} = \nabla\times\vec{A}$, $\vec{A}$ = VECTOR POTENTIAL

$\vec{E} = -\nabla\phi - \frac{\partial}{\partial t}\vec{A}$, ϕ = SCALAR POTENTIAL

GAUGE TRANSFORMATIONS

$$\left.\begin{aligned} \vec{A}' &= A + \nabla\chi \\ \phi' &= \phi - \frac{\partial\chi}{\partial t} \end{aligned}\right\} \quad \begin{aligned} \vec{E}' &= \vec{E} \\ \vec{B}' &= \vec{B} \end{aligned}$$

REWRITE MAXWELL'S EQUATIONS:

USING THAT

$$\nabla\times(\nabla\times\vec{A}) = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}$$

$$\begin{cases} \nabla\cdot\vec{E} = \nabla\cdot\left(-\nabla\phi - \frac{\partial}{\partial t}\vec{A}\right) = \frac{1}{\epsilon_0}\rho \\ \nabla\times\vec{B} = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A} = \frac{1}{c^2}\left(-\nabla\left(\frac{\partial}{\partial t}\phi\right) - \frac{\partial^2}{\partial t^2}\vec{A}\right) + \frac{1}{\epsilon_0 c^2}\vec{J} \end{cases}$$

$$\begin{cases} \nabla^2\phi = -\frac{1}{\epsilon_0}\rho - \frac{\partial}{\partial t}(\nabla\cdot\vec{A}) \\ \nabla\left(\nabla\cdot\vec{A} + \frac{1}{c^2}\frac{\partial}{\partial t}\phi\right) = \nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} + \frac{1}{\epsilon_0 c^2}\vec{J} \end{cases}$$

*OBS: IN THE LORENTZ GAUGE, $\nabla\cdot\vec{A} + \frac{1}{c^2}\frac{\partial}{\partial t}\phi = 0$

$$\Rightarrow \begin{cases} \nabla^2\phi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi = -\frac{1}{\epsilon_0}\rho = \Box\phi \\ \nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} = -\frac{1}{\epsilon_0}\frac{\vec{J}}{c^2} = \Box\vec{A} \end{cases}$$

$\Box$ = D'ALEMBERTIAN $= \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$

*OBS: IN THE FOUR-VECTOR NOTATION (EUCLIDEAN METRIC)

$$\vec{A} = \left(i\frac{\phi}{c}, A_x, A_y, A_z\right); \quad \vec{J} = (ic\rho, J_x, J_y, J_z) \quad \Rightarrow \boxed{\Box\vec{A} = -\mu_0\vec{J}}$$

ENERGY DENSITY

$$\mathcal{E} = \frac{1}{2}\left(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H}\right) = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0} B^2 \quad ; \quad \text{ENERGY} = \int d^3r\, \mathcal{E}(\vec{r},t)$$

POYNTING VETOR : $\vec{N} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$

*OBS: NOTICE WE CAN SET $\phi = 0$ BY MAKING A GAUGE TRANSFORMATION

$$\phi' = \phi - \frac{1}{c}\frac{\partial}{\partial t}\chi \quad \text{SUCH THAT} \quad \frac{1}{c}\frac{\partial}{\partial t}\chi = \phi$$

$$\Rightarrow \mathcal{E} = \frac{1}{2}\epsilon_0\left|-\frac{\partial}{\partial t}\vec{A}\right|^2 + \frac{1}{2\mu_0}\left|\nabla\times\vec{A}\right|^2$$

$$\vec{N} = -\frac{1}{\mu_0}\left(\frac{\partial}{\partial t}\vec{A}\right)\times\left(\nabla\times\vec{A}\right)$$

IF YOU DON'T WANT TO SET $\phi = 0$,

$$\mathcal{E} = \frac{1}{2}\epsilon_0\left|-\frac{\partial}{\partial t}\vec{A} - \nabla\phi\right|^2 + \frac{1}{2\mu_0}\left|\nabla\times\vec{A}\right|^2$$

$$\left|\nabla\phi + \frac{\partial}{\partial t}\vec{A}\right|^2 = \left|\nabla\phi\right|^2 + 2\left(\frac{\partial}{\partial t}\vec{A}\right)\cdot\nabla\phi + \left|\frac{\partial}{\partial t}\vec{A}\right|^2$$

- IN THE TRANSVERSE GAUGE $\nabla\cdot\vec{A} = 0 \Rightarrow$ THE CROSS TERM INTEGRATES TO ZERO

$$\text{ENERGY}_{\text{CROSS}} = \epsilon_0\int d^3r\left(\frac{\partial}{\partial t}\vec{A}\right)\cdot\nabla\phi = 2\epsilon_0\Big[\underbrace{\left|\frac{\partial}{\partial t}\vec{A}\right|\phi\,\Big|_{-\infty}^{\infty}}_{0 \text{ (NO CHARGES AT } \infty)} - \int d^3r\,\phi\,\frac{\partial}{\partial t}\underbrace{\left(\nabla\cdot\vec{A}\right)}_{0}\Big]$$

THUS, ENERGY CROSS = 0

$$\text{ENERGY}_{|\nabla\phi|^2} = \epsilon_0\int d^3r\,\left|\nabla\phi\right|^2 = \epsilon_0\int d^3r\,\nabla\phi\cdot\nabla\phi = \underbrace{\epsilon_0\phi\,|\nabla\phi|\,\Big|_{-\infty}^{\infty}}_{0} - \epsilon_0\int d^3r\,\phi\,\nabla^2\phi$$

$$\text{BUT } \nabla^2\phi = -\frac{1}{\epsilon_0}\rho - \frac{\partial}{\partial t}\underbrace{(\nabla\cdot\vec{A})}_{0} = -\frac{1}{\epsilon_0}\rho$$

$$\Rightarrow \text{ENERGY}_{|\nabla\phi|^2} = \int d^3r\,\phi(\vec{r},t)\,\rho(\vec{r},t)$$

FINALY, SINCE $\nabla^2 \phi = -\frac{1}{\epsilon_0} \rho$

AND RECALLING THAT $\nabla^2 \left(\frac{1}{|\vec{r}-\vec{r}'|} \right) = -4\pi \delta^{(3)}(\vec{r}-\vec{r}')$

$$\Rightarrow \phi(\vec{r},t) = +\frac{1}{4\pi\epsilon_0} \int d^3r' \, \frac{\rho(\vec{r}',t)}{|\vec{r}-\vec{r}'|}$$

THERE FORE,

$$\text{ENERGY} = \int d^3r \, \frac{1}{2\mu_0} |\nabla \times \vec{A}|^2 + \frac{1}{2}\epsilon_0 \left| \frac{\partial \vec{A}}{\partial t} \right|^2 + \underbrace{\frac{1}{4\pi\epsilon_0} \int d^3r \int d^3r' \, \frac{\rho(\vec{r},t)\rho(\vec{r}',t)}{|\vec{r}-\vec{r}'|}}_{\text{INSTANTANEOUS COULOMB INTERACTION}}$$

- WHAT IS THE CORRESPONDING GAUGE TRANSF. OR THE TRANSVERSE GAUGE?

$$\nabla \cdot \vec{A} = 0 = \nabla \cdot (\vec{A} + \nabla \chi) \Rightarrow \nabla^2 \chi = -\nabla \cdot \vec{A} \Rightarrow \chi = \frac{1}{4\pi} \int d^3r' \, \frac{\nabla \cdot \vec{A}(\vec{r}',t)}{|\vec{r}-\vec{r}'|}$$

<u>HOW ABOUT THE SOURCES?</u>

DEF: $\rho(\vec{r},t) = \sum_{i=1}^{N} q_i \delta^{(3)}(\vec{r}-\vec{r}_i(t))$ $\quad \rho \equiv$ CHARGE DENSITY OPERATOR

$$= \frac{1}{(2\pi)^3} \int d^3p \sum_i q_i e^{ip(\vec{r}-\vec{r}_i)}$$

, $\vec{r}_i$ ARE OPERATORS

NOTE $\sum_i q_i \phi(\vec{r}_i,t) = \int d^3r \, \phi(\vec{r},t) \sum_i q_i \delta^{(3)}(\vec{r}-\vec{r}_i) = \int d^3r \, \rho(\vec{r},t)\phi(\vec{r},t)$

DEF: $\vec{j}(\vec{r}) = \frac{1}{2} \sum_{i=1}^{N} \left(q_i \frac{\vec{P}_i}{m_i} \delta(\vec{r}-\vec{r}_i) + q_i \delta(\vec{r}-\vec{r}_i) \frac{\vec{P}_i}{m} \right)$ $\quad \vec{P}_i =$ CANONICAL MOMENTUM

NOTE THAT $-\sum_i \left(\frac{\vec{P}_i}{2m} q_i \vec{A}(\vec{r}_i,t) + \frac{q_i}{2m} \vec{A}(\vec{r}_i,t) \cdot \vec{P}_i \right)$

$$= -\int d^3r \, \vec{j}(\vec{r}) \cdot \vec{A}(\vec{r},t)$$

$\vec{j}(\vec{r},t)$ IS HERMITEAN BUT NO GAUGE INVARIANT (IT IS CALLED PARAMAGNETIC CURRENT)

$\vec{J}(\vec{r},t) = \vec{j} - \frac{\vec{A}(\vec{r},t)}{m} \rho(\vec{r},t) \rightarrow$ HERMITEAN AND GAUGE INVARIANT

SINGLE PARTICLE ~~DENSITY~~ CURRENT OPERATOR

(3-9)

$$\vec{J} = \frac{1}{2m}\left[\psi^* \vec{p}\,\psi - \psi \vec{p}\,\psi^* - 2q\vec{A}\,|\psi|^2\right]$$

RECALL $\vec{J} = \frac{\hbar}{2mi}\left(\psi^* \nabla\psi - \psi\nabla\psi^*\right)$

$= \frac{1}{2m}\left(\psi^*\vec{p}\,\psi - \psi\vec{p}\,\psi^*\right)$

GENERALIZING TO VECTOR POTENTIAL

= WITH $\vec{p} = -i\hbar\nabla$

LET $\psi = \sqrt{\rho}\, e^{i\theta}$, $\rho = \psi^*\psi$

$$\Rightarrow \nabla\cdot\vec{J} = \frac{1}{2m}\left[\nabla\psi^*\cdot\vec{p}\,\psi + \psi^*(\nabla\cdot\vec{p})\psi + \psi^*(\vec{p}\cdot\nabla\psi) + \cdots - 2q\nabla\cdot\vec{A}\,|\psi|^2 - 2qA\cdot\nabla|\psi|^2\right]$$

$$= \frac{i}{\hbar}\left[\psi^*(H\psi) - (H\psi)^*\psi\right]$$

BUT NOTICE THAT

$$\frac{\partial}{\partial t}|\psi|^2 = \psi^*\left(\frac{\partial}{\partial t}\psi\right) + \left(\frac{\partial}{\partial t}\psi^*\right)\psi = \frac{1}{i\hbar}\left[\psi^*(H\psi) + (H\psi)^*\psi\right]$$

$$= -\nabla\cdot\vec{J}$$

$$\Rightarrow \frac{\partial}{\partial t}|\psi|^2 + \nabla\cdot\vec{J} = \boxed{\frac{\partial}{\partial t}\rho + \nabla\cdot\vec{J} = 0}$$

CONTINUITY EQUATION

NOTICE:

$[\rho] = \frac{1}{\text{VOLUME}}$

$[\vec{J}] = \frac{1}{\text{TIME}\cdot\text{AREA}}$

FREE-FIELD HAMILTONIAN $= \int d^3r \, \frac{\epsilon_0}{2} |E^2| + \frac{1}{2\mu_0} |B|^2$

$(\rho = \vec{J} = 0)$

IN THE TRANSVERSE GAUGE $\nabla \cdot \vec{A} = 0$

FOR $\rho = 0 \Rightarrow \nabla^2 \phi = 0 \rightarrow$ CHOICE $\phi = 0$

WAVE EQUATION $\quad \Box \vec{A} = 0 \quad \rightarrow \quad \boxed{\nabla^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{A} = 0}$

JUST LIKE AN STRING

SOLUTION:

$$\vec{A}(\vec{r},t) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} c_{\vec{k}} \, \hat{e}_k \, e^{i(\vec{k}\cdot\vec{r} - \omega t)} + c.c.$$

$$\begin{cases} k \equiv \text{PROPAGATION DIRECTION} & \hat{k} \cdot \hat{e}_k = 0 \\ \hat{e}_k \equiv \text{POLARIZATION VECTOR} & \\ \omega = c|\vec{k}| & \end{cases}$$

THE INTEGRAL $\int d^3r \, \frac{1}{2\mu_0} |\vec{B}|^2$ WILL INVOLVE $\int (\nabla \times \vec{A}) \cdot (\nabla \times \vec{A}) \, d^3r$

$$= \sum_{\vec{k},\vec{k}'} \frac{1}{V} \int d^3r \left[\underbrace{c_k \left(\nabla \times \hat{e}_k e^{i(\vec{k}\cdot\vec{r} - \omega t)} \right)}_{e^{i(\vec{k}\cdot\vec{r}-\omega t)} (\nabla \times \hat{e}_k) - \hat{e}_k \times \nabla e^{i(\vec{k}\cdot\vec{r} - \omega t)}} + c.c. \right] \cdot \left[c_{\vec{k}'} \nabla \times \left(\hat{e}_{\vec{k}'} e^{i(\vec{k}'\cdot\vec{r} - \omega' t)} \right) + c.c. \right]$$

$$= -i (\hat{e}_k \times \vec{k}) \, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

RECALL $\quad \nabla \times (\psi \vec{A}) = \nabla \psi \times \vec{A} + \psi \nabla \times \vec{A}$

$$= \sum_{\vec{k},\vec{k}'} \frac{1}{V} \int d^3r \quad \begin{aligned} & c_{\vec{k}} c_{\vec{k}'} \left(-(\hat{e}_k \times \vec{k}) \cdot (\hat{e}_{k'} \times \vec{k}') \right) e^{i(\vec{k}\cdot\vec{r} - \omega t)} e^{i(\vec{k}'\cdot\vec{r} - \omega' t)} \\ & c_k^* c_{k'} \left(+(\hat{e}_k \times \vec{k}) \cdot (\hat{e}_{k'} \times \vec{k}') \right) e^{-i(\vec{k}\cdot\vec{r} - \omega t)} e^{i(\vec{k}'\cdot\vec{r} - \omega' t)} \\ & c_k c_{k'}^* \left(\qquad \right) e^{i(\vec{k}\cdot\vec{r} - \omega t)} e^{-i(\vec{k}'\cdot\vec{r} - \omega' t)} \\ & c_k^* c_{k'}^* \left(\qquad \right) e^{-i(\vec{k}\cdot\vec{r} - \omega t)} e^{-i(\vec{k}'\cdot\vec{r} - \omega' t)} \end{aligned}$$

5

$$\begin{cases} \frac{1}{V}\int d^3r\, e^{i\vec{k}\cdot\vec{r}}\, e^{i\vec{k}'\cdot\vec{r}} = \delta_{\vec{k},-\vec{k}'} \\ \frac{1}{V}\int d^3r\, e^{i\vec{k}\cdot\vec{r}}\, e^{-i\vec{k}'\cdot\vec{r}} = \delta_{\vec{k},\vec{k}'} \end{cases}$$

$$\Rightarrow \int (\nabla\times\vec{A})\cdot(\nabla\times\vec{A})\, d^3r = \sum_k C_{\vec{k}} C_{-\vec{k}} (-k^2) e^{-2i\omega t} + C_k^* C_{\vec{k}} (+k^2) + C_k C_k^* (+k^2) + C_k^* C_{-k}^* (-k^2) e^{2i\omega t}$$

DEFINING $C_{\vec{k}}(t) = C_{\vec{k}}^0 e^{-i\omega t}$, $C_k^*(t) = C_k^* e^{i\omega t}$

$$\Rightarrow \int(\nabla\times A)\cdot(\nabla\times\vec{A}) = \sum_k k^2 \left[C_k(t) C_k(t) + C_k^*(t) C_k(t) + c.c. \right]$$

FOR $\frac{\epsilon_0}{2}\int d^3r\, |E^2|$ $\rightarrow$ $\int d^3r \frac{1}{V}\sum_{kk'} \left(C_k \hat{e}_k e^{i(kr-\omega t)} (-i\omega) + c.c. \right) \cdot \left(C_{k'} \hat{e}_{k'} (-i\omega') \cdots \right)$

$$= +\omega^2 C_{\vec{k}} C_{-\vec{k}}(t) + \omega^2 \left(C_k^*(t) C_k(t) + C_k C_k^* \right) + \omega^2 C_k^*(t) C_k^*(t)$$

$$\Rightarrow H = \frac{\epsilon_0}{2} \sum_k 2\omega_k^2 \left(C_k^* C_k + C_k C_k^* \right)$$

where

$$\ddot{C}_k = -\omega_k^2 C_k$$

~~DEFINING $Q_k(t) = \sqrt{\epsilon_0}\, C_k(t)$~~

DEFINING $Q_k = \sqrt{\epsilon_0}(C_k + C_k^*)$

~~$\Rightarrow$ $H =$~~ $P_k = -i\sqrt{\epsilon_0}\,\omega_k (C_k - C_k^*)$

$$\Rightarrow H = \sum_k \frac{1}{2} \left(P_k^2 + \omega^2 Q_k^2 \right) \rightarrow \text{HARMONIC OSCILLATOR}$$

$$H = \epsilon_0 \sum \omega_k^2 \left(c_k^* c_k + c_k c_k^* \right) \quad , \text{ with } \ddot{c}_k = -\omega_k^2 c_k$$
$$c_k(t) = c_k(0)\, e^{-i\omega t}$$

$$Q_k = \sqrt{\epsilon_0}\,(c_k + c_k^*)$$
$$P_k = -i\sqrt{\epsilon_0}\,\omega_k\,(c_k - c_k^*)$$
$$\Rightarrow \begin{cases} c_k = \dfrac{1}{2\sqrt{\epsilon_0}}\left(Q_k + i\dfrac{P_k}{\omega_k}\right) \\ c_k^* = \dfrac{1}{2\sqrt{\epsilon_0}}\left(Q_k - i\dfrac{P_k}{\omega_k}\right) \end{cases}$$

$$\Rightarrow H = \sum_k \epsilon_0 \omega_k^2 \left(\frac{2}{4\epsilon_0}\left(Q_k^2 + \frac{P_k^2}{\omega_k^2}\right)\right) = \sum_k \frac{1}{2}\left(P_k^2 + \omega_k^2 Q_k^2\right)$$

$$\begin{cases} \dot{P}_k = -i\sqrt{\epsilon_0}\,\omega_k\left[-i\omega_k c_k - i\omega_k c_k^*\right] = -\omega_k^2 Q_k \\ \dot{Q}_k = P_k \end{cases}$$

$$* \ \frac{\partial H}{\partial Q_k} = \omega_k^2 Q_k = -\dot{P}_k \qquad * \ \frac{\partial H}{\partial P_k} = P_k = \dot{Q}_k$$

$\Rightarrow$ Q_k AND P_k ARE CONJUGATE VARIABLES

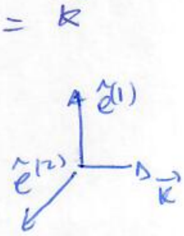
~~QUANTIZATION OF THE RADIATION FIELD~~

~~POSTULATE THAT Q_k AND P_k ARE OPERATORS~~

~~WITH $[\hat{Q}_k, \hat{P}_{k'}] = i\hbar\,\delta_{\vec{k},\vec{k}'}$~~

POLARIZATION : $\hat{e}_k = a_{1k}\hat{e}_1 + a_{2k}\hat{e}_2$, $\hat{e}_1 \times \hat{e}_2 = \hat{k}$

$\hat{e}_1, \hat{e}_2, \hat{k}$ = FORM A RIGHT-HANDED BASIS



REWRITE $$\vec{A} = \frac{1}{\sqrt{V}} \sum_{\vec{k}} c_{\vec{k}}\,\hat{e}_k\, e^{i(\vec{k}\cdot\vec{r} - \omega t)} + c.c.$$

$$= \frac{1}{\sqrt{V}} \sum_{\vec{k}} c_{\vec{k}}\,(a_1\hat{e}_1 + a_2\hat{e}_2)\, e^{i(\vec{k}\cdot\vec{r} - \omega t)} + c.c.$$

$$= \frac{1}{\sqrt{V}} \sum_{k,\alpha} \underbrace{c_{\vec{k},\alpha}\, e^{-i\omega t}}_{c_{\vec{k},\alpha}(t)}\, \hat{e}_\alpha\, e^{i\vec{k}\cdot\vec{r}}$$

$\Rightarrow \quad H = \sum_{\vec{k},\alpha} \frac{1}{2}\left(P_{\vec{k},\alpha}^2 + \omega_k^2 Q_{\vec{k},\alpha}^2\right)$ ⑦

with $\begin{cases} \dfrac{\partial H}{\partial Q_{\vec{k}\alpha}} = -\dot{P}_{\vec{k}\alpha} \\ \dfrac{\partial H}{\partial P_{\vec{k},\alpha}} = \dot{Q}_{\vec{k},\alpha} \end{cases}$ $\Rightarrow$ RADIATION FIELD CAN BE REGARDED AS A COLLECTION OF INDEPENDENT HARMONIC OSCILLATORS LABELED BY $\vec{k}, \alpha$

CIRCULAR POLARIZATION

$$\begin{cases} \hat{e}^+ = -\frac{1}{\sqrt{2}}\left(\hat{e}^{(1)} + i\,\hat{e}^{(2)}\right) \\ \hat{e}^- = \frac{1}{\sqrt{2}}\left(\hat{e}^{(1)} - i\,\hat{e}^{(2)}\right) \end{cases}$$

CORRESPONDING PLANE WAVE: $\frac{1}{\sqrt{V}} e^{i(\vec{k}\cdot\vec{r} - \omega t)}\,\hat{e}^{(\pm)}$

* THIS IS INTERESTING BECAUSE CIRCULAR POLARIZED PHOTONS HAVE WELL-DEFINED ANGULAR MOMENTUM: $\pm\hbar$ IF THE POLARIZATION IS RIGHT-HANDED (+) OR LEFT-HANDED (−)

QUANTIZATION OF THE RADIATION FIELD

POSTULATE THAT $Q_{\vec{k},\alpha}$ AND $P_{\vec{k},\alpha}$ ARE NOW OPERATORS SATISFYING

$$[Q_{\vec{k}\alpha}, P_{\vec{k}',\alpha'}] = i\hbar\, \delta_{\vec{k},\vec{k}'}\, \delta_{\alpha,\alpha'}$$

FOR CONTINUOUS VARIABLES $\Rightarrow$ DIRAC'S DELTA

$$[Q_k, Q_{k'}] = [P_k, P_{k'}] = 0$$

$\Rightarrow$ USUAL CREATION / ANNIHILATION OPER.

$$a_{\vec{k}\alpha} = \frac{1}{\sqrt{2\hbar\omega_k}}\left(\omega_k Q_{\vec{k}\alpha} + i P_{\vec{k}\alpha}\right)$$

$$a^+_{\vec{k}\alpha} = \frac{1}{\sqrt{2\hbar\omega}}\left(\omega \vec{Q}_{\vec{k},\alpha} - i P_{\vec{k},\alpha}\right) \Rightarrow \begin{cases} P_{\vec{k},\alpha} = \sqrt{\dfrac{\hbar\omega_k}{2}}\, i\left(a^+_{k\alpha} - a_{k\alpha}\right) \\ Q_{\vec{k},\alpha} = \sqrt{\dfrac{\hbar}{2\omega_k}}\left(a^+_{k\alpha} + a_{k\alpha}\right) \end{cases}$$

$$\Rightarrow \left[a, a^{\dagger}\right] = \left(\frac{1}{\sqrt{2\hbar\omega_k}}\right)^2 \left\{ -i\omega_k \overbrace{\left[Q_k, P_{k'}\right]}^{i\hbar\delta_{k,k'}\delta_{\alpha\alpha'}} + i\omega_k \overbrace{\left[P_k, Q_{k'}\right]}^{-i\hbar\delta_{kk'}\delta_{\alpha\alpha'}} \right\}$$

8

$$\boxed{\left[a_{\vec{k}\alpha}, a^{\dagger}_{\vec{k}'\alpha'}\right] = \delta_{\vec{k},\vec{k}'}\,\delta_{\alpha,\alpha'}}$$

$$\left[a_{\vec{k}\alpha}, a_{\vec{k}'\alpha'}\right] = \left[a^{\dagger}_{\vec{k}\alpha}, a^{\dagger}_{\vec{k}'\alpha'}\right] = 0$$

$$H = \sum_{\vec{k},\alpha} \frac{1}{2}\left(P_k^2 + \omega_k^2 Q_{\vec{k}\alpha}^2\right) = \sum_{\vec{k}\alpha} \frac{1}{2}\left(\frac{\hbar\omega_k}{2}\left(-a^{\dagger}a^{\dagger} + aa^{\dagger} + a^{\dagger}a - aa\right) + \frac{\hbar\omega_k}{2}\left(a^{\dagger}a^{\dagger} + aa^{\dagger} + a^{\dagger}a + aa\right)\right)$$

$$= \sum_{\vec{k},\alpha} \frac{\hbar\omega_k}{2}\left(a^{\dagger}_{\vec{k}\alpha}a_{\vec{k}\alpha} + a_{\vec{k}\alpha}a^{\dagger}_{\vec{k}\alpha}\right)$$

$$\boxed{H = \sum_{\vec{k},\alpha} \hbar\omega_k \left(a^{\dagger}_{\vec{k}\alpha}a_{\vec{k}\alpha} + \frac{1}{2}\right)} = \sum_{\vec{k},\alpha} \hbar\omega_k\left(N_{\vec{k}\alpha} + \frac{1}{2}\right)$$

$\downarrow$ CONSTANT WHICH CAN BE SET TO ZERO

NUMBER OPERATOR $\quad N_{\vec{k}\alpha} = a^{\dagger}_{\vec{k},\alpha}\, a_{\vec{k},\alpha}$

AS IN THE USUAL H.O.:

$$\left[a_{\vec{k}\alpha}, N_{\vec{k}'\alpha'}\right] = \delta_{\vec{k},\vec{k}'}\,\delta_{\alpha,\alpha'}\, a_{\vec{k},\alpha}$$

$$\left[a^{\dagger}_{\vec{k}\alpha}, N_{\vec{k}'\alpha'}\right] = -\delta_{\vec{k},\vec{k}'}\,\delta_{\alpha,\alpha'}\, a^{\dagger}_{\vec{k}',\alpha}$$

EIGEN STATES: $N|n\rangle = n|n\rangle$

$$\Rightarrow \begin{cases} N a|n\rangle = (n-1)\, a|n\rangle \\ N a^{\dagger}|n\rangle = (n+1)\, a^{\dagger}|n\rangle \end{cases}$$

$$a^{\dagger}|n\rangle = \sqrt{n+1}\;|n\rangle$$

$$a|n\rangle = \sqrt{n}\;|n-1\rangle$$

$$n = \langle n|N|n\rangle \geq 0$$

EIGEN-STATES

9

$$|m_{\vec{k}_1,\alpha_1}, m_{\vec{k}_2,\alpha_2}, \dots\rangle = \prod_{k_i \alpha_i} \frac{\left(a^{+}_{k_i,\alpha_i}\right)^{m_{\vec{k}_i\alpha_i}}}{\sqrt{m_{\vec{k}_i\alpha_i}}} |0\rangle$$

VACUUM

$$H|m_{k_1,\alpha_1}, \dots\rangle = \left(\sum_i m_{\vec{k}_i\alpha_i} \hbar\omega_{k_i}\right) |m_{k_1\alpha_1}\, m_{k_2\alpha_2} \dots\rangle$$

THE FACTOR $\frac{1}{2}$ WAS DROPPED OUT

VECTOR POTENTIAL OPERATOR

$$\vec{A} = \frac{1}{\sqrt{V}} \sum_{k,\alpha} C_{\vec{k},\alpha}(t)\, \hat{e}_\alpha\, e^{i\vec{k}\cdot\vec{r}} + c.c.$$

REMEMBER $C_{\vec{k},\alpha} = \frac{1}{2\sqrt{\epsilon_0}}\left(Q_{k\alpha} + \frac{i}{\omega_{k\alpha}} P_{k\alpha}\right) = \frac{1}{2\sqrt{\epsilon_0}}\left(\sqrt{\frac{\hbar}{2\omega_k}}\left(a^{+}_{k\alpha} + a_{k\alpha}\right) + \frac{i}{\omega_k}\sqrt{\frac{\hbar\omega_k}{2}}\frac{(a^{+} - a)}{i}\right)$

$$= \sqrt{\frac{\hbar}{2\omega_k\epsilon_0}}\, a_{\vec{k},\alpha}$$

$$\Rightarrow \boxed{\vec{A} = \frac{1}{\sqrt{V}} \sum_{\vec{k},\alpha} \sqrt{\frac{\hbar}{2\omega_k\epsilon_0}}\, a_{\vec{k},\alpha}\, \hat{e}_\alpha\, e^{i\vec{k}\cdot\vec{r}} + h.c.}$$

$\hat{e}_\alpha \cdot \vec{k} = 0$

$\hat{e}_\alpha \cdot \hat{e}_\alpha^* = 1$

$\hat{e}_1 \cdot \hat{e}_2^* = 0$

HEISENBERG REPRESENTATION

SINCE $\ddot{a}_{\vec{k}\alpha} = -\omega_k^2 a_{k\alpha}$

ELECTRIC AND MAGNETIC FIELD ~~VECTOR POTENTIAL~~ OPERATOR

$$\vec{E} = -\frac{\partial}{\partial t}\vec{A} = \frac{-1}{\sqrt{V}} \sum_{\vec{k},\alpha} \sqrt{\frac{\hbar\omega_k}{2\epsilon_0}} \left(-i a_{k,\alpha}\right) \hat{e}_\alpha\, e^{i\vec{k}\cdot\vec{r}} + h.c.$$

$$\vec{B} = \nabla\times\vec{A} = \frac{i}{\sqrt{V}} \sum_{k,\alpha} \sqrt{\frac{\hbar}{2\omega_k\epsilon_0}} \left(-i a_{\vec{k}\alpha}\right)\left(\hat{e}_k \times \vec{k}\right) e^{i\vec{k}\cdot\vec{r}} + h.c.$$

POYNTING'S VECTOR OPERATOR

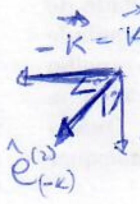
(10)

$$\vec{N} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{1}{V}\sum_{\vec{k},\vec{k}'}\sum_{\alpha\alpha'}\frac{\hbar}{2\epsilon_0 \mu_0}\Big[(-a_{k\alpha}a_{k'\alpha'})\hat{e}_\alpha\times(\hat{e}_{\alpha'}\times\vec{k}')e^{i(\vec{k}+\vec{k}')\cdot\vec{r}} + (a_{k\alpha}a^+_{k'\alpha'})\hat{e}_\alpha\times(\hat{e}_{\alpha'}\times\vec{k}')e^{i(\vec{k}-\vec{k}')\cdot\vec{r}} + (a^+_{k\alpha}a_{k'\alpha'})\hat{e}_\alpha\times(\hat{e}_{\alpha'}\times\vec{k}')e^{i(-\vec{k}+\vec{k}')\cdot\vec{r}} + (+a^+_{k\alpha}a^+_{k'\alpha'})\hat{e}_\alpha\times(\hat{e}_{\alpha'}\times\vec{k}')e^{-i(\vec{k}+\vec{k}')\cdot\vec{r}}\Big]$$

MOMENTUM OPERATOR OF THE RADIATION FIELD

$$\vec{P} = \int d^3r\,\frac{\vec{N}}{c^2} = \sum_{kk'}\sum_{\alpha\alpha'}\frac{\hbar}{2\epsilon_0\mu_0 c^2}\Big[(-a_{k\alpha}a_{k'\alpha'})\hat{e}_\alpha\times(\hat{e}_{\alpha'}\times\vec{k}')\delta_{\vec{k},-\vec{k}'} + (a_{k\alpha}a^+_{k'\alpha'})\ {}''\ \delta_{\vec{k},\vec{k}'} + \cdots\Big]$$

= TOTAL MOMENTUM CARRIED BY THE FIELD



$$\hat{e}_\alpha\times(\hat{e}_{\alpha'}\times\vec{k}') = (\hat{e}_\alpha\cdot\vec{k}')\hat{e}_{\alpha'} - (\hat{e}_\alpha\cdot\hat{e}_{\alpha'})\vec{k}'$$

• FOR $\vec{k}=\vec{k}'$

$$\Rightarrow -\delta_{\alpha,\alpha'}\vec{k}$$

• FOR $\vec{k}=-\vec{k}'$

$$\Rightarrow \begin{cases} -\vec{k} & \text{IF } \alpha=1 \\ +\vec{k} & \text{IF } \alpha=2 \end{cases}$$

THEOREM: WHENEVER THERE IS AN ENERGY FLOW $\vec{N}$

$\Rightarrow$ THERE IS ALSO ~~A~~ MOMENTUM DENSITY ~~[illegible]~~ $= \frac{1}{c^2}\vec{N}$

$$\Rightarrow \sum_{\alpha'}\hat{e}_\alpha\times(\hat{e}_{\alpha'}\times\vec{k}') = 0 \ \text{, IF } \vec{k}'=-\vec{k}$$

$$\Rightarrow \vec{P} = \int d^3r\,\frac{\vec{N}}{c^2} = \sum_{k\alpha}\hbar\left[a_{k\alpha}a^+_{k\alpha}\vec{k} + a^+_{k\alpha}a_{k\alpha}\vec{k}\right]$$

$$\boxed{\vec{P} = \sum_{\vec{k},\alpha}\hbar\vec{k}\left[N_{\vec{k},\alpha}+\tfrac{1}{2}\right]} = \sum_{\vec{k},\alpha}\hbar\vec{k}\left[N_{\vec{k}\alpha}\right]$$

SINCE $\sum\frac{1}{2}\hbar\vec{k} = 0$ BECAUSE THERE IS ALWAYS A WAVE-VECTOR $-\vec{k}$ ASSOCIATED WITH $\vec{k}$

THEREFORE, WE ASSOCIATE TO EACH PHOTON (11)

A MOMENTUM $\hbar\vec{k}$ $\Rightarrow$ TOTAL MOMENTUM $= \vec{P} = \sum_{k\alpha} \hbar\vec{k}\, N_{\vec{k}\alpha}$

WITH ENERGY $\hbar\omega_k$ $\Rightarrow$ TOTAL ENERGY $= H = \sum_{k,\alpha} \hbar\omega_k\, N_{\vec{k}\alpha}$

$$\boxed{\begin{aligned} H\, a^+_{k\alpha}|0\rangle &= \hbar\omega_k\, a^+_{k\alpha}|0\rangle \\ \vec{P}\, a^+_{k\alpha}|0\rangle &= \hbar\vec{k}\, a^+_{k\alpha}|0\rangle \end{aligned}}$$

MASS OF THE PHOTON $\Rightarrow$ $m^2 = \frac{1}{c^4}\left(E^2 - P^2c^2\right) = \frac{1}{c^4}\left(\hbar^2\omega^2 - \hbar^2\overset{\omega^2}{k^2c^2}\right) = 0$

ANGULAR MOMENTUM

$$\vec{L} = \int d^3r\, \vec{r}\times\frac{\vec{N}}{c^2} = L^{(0)} + L^{(s)} = \frac{1}{\mu_0 c^2}\int d^3r\, \vec{r}\times\left(\vec{E}\times\nabla\times\vec{A}\right)$$

with $\vec{L}^{(0)} = \frac{1}{\mu_0 c^2}\int d^3r \sum_i E_i\left(\vec{l} A_i\right)$ (ORBITAL), where $\vec{l}\phi = -i\vec{r}\times\nabla\phi$

$\vec{L}^{(s)} = \frac{1}{\mu_0 c^2}\int d^3r\, \vec{E}\times\vec{A}$ (SPIN)

POLARIZATION

$$\hat{e}^{(+)} = \frac{1}{\sqrt{2}}\left(\hat{e}^{(1)} + i\hat{e}^{(2)}\right)$$

$$a^+_{k,\pm} = \mp\frac{1}{\sqrt{2}}\left(a^+_{k,1} \mp i\, a^+_{k,2}\right)$$

$$\hat{e}^{(-)} = \frac{1}{\sqrt{2}}\left(\hat{e}^{(1)} - i e^{(2)}\right)$$

$a^+_{k,\pm}|0\rangle = |\vec{k},\pm\rangle$

$= $ STATE OF A SINGLE PHOTON WITH POLARIZATION $\pm$ AND MOMENTUM $\hbar\vec{k}$

NOTICE THAT

$$\vec{L}^{(s)} = -\epsilon_0 \int d^3r \; \frac{\partial}{\partial t}\vec{A} \times \vec{A}$$

IF THE POLARIZATION IS LINEAR $\Rightarrow \frac{\partial}{\partial t}\vec{A} \parallel \vec{A} \Rightarrow \vec{L}^{(s)} = 0$

IN GENERAL

$$\vec{A} = \frac{1}{\sqrt{V}} \sum_{k,\alpha} \sqrt{\frac{\hbar}{2\omega_k \epsilon_0}} \; a_{\vec{k},\alpha} \, \hat{e}_\alpha \, e^{i\vec{k}\cdot\vec{r}} + h.c.$$

DEF: $\left\{ \begin{aligned} a_{\vec{k},+} &= -\frac{1}{\sqrt{2}}\left(a_{\vec{k},1} - i\, a_{\vec{k},2}\right) \\ a_{\vec{k},-} &= \frac{1}{\sqrt{2}}\left(a_{\vec{k},1} + i\, a_{\vec{k},2}\right) \end{aligned} \right. \Rightarrow \left\{ \begin{aligned} a_{\vec{k},1} &= \frac{1}{\sqrt{2}}\left(-a_{\vec{k}+} + a_{\vec{k}-}\right) \\ a_{\vec{k},2} &= -\frac{i}{\sqrt{2}}\left(a_{\vec{k}+} + a_{\vec{k}-}\right) \end{aligned} \right.$

$$\Rightarrow \vec{A} = \frac{1}{\sqrt{V}} \sum_{k} \sqrt{\frac{\hbar}{2\omega_k \epsilon_0}} \left(\frac{1}{\sqrt{2}}\left(-a_{k+} + a_{k-}\right)\hat{e}_1 - \frac{i}{\sqrt{2}}\left(a_{k+} + a_{k-}\right)\hat{e}_2 \right) e^{i\vec{k}\cdot\vec{r}} + h.c.$$

$$= \frac{1}{\sqrt{V}} \sum_{k} \sqrt{\frac{\hbar}{2\omega_k \epsilon_0}} \left(a_{k+}\left(\frac{-\hat{e}_1 - i\hat{e}_2}{\sqrt{2}}\right) + a_{k-}\left(\frac{\hat{e}_1 - i\hat{e}_2}{\sqrt{2}}\right) \right) e^{i\vec{k}\cdot\vec{r}} + h.c.$$

$$= \frac{1}{\sqrt{V}} \sum_{k} \sqrt{\frac{\hbar}{2\omega_k \epsilon_0}} \left(a_{k+}\, \hat{e}^{(+)*} + a_{k-}\, \hat{e}^{(-)*} \right) e^{i\vec{k}\cdot\vec{r}} + h.c.$$

$$= \frac{1}{\sqrt{V}} \sum_{k} \sqrt{\frac{\hbar}{2\omega_k \epsilon_0}} \left(a^{+}_{\vec{k},+}\, \hat{e}^{(+)*} + a^{+}_{\vec{k},-}\, \hat{e}^{(-)*} \right) e^{-i\vec{k}\cdot\vec{r}} + h.c.$$

$$\Rightarrow \vec{E} = -\frac{\partial}{\partial t}\vec{A} = -\frac{i}{\sqrt{V}} \sum_{k} \sqrt{\frac{\hbar\omega_k}{2\epsilon_0}} \left(a^{+}_{\vec{k},+}\, \hat{e}^{(+)*} + a^{+}_{\vec{k},-}\, \hat{e}^{(-)*} \right) e^{-i\vec{k}\cdot\vec{r}} + h.c.$$

$$a^{+}_{\vec{k},\pm} = \mp\frac{1}{\sqrt{2}}\left(a^{+}_{\vec{k},1} \mp i\, a^{+}_{\vec{k},2}\right)$$

$$\Rightarrow \frac{\partial}{\partial t} a^{+}_{\vec{k},\pm} = i\,\omega_k\, a^{+}_{\vec{k},\pm}$$

$$\Rightarrow L^{(s)} = \epsilon_0 \int d^3r \ \vec{E} \times \vec{A}$$ (13)

$$= \sum_{k(k')} \epsilon_0 \sqrt{\frac{\hbar\omega_k}{2\epsilon_0}} \sqrt{\frac{\hbar}{2\omega_{k'}\epsilon_0}} \Big[(-i) \left(a^+_{k,+} \hat{e}^{*}_{(+)} + a^+_{k,-} \hat{e}^{*}_{(-)}\right) \times \left(a^+_{-k,+} \hat{e}^{*}_{(+)} + a^+_{-k-} \hat{e}^{*}_{(-)}\right)$$

$$+ (-2i) \left(a^+_{k+} \hat{e}^{*}_{(+)} + a^+_{k-} \hat{e}_{(-)}\right) \times \left(a_{k+} \hat{e}^{*}_{(+)} + a_{k-} \hat{e}_{(-)}\right)$$

$$+ \ h.c. \Big]$$

$$= \sum_{k} \frac{\hbar}{2} \Big\{ 0 + (-i) \left[a^+_{k+} a_{k+} (+i\hat{k}) + a^+_{k-} a_{k-} (-i\hat{k}) \right]$$

$$+ (i) \left[a_{k+} a^+_{k+} (-i\hat{k}) + a_{k-} a^+_{k-} (+i\hat{k}) \right] + 0 \Big\}$$

$$= \sum_k \frac{\hbar}{2} \left\{ 2 a^+_{k-} a_{k-} + 2 a^+_{k+} a_{k+} \right(\hat{k}$$

$$\Rightarrow \vec{L}^{(s)}_0 = \sum_k \hbar \left(N_{k,+} - N_{k,-} \right) \hat{k}$$

FOR A SINGLE PHOTON

$$\hat{k} \cdot \vec{L}^{(s)} \ |\vec{k}, \pm\rangle = \pm \hbar \ |k, \pm\rangle$$

$\Rightarrow$ THUS, THE ONLY EIGEN-VALUES ARE $\pm \hbar$

$\Rightarrow$ PHOTON HAS SPIN 1

BUT THERE IS NO PROJECTION $L_k = 0$

THIS IS BECAUSE THE PHOTON TRAVEL WITH SPEED $= c$

NOTE $\vec{L} \propto \hat{k} \Rightarrow$ SPIN IS PARALLEL OR ANTIPARALLEL TO $\vec{k}$

IF. $m_{photon} \neq 0 \Rightarrow$ GO TO ITS REFERENCE FRAME $(k=0) \Rightarrow$ PROJECTION $L_k = 0$ WOULD APPEAR.

LAST CLASS FREE FIELD

$$\vec{A} = \frac{1}{\sqrt{V}} \sum_{\vec{k}} \sqrt{\frac{\hbar}{2\omega_k \epsilon_0}} \left(a_{\vec{k},1}(t)\hat{e}_1 + a_{\vec{k},2}(t)\hat{e}_2 \right) e^{i(\vec{k}\cdot\vec{r})} + h.c.$$

$$\frac{d}{dt} a_{\vec{k}\alpha}(t) = -i\omega_k a_{\vec{k},\alpha} \rightarrow \boxed{a_{\vec{k}\alpha}(t) = a_{k\alpha}(0)\, e^{-i(\omega_k t + \phi_\alpha)}}$$

$$\left[a_{\vec{k}\alpha}, a^{\dagger}_{\vec{k}'\alpha'} \right] = \delta_{\vec{k},\vec{k}'} \delta_{\alpha,\alpha'}$$ ALL OTHERS COMMUTATORS $= 0$

$$\vec{E} = -\frac{\partial}{\partial t}\vec{A} \qquad \vec{B} = \nabla \times \vec{A}$$

ZERO POINT ENERGY

FREE FIELD ENERGY: $H = \int \left(\frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0} B^2 \right) d^3r = \sum_{\vec{k}} \hbar\omega_k \left(a^{\dagger}_{k1} a_{k1} + a^{\dagger}_{k2} a_{k2} \right) + E_0$

$\boxed{\omega_k = ck}$ LINEAR DISPERSION

LINEAR MOMENTUM $= \vec{P} = \int d^3r \frac{\vec{N}}{c^2} = \int d^3r \frac{\vec{E}\times\vec{B}}{\mu_0 c^2}$

$$= \sum_{\vec{k}} \hbar\vec{k} \left(a^{\dagger}_{\vec{k}1} a_{\vec{k}1} + a^{\dagger}_{\vec{k}2} a_{\vec{k},2} \right)$$

SPIN: $\vec{L}^{(s)} = -\epsilon_0 \int d^3r \frac{\partial}{\partial t}\vec{A} \times \vec{A} = \sum_{\vec{k}} \hbar\hat{k} \left(a^{\dagger}_{k+} a_{k+} - a^{\dagger}_{k-} a_{k-} \right)$

$\Rightarrow L^{(s)}_{\hat{k}} = \pm\hbar$ (HELICITY) $\quad \begin{cases} a^{\dagger}_{k1} = \frac{1}{\sqrt{2}}\left(-a^{\dagger}_{k+} + a^{\dagger}_{k-} \right) \\ a^{\dagger}_{k2} = \frac{i}{\sqrt{2}}\left(a^{\dagger}_{k+} + a^{\dagger}_{k-} \right) \end{cases}$

$\vec{L}^{(s)} \parallel \vec{k}$

WHY PHOTONS HAVE ONLY 2 PROJECTIONS OF SPIN? $\rightarrow$ SEE E. WIGNER RMP **29**, 255 (1957)

$\Rightarrow$ PHOTONS HAVE ENERGY $\hbar\omega$, MOMENTUM $\hbar\vec{k}$, AND SPIN-1, WITH PROJECTIONS $\pm\hbar$ ALONG THE DIRECTION OF $\vec{k}$

$\rightarrow$ THIS IS HELICITY. THUS, IT IS AN INTRINSIC PROPERTY OF THE PHOTON. ACTUALLY, IT IS INTRINSIC PROPERTY OF MASSLESS PARTICLES WITH SPIN. NOTICE THAT THERE IS NO ZERO-HELICITY

IF THE PARTICLE WERE MASSIVE, WE COULD GO INTO ITS REST FRAME AND MEASURE THE HELICITY. THIS WOULD BREAK SPACE ISOTROPY

ALSO $\hat{k}\cdot \vec{L}^{(0)}|k\,\pm\rangle = 0$ $\Rightarrow$ ORBITAL MOMENTUM IS ORTHOGONAL TO LINEAR MOMENTUM

FLUCTUATIONS

$\langle 0|\vec{E}|0\rangle = 0$ $\rightarrow$ ONLY APPEARS $\langle 0|a|0\rangle$ or $\langle 0|a^{+}|0\rangle$ WHICH ARE BOTH ZERO

HOWEVER $\langle 0|\vec{E}\cdot\vec{E}|0\rangle = \infty$ BECAUSE IT APPEARS $\langle 0|aa^{+}|0\rangle = 1$

THERE IS SOMETHING IN THE VACUUM

$\Rightarrow \boxed{\langle E^2\rangle - \langle E\rangle^2 = \infty}$ SUMMING OVER ALL k $\Rightarrow V \rightarrow \infty$

THIS IS ANALOGOUS TO

$$\langle \vec{r}|\vec{p}|\vec{r}\rangle = 0 = \left|\int d\vec{p}'\, p' e^{i\vec{p}'\cdot\vec{r}}\right|^2 \rightarrow \langle r|\vec{p}\cdot\vec{p}|r\rangle \rightarrow \infty$$

LOCALIZED PARTICLE $\rightarrow$ ALL PHASES (ALL $\vec{p}'$)

REDEFINITION:
$$\begin{cases} a_{k\alpha} = e^{i(\phi_{k\alpha} - \omega t)}\sqrt{N_{k\alpha}} \\ a^{+}_{k\alpha} = \sqrt{N_{k\alpha}}\, e^{-i(\phi_{k\alpha} - \omega t)} \end{cases}$$
$\phi_{k,\alpha}$ = PHASE

$$\Rightarrow A(\vec{r},t) = \frac{1}{\sqrt{V}}\sum_{k,\alpha}\sqrt{\frac{\hbar}{2\omega_k\epsilon_0}}\, e^{i(\phi_{k\alpha} - \omega t + \vec{k}\cdot\vec{r})}\sqrt{N_{k\alpha}}\,\hat{e}_\alpha + h.c.$$

AS $[a, a^{+}] = 1 \Rightarrow e^{i\phi} N e^{-i\phi} - N = 1 \rightarrow [e^{i\phi}, N] = e^{i\phi}$

EXPANDING THE EXPONENTIAL $\Rightarrow$ $\boxed{[N, \phi] = i}$ CONJUGATE VARIABLES

WHEN THE CLASSICAL DESCRIPTION IS GOOD?

$\langle 0|E^2|0\rangle_{QUANTUM} \sim \frac{\hbar\omega}{\epsilon_0}\times\frac{1}{V}$ FOR SMALL VOLUME OF ORDER λ^3

$\overline{E^2}_{CLASSICAL} \sim \rho\times\frac{\hbar\omega}{\epsilon_0}$, ρ = DENSITY OF PHOTONS $\Rightarrow$ $E^2_{CLASS} \gg E_{QUANT}$ $\Rightarrow \rho \gg 1/\lambda^3$ HIGH DENSITY

$$[e^{i\phi}, N] = e^{i\phi}$$

14.Q

$$e^{i\phi} = 1 + i\phi + \frac{1}{2!}(i\phi)^2 + \cdots$$

$$[e^{i\phi}, N] = (1, N) + i[\phi, N] + \frac{1}{2!}[-\phi^2, N] + \frac{1}{3!}[-i\phi^3, N] + \cdots$$

$$= 0 + i[\phi, N] - \frac{1}{2}\{\phi[\phi,N] + [\phi,N]\phi\} + \cdots$$

$$= e^{i\phi} = 1 + i\phi + \frac{1}{2!}(i\phi)^2 + \cdots$$

$$\Rightarrow \begin{cases} i[\phi, N] = 1 \rightarrow [N, \phi] = i \\ -\frac{1}{2}\{\phi(-i) + (-i)\phi\} = i\phi \end{cases}$$

$$\frac{1}{3!}[-i\phi^3, N] = -\frac{i}{3!}\{\phi^2[\phi,N] + [\phi^2,N]\phi\} = -\frac{i}{3!}[\phi^2(-i) + 2(-i)\phi^2] = -\frac{1}{2!}\phi^2$$

$$\underbrace{(i)^n (-i) n \phi^{n-1}}$$

$$\frac{1}{n!}[(i\phi)^n, N] = \frac{i^n}{n!}\{\phi^{n-1}[\phi, N] + [\phi^{n-1}, N]\phi\}$$

$$= \frac{i^n}{n!}\{\phi^{n-1}(-i) + (-i)(n-1)\phi^{n-2}\phi\}$$

$$= \frac{-i^{n+1}}{(n-1)!}\phi^{n-1} = \frac{(i\phi)^{n-1}}{(n-1)!}$$

$$\Rightarrow \boxed{[N, \phi] = i}$$

EMISSION AND ABSORPTION OF PHOTONS BY ATOMS

⇒ THEN, FOR MOST APPLICATIONS, WE CONSIDER THE INTERACTION BETWEEN LIGHT AND ELECTRONS (CHARGE $-e$)

$$H = \underbrace{\sum_i \frac{1}{2m}\left(\vec{p}_i + e\vec{A}\right)^2}_{\text{KINETIC ENERGY}} + V(\vec{r}_i) + \underbrace{\int d^3r \; \frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2}_{H_{em}} + H_{SPIN}$$

$V(\vec{r}_i)$ ↓ EXTERNAL POTENTIAL (INFINITELY HEAVY NUCLEUS) + COULOMB INTERACTION

$$H_{SPIN} = -\sum_i \vec{\mu}_i \cdot \vec{B}(\vec{r}_i, t) = -\gamma \sum_i \vec{S}_i \cdot (\nabla \times \vec{A})$$

$$H = H_0 + H_{INT}$$

$$H_0 = \sum_i \frac{1}{2m} p_i^2 + V(\vec{r}_i) + H_{em}$$

$$H_{INT} = \sum_i \frac{1}{2m}\left(\vec{p}_i \cdot \vec{A}(\vec{r}_i, t) + \vec{A}(\vec{r}_i, t) \cdot \vec{p}_i\right) + \frac{e^2}{2m} \vec{A} \cdot \vec{A} + H_{SPIN}$$

IN THE TRANSVERSE GAUGE $\nabla \cdot \vec{A} = 0$

$$\Rightarrow \vec{p} \cdot (\vec{A}\psi) = \frac{\hbar}{i} \underbrace{\nabla \cdot \vec{A}}_{0}\, \psi + \frac{\hbar}{i} \vec{A} \cdot \nabla \psi = \vec{A} \cdot \vec{p}\psi$$

$$\Rightarrow H_{INT} = \sum_i \frac{e}{m} \vec{A}(\vec{r}_i, t) \cdot \vec{p}_i + \frac{e^2}{2m} \vec{A} \cdot \vec{A} + H_{SPIN}$$

→ TREAT H_{INT} AS A PERTURBATION OF H_0

H_0 ⇒ MATTER + $\underbrace{\text{FREE FIELD}}$

PHOTONS WITH WELL DEFINED ENERGY, $\vec{P}$, $\vec{L}$

LET'S NOT CONSIDER H_{SPIN}

1st ORDER OF TIME-DEPENDEN PERT. THEORY

$$\frac{1}{i\hbar} \langle \text{FINAL} | H_{INT} | \text{INITIAL} \rangle e^{i\omega_{fi} t}$$

WHAT IS THE DIFFERENCE WITH RESPECT TO THE CLASSICAL CASE? (WHEN LIGHT WAS TREATED CLASSICALLY?)

IT IS THE STATES OF THE RADIATION FIELD

$$|\text{INITIAL}\rangle = |i\rangle_{ELECTRONS} \otimes |\{ n_{\vec{k},\alpha} \}\rangle$$

$$|\text{FINAL}\rangle = |f\rangle_{ELECTRONS} \otimes |\{ n_{\vec{k}\alpha} \pm 1 \}\rangle$$

WHEN TREATED CLASSICALLY, THE STATE OF THE RADIATION FIELD REMAINED UNCHANGED DURING THE TRANSITION!!

ABSORPTION OF 1 PHOTON

$$\langle f; n_{\vec{k}\alpha} - 1 | H_{INT} | i; n_{\vec{k}\alpha} \rangle = \sum_K \frac{e}{m} \sqrt{\frac{\hbar}{2\omega_k \epsilon_0}} \langle n_{K\alpha} - 1 | \frac{a_{K\alpha}}{\sqrt{V}} | n_{K\alpha} \rangle e^{-i\omega_k t}$$

$$* \sum_i e^{i\vec{k}\cdot\vec{r}_i} \langle f | \hat{e}_\alpha \cdot \vec{p}_i | i \rangle e^{i\omega t}$$

$$= \frac{e}{m} \sqrt{\frac{\hbar n_{K\alpha}}{2\omega_k \epsilon_0 V}} e^{-i\omega_k t} \sum_i e^{i\vec{k}\cdot\vec{r}_i} \langle f | \hat{e}_\alpha \cdot \vec{p}_i | i \rangle$$

VS. PREVIOUSLY, IN THE CLASSICAL TREATMENT. $\rightarrow \frac{e}{m} A^{(CLASSICAL)}_{\vec{k},\alpha} \frac{e^{i(\vec{k}\cdot\vec{r} - \omega t)}}{\sqrt{V}} \langle f | \hat{e}_\alpha \cdot \vec{p} | i \rangle$

$$\Rightarrow \vec{A}^{(CLASSICAL)}_{K\alpha} \longleftrightarrow \sqrt{\frac{\hbar n_{\vec{k}\alpha}}{2\omega_k \epsilon_0}} \hat{e}_\alpha$$

GOOD DESCRIPTION OF CLASSICAL THEORY FOR $\langle n_{\vec{k}\alpha} \rangle \gg 1$

EMISSION OF A SINGLE PHOTON

(7)

MATRIX ELEMENT $\langle f; m_{\vec{k}\alpha}+1 \mid H_{INT} \mid i; m_{\vec{k}\alpha} \rangle$

$$= \frac{e}{m}\sqrt{\frac{\hbar\,(m_{\vec{k}\alpha}+1)}{2\omega_k \epsilon_0 V}} \sum_i e^{-i(\vec{k}\cdot\vec{r}_i - \omega_k t)} \langle f \mid \hat{e}_\alpha \cdot \vec{p}_i \mid i \rangle$$

$$\vec{A}^{(\text{CLASSICAL, EMISSION})}_{\vec{k}\alpha} \longleftrightarrow \sqrt{\frac{\hbar\,(m_{\vec{k}\alpha}+1)}{2\omega_k \epsilon_0}}\ \hat{e}_\alpha$$

AGAIN,
GOOD DESCRIPTION OF CLASSICAL THEORY WHEN $\langle m_{\vec{k}\alpha}\rangle \gg 1$

ENORMOUS DIFFERENCE WHEN $A_{\vec{k}\alpha}^{(\text{CLASS, EMISS})} = 0$

WHEN $A_{\vec{k}\alpha} \Rightarrow 0 \Rightarrow$ NO EMISSION OF PHOTONS!

$\Rightarrow$ ONLY STIMULATED EMISSION!
CLASSICAL TREATMENT ONLY ALLOWS FOR <u>STIMULATED</u> EMISSION

FOR THE QUANTUM LIGHT, HOWEVER, THERE IS EMISSION EVEN WHEN $m_{\vec{k}\alpha} = 0$ (SPONTANEOUS EMISSION)

- GREAT TRIUMPH OF THIS THEORY IS TO TREAT IN THE SAME FOOT SPONTANEOUS ($m_{\vec{k}\alpha} = 0$) AND STIMULATED ($m_{\vec{k}\alpha} \neq 0$) EMISSION

LET'S PERFORM THE PERTURBATION THEORY

$$|C^{(1)}_{fi}(t)|^2 = \frac{1}{\hbar^2}\left| \int_{-\infty}^{t} dt'\, e^{i\omega_{fi}t'} \langle F \mid H_{INT} \mid I \rangle \right|^2$$

ONLY THE $\oplus$ SIGN (EMISSION)

$$= \frac{1}{\hbar^2} \left| \langle f \mid \text{STATIC} \mid i \rangle \right|^2 * \left| \int_{-\infty}^{t} dt'\, e^{i(E_f - E_i \oplus \hbar\omega_k)\frac{t'}{\hbar}} \right|^2$$

$2\pi\hbar\,\delta(E_f - E_i \mp \hbar\omega_k)(t - t_0)$

t_0

AT t_0, SYSTEM IS IN STATE $|I\rangle$

BUT RECALL THE EMITTED PHOTON IS IN A CONTINUUM OF STATES

$\frac{V\, d^3k}{(2\pi)^3}$ AS $\omega_k = ck$

$$d^3k = k^2\, dk\, d\Omega = \frac{\omega^2}{c^2}\frac{d\omega}{c}\, d\Omega \equiv \frac{(2\pi)^3}{V}\rho\, \overbrace{d(\hbar\omega)}^{dE}$$

$$\Rightarrow \boxed{\rho = \frac{V}{(2\pi)^3}\frac{\omega^2}{\hbar c^3}\, d\Omega}$$

$$\Rightarrow \text{TRANSITION RATE} = \frac{d}{dt}\left(\int |C_{fi}^{(1)}(t)|^2 \frac{d^3k}{\frac{(2\pi)^3}{V}}\right) = W_{fi}$$

$$W_{fi} = \frac{2\pi}{\hbar}\left|\langle f| H^{INT}_{STATIC} |i\rangle\right|^2 \rho \Big|_{\hbar\omega = E_f - E_i}$$

$$= \frac{2\pi}{\hbar}\frac{e^2}{m^2}\frac{\hbar(n_{k\alpha}+1)}{2\omega\epsilon_0 V}\left|\sum_i \langle f| e^{i\vec{k}\cdot\vec{r}_i}\, \hat{e}_\alpha\cdot\vec{p}_i |i\rangle\right|^2 \frac{V}{(2\pi)^3}\frac{\omega^2 d\Omega}{\hbar c^3}$$

FOR SPONTANEOUS EMISSION: $n_{k\alpha} = 0$

ELECTRIC DIPOLE APPROXIMATION: $\underbrace{\lambda_{PHOTON}}_{\sim \frac{1}{k}} \gg a_0 \Rightarrow \vec{k}\cdot\vec{r}_i \ll 1$

$$\Rightarrow e^{i\vec{k}\cdot\vec{r}_i} \simeq 1$$

$\Rightarrow$ THE ~~H~~ SPIN IS NEGLIGIBLE IN THIS APPROXIMATION

~~BECAUSE ...~~

COMPARE $\langle \hat{e}\cdot\vec{p}\rangle$ WITH $\langle F| \vec{\mu}\cdot\vec{B} |I\rangle = \langle F| \vec{\mu}\cdot \nabla\times\vec{A} |I\rangle$

$\alpha = \langle F| \frac{e}{m} A\, \hat{e}\cdot\vec{p}_i |I\rangle$ $\quad\langle F| \vec{\mu}\cdot\vec{B} |I\rangle = \langle F| \vec{\mu}\cdot(\vec{k}\times\hat{e}_\alpha) A |I\rangle$

$$\sim \langle F| \underbrace{\frac{e\hbar}{2m}}_{\mu_B} k\, A |I\rangle = \beta$$

$$\frac{\beta}{\alpha} \propto \frac{\langle \hbar k\rangle}{\langle p_i\rangle} = \frac{\text{LIGHT MOMENTUM}}{\text{ELECTRON MOMENTUM}} \sim \frac{1/\lambda_{PHOTON}}{1/a_0} = \frac{a_0}{\lambda_{PHOTON}} \ll 1$$

$$\frac{d}{dt}\left|\int_{t_0}^{t} dt'\, e^{i\omega t'}\right|^2 = \frac{d}{dt}\left(\int_{t_0}^{t} dt'\, e^{i\omega t'} * \int_{t_0}^{t} dt''\, e^{-i\omega t''}\right)$$ 18.a

$$= e^{i\omega t}\int_{t_0}^{t} dt'\, e^{-i\omega t'} + e^{-i\omega t}\int_{t_0}^{t} dt'\, e^{i\omega t'}$$

MAKE $t_0 \to -\infty$

$$\int_{-\infty}^{t} dt'\, e^{-i\omega t'} = \lim_{\epsilon\to 0^+}\int_{-\infty}^{t} e^{-i\omega t' + \epsilon t'} dt'$$

$$= \lim_{\epsilon\to 0^+} \left.\frac{e^{-i\omega t' + \epsilon t'}}{-i\omega+\epsilon}\right|_{-\infty}^{t}$$

$$= \lim_{\epsilon\to 0^+}\frac{e^{-i\omega t}}{-i\omega+\epsilon} = \frac{i e^{-i\omega t}}{\omega + i0^+}$$

$$\Rightarrow f = e^{i\omega t} * \frac{i e^{-i\omega t}}{\omega + i0^+} + e^{-i\omega t} * \frac{i e^{i\omega t}}{-\omega + i0^+} = i\left[\frac{1}{\omega+i0^+} - \frac{1}{\omega - i0^+}\right]$$

$$= i\left[\frac{\omega - i\epsilon - (\omega + i\epsilon)}{\omega^2+\epsilon^2}\right] = \lim_{\epsilon\to 0}\frac{2\epsilon}{\omega^2+\epsilon^2} = \begin{cases} 0, & \text{for } \omega \neq 0 \\ \infty, & \text{for } \omega = 0\end{cases}$$

~~$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} f(\omega) * \frac{2\epsilon}{\omega^2+\epsilon^2}\, d\omega$~~

$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\frac{2\epsilon}{\omega^2+\epsilon^2}\, d\omega = \lim_{\epsilon\to 0}\frac{2\epsilon}{\epsilon^2}\int_{-\infty}^{\infty}\frac{d\omega}{\left(\frac{\omega}{\epsilon}\right)^2+1}$$

$$= \lim_{\epsilon\to 0} 2\underbrace{\int_{-\infty}^{\infty}\frac{dx}{x^2+1}}_{\pi} = 2\pi$$

$$\Rightarrow \frac{d}{dt}\left|\int_{t_0}^{t} dt'\, e^{i\omega t'}\right|^2 = 2\pi\,\delta(\omega)$$

No $t_0 \to -\infty$

18.6

$$\int_{t_0}^{t} dt'\, e^{-i\omega t'} = \left(\frac{1}{-i\omega}\right)\left[e^{-i\omega t} - e^{-i\omega t_0}\right]$$

$$\to f = \frac{i}{\omega}\left[1 - e^{i\omega(t-t_0)}\right] + \left(\frac{-i}{\omega}\right)\left[1 - e^{-i\omega(t-t_0)}\right]$$

$$= \frac{i}{\omega}\left[e^{-i\omega(t-t_0)} - e^{i\omega(t-t_0)}\right] = \frac{2}{\omega}\sin(\omega(t-t_0))$$

FOR $t-t_0 \to \infty$

$$\int_{-\infty}^{\infty} d\omega\, \frac{2}{\omega}\sin(\omega\Delta t) = 2\int dx\, \frac{\sin x}{x} = 2\pi$$

$$\int_{\omega_0}^{\infty} d\omega\, \frac{2}{\omega}\sin(\omega\Delta t) = 2\int_{\omega_0\Delta t}^{\infty} dx\, \frac{\sin x}{x} \xrightarrow{\Delta t\to\infty} 0, \text{ for } \omega_0>0$$

$$\to 2\pi \text{ for } \omega_0<0$$

$$\frac{d}{dt}\left|\int_{-\infty}^{t} e^{i\epsilon\frac{t'}{\hbar}}\right| = 2\pi\,\delta\!\left(\frac{\epsilon}{\hbar}\right) = 2\pi\hbar\,\delta(\epsilon)$$

FURTHERMORE, ONLY $1e^-$ CONTRIBUTES IN THE TRANSITION

$$W_{fi,\alpha} = \frac{1}{8\pi^2}\frac{e^2\omega_{fi}}{m^2\hbar\epsilon_0 c^3}\left|\langle f|\hat{e}_\alpha\cdot\vec{p}|i\rangle\right|^2 d\Omega$$

BUT $\vec{p} = \frac{im}{\hbar}\left[H_0,\vec{r}\right]$

$$\Rightarrow \langle f|\hat{e}_\alpha\cdot\vec{p}|i\rangle = i\, m\, \omega_{fi}\, \hat{e}_\alpha\cdot\langle f|\vec{r}|i\rangle$$

$$\Rightarrow W_{fi,\alpha} = \frac{1}{8\pi^2}\frac{e^2\omega_{fi}^3}{\hbar\epsilon_0 c^3}\left|\hat{e}_\alpha\cdot\vec{r}_{fi}\right|^2 d\Omega$$

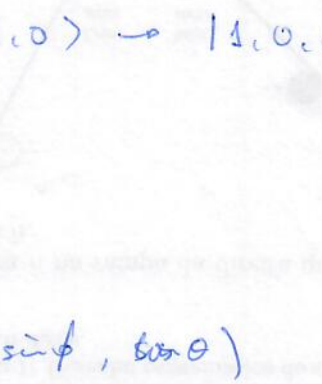
$$= \underbrace{\left(\frac{e^2}{4\pi\epsilon_0\hbar c}\right)}_{\alpha \cong 1/137}\frac{\omega_{fi}^3}{2\pi c^2}\left|\hat{e}_\alpha\cdot\vec{r}_{fi}\right|^2 d\Omega$$

EXAMPLE: HYDROGEN ATOM $|2,1,0\rangle \to |1,0,0\rangle$

$$\Rightarrow \hat{e}_\alpha\cdot\vec{r}_{fi} = \hat{e}_\alpha^{(z)}\langle f|z|i\rangle$$

LET US DETERMINE $\hat{e}_\alpha$

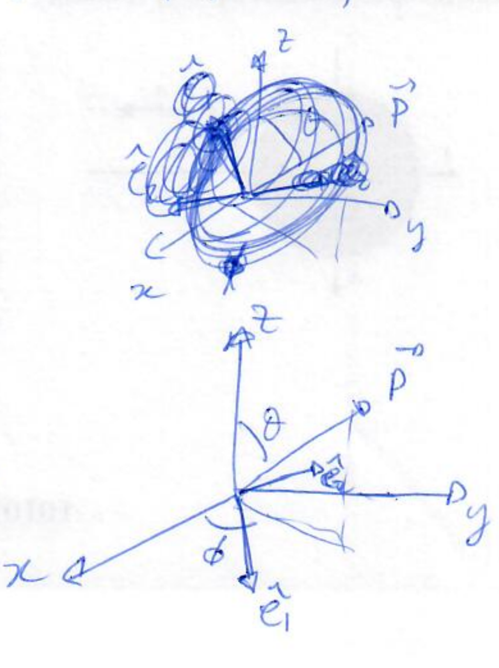
PHOTON WAVE VECTOR $\vec{k} = k(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$



$$\hat{e}_1 = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$$

$$\hat{e}_2 = (-\sin\phi, \cos\phi, 0)$$

TOTAL TRANSITION RATE



$$W_{fi} = \int d\Omega \sum_\beta \alpha\frac{\omega_{fi}^3}{2\pi c^2}\left|\hat{e}_\beta\cdot\vec{r}_{fi}\right|^2$$

$$W_{fi} = \frac{\alpha\, \omega_{fi}^3}{2\pi c^2} \int d\Omega \left| \langle 100 | e_i^z\, r\cos\theta | 210 \rangle \right|^2 \quad (20)$$

$$= \frac{\alpha\, \omega_{fi}^3}{2\pi c^2} * \underbrace{\int d\Omega \left(e_i^z\right)^2}_{2\pi \int_0^\pi \sin^2\theta \, d(\cos\theta) \; = 2\pi \times \frac{4}{3}} * \underbrace{\left| \langle 100 | r\cos\theta | 210 \rangle \right|^2}_{\left| \underbrace{\int_0^\infty r^2 dr\, R_{10}^* \, r\, R_{21}}_{\frac{2^8}{\sqrt{6}\, 3^4} a_0} * \underbrace{\int d\Omega\, Y_{00}^* \cos\theta\, Y_{10}}_{\frac{1}{\sqrt{3}}} \right|^2}$$

$$= \frac{\alpha\, \omega_{fi}^3}{2\pi c^2} * \frac{8\pi}{3} * \frac{2^{15}}{3^{10}} a_0^2 = \frac{\alpha\, \omega_{fi}^3}{c^2} \frac{2^{17}}{3^{11}} a_0^2 = 6.268 \cdot 10^8 \; s^{-1}$$

$$= \left(\frac{2}{3}\right)^8 \alpha^5 \frac{mc^2}{\hbar}$$

$$\Rightarrow \text{LIFETIME} = \boxed{\tau = \frac{1}{W} = 1.5932 \, ns}$$

EXPERIMENT: (1.600 ± 0.004) ns

$\Rightarrow$ LINE WIDTH: ENERGY-TIME UNCERTAINTY RELATION: $\Delta E_{2P} \simeq \frac{\hbar}{\tau_{2P}} \simeq \boxed{4 \cdot 10^{-7} eV}$

$\Rightarrow$ FINITE WIDTH $\Rightarrow \boxed{\frac{\Delta\lambda}{\lambda} \simeq \frac{\Delta E_{2P}}{\hbar\omega_{2P}} = 4\, 10^{-8}}$

PLANCK'S RADIATION LAW

LET US CONSIDER THE TRANSITION OF TYPE $I \rightleftarrows F + \gamma$

IN THERMAL EQUILIBRIUM $\underbrace{N(F)}_{\text{POPULATION IN STATE } F} W_{ABS} = N(I)\, W_{EMISS\,(\text{STIMULATED + SPONTANEOUS})}$

AND $\frac{N(F)}{N(I)} = \frac{N e^{-E_F/k_BT}/Z}{N e^{-E_I/k_BT}/Z} = e^{+\hbar\omega/k_BT}$ $\qquad \hbar\omega = E_I - E_F$

MOREOVER: $$\frac{W_{EMISS}}{W_{ABS}} = \frac{(n_{\vec{k}\alpha}+1) \left| \sum_i \langle F | e^{-i\vec{k}\cdot\vec{r}_i} \hat{e}_\alpha \cdot \vec{p}_i | I \rangle \right|^2}{n_{\vec{k}\alpha} \left| \sum_i \langle I | e^{i\vec{k}\cdot\vec{r}_i} \hat{e}_\alpha \cdot \vec{p}_i | F \rangle \right|^2}$$

$$\langle F | e^{-i\vec{k}\cdot\vec{r}} \hat{e}_\alpha \cdot \vec{p} | I \rangle = \left(\langle I | \vec{p}\cdot\hat{e}_\alpha e^{i\vec{k}\cdot\vec{r}} | F \rangle \right)^* \underset{\vec{k}\cdot\hat{e}_\alpha = 0}{=} \left(\langle I | e^{i\vec{k}\cdot\vec{r}} \hat{e}_\alpha \cdot \vec{p} | F \rangle \right)^*$$

$$\Rightarrow \boxed{W_{EMISS} = \frac{n_{k\alpha}+1}{n_{k\alpha}} W_{ABS}} \Rightarrow \frac{n_{k\alpha}+1}{n_{k\alpha}} = e^{\frac{\hbar\omega}{k_BT}} \Rightarrow \underbrace{\boxed{n_{k\alpha} = \frac{1}{e^{\frac{\hbar\omega}{k_BT}} - 1}}}_{\text{BOSE-EINSTEIN}}$$

$\Rightarrow$ INSIDE A BLACK-BODY

POLARIZATIONS

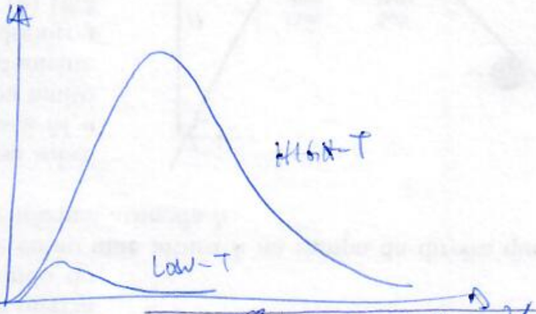
$$\frac{\text{ENERGY OF RADIATION FIELD}}{\text{VOLUME FOR ANGULAR FREQ } (\omega, \omega+d\omega)} = U(\omega)\,d\omega = \frac{\hbar\omega\, n_{k\alpha} \times 2 \times d^3k}{L^3 \left(\frac{2\pi}{L}\right)^3}$$

$$\Rightarrow U(\omega)\,d\omega = \frac{1}{4\pi^3}\left(\frac{\hbar\omega}{e^{\frac{\hbar\omega}{k_BT}}-1}\right) \times 4\pi k^2 dk$$

$\omega = ck$

$$\Rightarrow U(\omega)\,d\omega = \frac{1}{4\pi^3}\left(\;\right) \times 4\pi \frac{\omega^2}{c^2}\frac{d\omega}{c} = \frac{\hbar}{\pi^2 c^3}\left(\frac{\omega^3}{(2\pi)^3}\right)(2\pi)^3 \times \frac{d\omega}{e^{\frac{\hbar\omega}{k_BT}}-1}$$

$$\Rightarrow \boxed{U(\nu) = U(\omega)\frac{d\omega}{d\nu} = \frac{8\pi h}{c^3}\frac{\nu^3}{e^{\frac{h\nu}{k_BT}}-1}}$$



$$\frac{\partial U}{\partial \nu} \Rightarrow = \frac{3\nu^2}{(\;)} - \frac{\nu^3}{(\;)^2}\frac{h}{k_BT}e^{\frac{h\nu}{k_BT}}$$

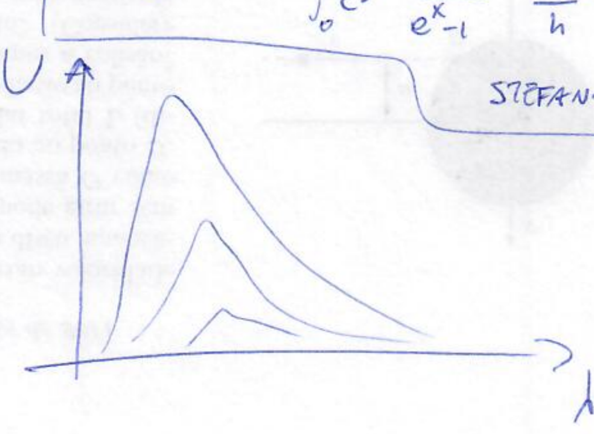
$$= \frac{\nu^2}{(\;)^2}\left[3e^{(\,)} - 3 - \frac{h\nu}{k_BT}e^{(\,)}\right] = 0$$

$$\Rightarrow \left(3 - \frac{h\nu^*}{k_BT}\right)e^{(\,)} = 3$$

$$\boxed{P = \int_0^\infty U(\nu)\,d\nu = \int_0^\infty \frac{8\pi h}{c^3}\frac{\left(\frac{k_BTx}{h}\right)^3}{e^x-1}\frac{k_BT\,dx}{h} = \sigma T^4}$$

STEFAN-BOLTZMANN

$c = \lambda\nu$



$$U(\lambda) = U(\nu)\left|\frac{d\nu}{d\lambda}\right| = \frac{8\pi h}{\lambda^5}\frac{c}{e^{\frac{hc}{\lambda k_BT}}-1}$$

$$\frac{\partial U}{\partial \lambda} \Rightarrow =$$

SCATTERING OF LIGHT BY ATOMIC ELECTRONS

- TOTAL NUMBER OF PHOTONS IS CONSERVED

$$H_{INT} = \sum_i \frac{e}{m} \vec{A}(\vec{r}_i,t)\cdot\vec{P}_i + \frac{e^2}{2m} \vec{A}(r_i,t)\cdot A(\vec{r}_i,t) + \underbrace{H_{SPIN}}_{-\sum_i \mu_i \cdot \nabla\times\vec{A}_i}$$

1st ORDER IN TIME-DEP. PERT. THEORY

→ ONLY $\vec{A}\cdot\vec{A}$ CONTRIBUTES (BECAUSE IS THE ONLY ONE CONSERVING THE # OF PHOTONS)

MATRIX ELEMENT: $\langle F | \frac{e^2}{2m}\sum_i \vec{A}_i\cdot\vec{A}_i | I \rangle$

$$\vec{A} = \frac{1}{\sqrt{V}} \sum_{k\alpha} \sqrt{\frac{\hbar}{2\omega_k \epsilon_0}}\, a_{\vec{k}\alpha}\, e^{i\vec{k}\cdot\vec{r}}\, e_\alpha + h.c.$$

$$\vec{A}_i\cdot\vec{A}_i = \frac{1}{V}\sum_{\vec{k}\vec{k}'}\sum_{\alpha\alpha'} \frac{\hbar}{2\epsilon_0}\times\frac{1}{\sqrt{\omega_k\omega_{k'}}}\left\{ a_{k\alpha}a^+_{k'\alpha'} e^{i(\vec{k}-\vec{k}')\cdot\vec{r}_i}\, \hat{e}_\alpha\cdot\hat{e}^*_{\alpha'} + a^+_{k\alpha}a_{k'\alpha'} e^{-i(\vec{k}-\vec{k}')\cdot\vec{r}_i}\, \hat{e}^*_\alpha\cdot\hat{e}_{\alpha'} \right\}$$

$$+ \left(aa + a^+a^+\right)$$

↳ NON CONTRIBUTING TERMS IN 1st ORDER

INITIAL AND FINAL STATES: PHOTON $(\vec{q},\beta) \xrightarrow[\text{TO}]{\text{SCATTERED}}$ PHOTON $(\vec{q}',\beta')$

$$|I\rangle = |i\rangle \otimes | n_{k_1,1}, n_{k_1,2}, \ldots n_{\vec{q}'\beta'}, \ldots, n_{\vec{q}\beta}, \ldots \rangle$$

$$|F\rangle = |f\rangle \otimes | n_{\vec{k}_1,1}, n_{\vec{k}_1,2}, \ldots n_{\vec{q}'\beta'}+1, \ldots, n_{\vec{q}\beta}-1, \ldots \rangle$$

THEN, WE HAVE $\langle \{n_F\} | a_{k\alpha}a^+_{k'\alpha'} + a^+_{k\alpha}a_{k'\alpha'} | \{n_I\} \rangle$

$$= \sqrt{(n_{q'\beta'}+1)\, n_{q\beta}}\; \delta_{k'q'}\delta_{\alpha'\beta'}\delta_{kq}\delta_{\alpha\beta} + \sqrt{n_{q\beta}(n_{q'\beta'}+1)}\times \delta_{k'q}\delta_{\alpha'\beta}\delta_{kq'}\delta_{\alpha\beta'}$$

$$\text{THEN,}\quad C^{(1)} = \frac{1}{i\hbar}\int_{t_0}^{t} dt'\, e^{i\omega_{FI}t'} \langle f|\, \frac{e^2\hbar}{4mV\epsilon_0} \times \frac{\mathcal{I}}{\sqrt{\omega_q\omega_{q'}}} \times \left(e^{i(\vec{q}-\vec{q}')\cdot\vec{r}_j}\, \hat{e}_p\cdot\hat{e}^*_{p'} + e^{-i(\vec{q}'-\vec{q})\cdot\vec{r}_j}\, \hat{e}_p\cdot\hat{e}^*_{p'} \right) |i\rangle$$

(23)

FOR BOUNDED ELECTRONS AND LOW-ENERGY PHOTONS

$\Rightarrow$ ELECTRIC DIPOLE APROXIMATION

$$|\vec{q}-\vec{q}'| << \frac{1}{a_0} \Rightarrow e^{i\Delta\vec{q}\cdot\vec{r}} \simeq 1$$

$$\Rightarrow \boxed{C^{(1)} = \frac{1}{i\hbar}\left(\frac{e^2\hbar}{2mV\epsilon_0}\right) \times \frac{\sqrt{(n_q+1)\,n_{q'}}}{\sqrt{\omega_q\omega_{q'}}} \times \underbrace{N}_{\#\text{ OF ELECTRONS}} \times \langle f|i\rangle\, \hat{e}_p\cdot\hat{e}^*_{p'} \int_{t_0}^{t} dt'\, e^{i\omega_{FI}t'}}$$

$$\boxed{\hbar\omega_{FI} = E_f + \hbar\omega_{q'} - (E_i + \hbar\omega_q)}$$

2ND ORDER IN PERTURBATION THEORY

$$C^{(2)} = \left(\frac{1}{i\hbar}\right)^2 \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \sum_J e^{i\omega_{FJ}t'} e^{i\omega_{JI}t''} \langle F|\sum_i \frac{e\vec{A}_i\cdot\vec{P}_i}{m}|J\rangle\langle J|\sum_l \frac{e\vec{A}_l\cdot\vec{P}_l}{m}|I\rangle$$

TIME-INTEGRAL IN t'':

$$\int_{t_0}^{t'} dt''\, e^{i\omega_{JI}t''}\,, \qquad \text{WE WILL TAKE } t_0 \to -\infty$$

$\Rightarrow$ NEED TO REGULARIZE THE INTEGRAL

$$\lim_{\epsilon\to 0^+} \int_{-\infty}^{t'} dt''\, e^{(iE_{JI}+\epsilon)\frac{t''}{\hbar}} = \frac{\hbar}{iE_{JI}+\epsilon}\, e^{(iE_{JI}+\epsilon)t'}$$

$$\Rightarrow C^{(2)} = \left(\frac{1}{i\hbar}\right)^2 \times \frac{\hbar}{i} \int_{-\infty}^{t} dt' \sum_J e^{i(E_{FJ}+E_{JI}-i\epsilon)t'}\, \frac{\langle F|V|J\rangle\langle J|V|I\rangle}{E_J - E_i - i\epsilon}$$

$$\Rightarrow C^{(2)} = \frac{-1}{i\hbar} \sum_J \frac{\langle F|V|J\rangle\langle J|V|I\rangle}{E_J - E_I + i\epsilon} \underbrace{\int_{-\infty}^{t} dt' \, e^{i(E_{FI} - i\epsilon)\frac{t'}{\hbar}}}_{\text{SAME TIME INTEGRAL AS IN 1st ORDER}}$$

LET US FOCUS ON THE MATRIX ELEMENT

$\langle F|V|J\rangle\langle J|V|I\rangle \rightarrow$ WHICH ARE THE CONTRIBUTING INTERMEDIATE STATES $|J\rangle$?

$$\langle F| (a_{k\alpha} + a^{+}_{k\alpha}) |J\rangle\langle J| (a_{k'\alpha'} + a^{+}_{k'\alpha'}) |I\rangle$$

AS $|F\rangle = |f\rangle \otimes |\{m_{q\beta} - 1, m_{q'\beta'} + 1\}\rangle$

$|I\rangle = |i\rangle \otimes |\{m_{q\beta}, m_{q'\beta'}\}\rangle$

$\Rightarrow$ CONTRIBUTION COMES IN TWO CASES.

(a) $\langle\{m_{q\beta}-1, m_{q'\beta'}+1\}| a_{k\alpha} |\{m_{q\beta}, m_{q'\beta'}+1\}\rangle\langle\{m_{q\beta}, m_{q'\beta'}+1\}| a^{+}_{k'\alpha'} |\{m_{q\beta}, m_{q'\beta'}\}\rangle$

(b) $\langle\{m_{q\beta}-1, m_{q'\beta'}+1\}| a^{+}_{k\alpha} |\{m_{q\beta}-1, m_{q'\beta'}\}\rangle\langle\{m_{q\beta}-1, m_{q'\beta'}\}| a_{k'\alpha'} |\{m_{q\beta}, m_{q'\beta'}\}\rangle$

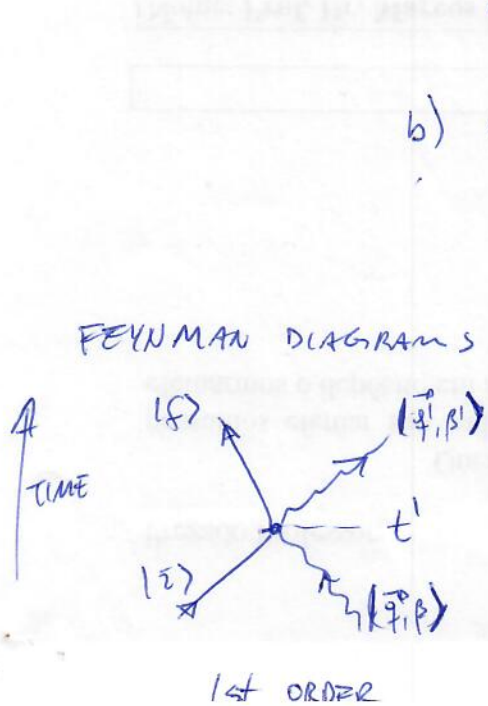
INTERPRETATION: a) EMITS PHOTON $(\vec{q}'\beta')$ AT TIME t'' $\begin{pmatrix}\vec{k}' = \vec{q}' \\ \beta' = \beta\end{pmatrix}$

ABSORBS " $(\vec{q}\beta)$ AT TIME t' $\begin{pmatrix}\vec{k} = \vec{q} \\ \alpha = \beta\end{pmatrix}$

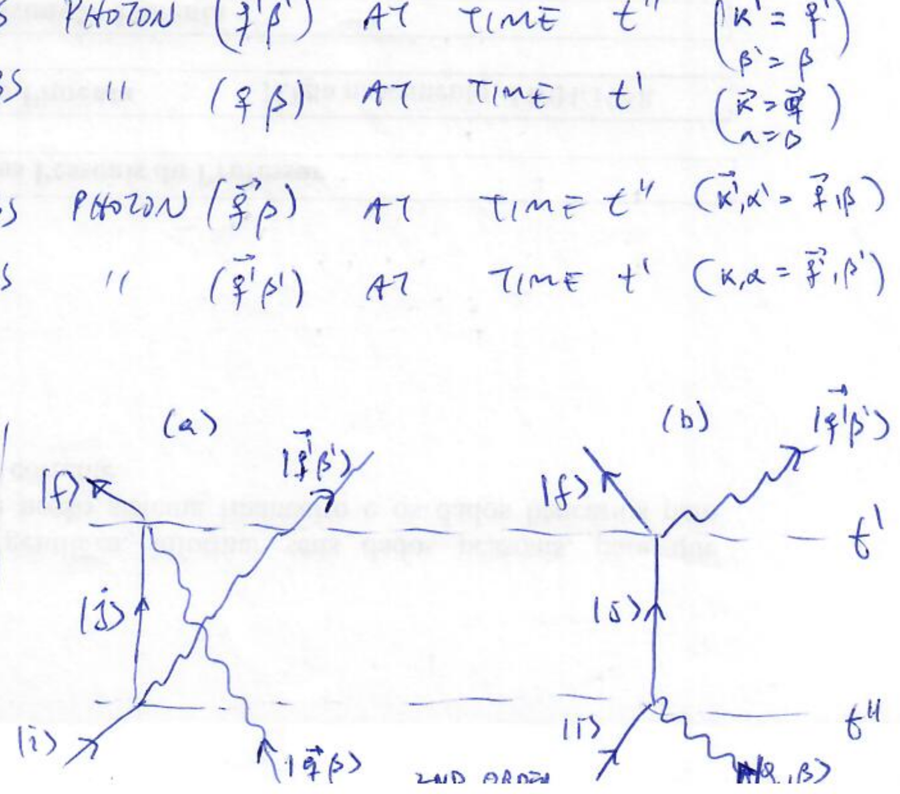
b) ABSORBS PHOTON $(\vec{q}\beta)$ AT TIME t'' $(\vec{k}',\alpha' = \vec{q},\beta)$

EMITS " $(\vec{q}'\beta')$ AT TIME t' $(k,\alpha = \vec{q}',\beta')$

FEYNMAN DIAGRAMS



1st ORDER



2ND ORDER

THE MATRIX ELEMENTS THEN BECOME $(e^{i\vec{q}\cdot\vec{r}} \approx 1)$

$$\sum_J \frac{\langle F|V|J\rangle\langle J|V|I\rangle}{E_J - E_I + i\epsilon} = \left(\frac{e}{m}\right)^2 \times \frac{\hbar}{2V\epsilon_0} \times \frac{\sum_j}{\sqrt{\omega_q \omega_{q'}}} \times \left[\frac{\langle f|\hat{e}_\beta\cdot\sum_\ell \vec{P}_\ell|j\rangle\langle j|\hat{e}^*_{\beta'}\cdot\sum_\ell\vec{P}_\ell|i\rangle}{E_j + \hbar\omega_{q'} - E_i - i\epsilon} \right.$$

$\times \sqrt{n_q}$

$$\left. + \frac{\langle f|\hat{e}^*_{\beta'}\cdot\sum_\ell\vec{P}_\ell|j\rangle\langle j|\hat{e}_\beta\cdot\sum_\ell\vec{P}_\ell|i\rangle}{E_j - \hbar\omega_q - E_i - i\epsilon} \right]$$

PUTTING ALL TOGETHER,

$$c^{(1)}+c^{(2)} = \frac{1}{i\hbar}\left(\frac{e^2\hbar}{2mV\epsilon_0}\right) \times \frac{\sqrt{(n_{q\beta'}+1)\, n_{q\beta}}}{\sqrt{\omega_q\omega_{q'}}} \left[N\, \delta_{fi}\, \hat{e}_\beta\cdot\hat{e}^*_{\beta'} - \frac{1}{m}\sum_j \frac{\langle f|\hat{e}_\beta\cdot\vec{P}|j\rangle\langle j|\hat{e}^*_{\beta'}\cdot\vec{P}|i\rangle}{E_j+\hbar\omega_{q'}-E_i-i\epsilon} - \frac{1}{m}\sum_j \frac{\langle f|\hat{e}^*_{\beta'}\cdot\vec{P}|j\rangle\langle j|\hat{e}_\beta\cdot\vec{P}|i\rangle}{E_j-\hbar\omega_q-E_i-i\epsilon}\right] \times \int_{-\infty}^{t} dt\, e^{i(E_{FI}-i\epsilon)t/\hbar}$$

$$|c^{(1)}+c^{(2)}|^2 = \frac{1}{\hbar^2}\left(\frac{e^2\hbar}{2mV\epsilon_0}\right)^2 \frac{(n_{q\beta'}+1)\,n_{q\beta}}{\omega_q\omega_{q'}} \left| \cdots \right|^2 \times 2\pi\hbar\,\delta(E_F-E_I)\,t$$

$E = \hbar c k$

FERMI'S GOLDEN RULE: EMITTED PHOTON IS IN A CONTINUUM

$$\sum_{q'} \to \int \frac{d^3 q}{\frac{(2\pi)^3}{V}} = \int \frac{V}{(2\pi)^3}\, d\Omega\, k^2 dk = \int \frac{V}{(2\pi)^3} \frac{E^2}{\hbar^2 c^2} \times \frac{dE}{\hbar c}\, d\Omega$$

$$\cancel{\Rightarrow |c^{(1)}+c^{(2)}|^2 \to \frac{1}{\hbar^2}\left(\frac{e^2\hbar}{2mV\epsilon_0}\right)^2 \frac{(n_{q\beta'}+1)\,n_{q\beta}}{\omega_q\omega_{q'}} \left|\cdots\right|^2 \frac{2\pi\hbar V}{(2\pi)^3} \frac{\hbar^2\omega_{q'}^2}{\hbar^2 c^2} \frac{\hbar\, d\omega_{q'}}{\hbar c}\, t\, d\Omega}$$

TRANSITION RATE:

$$\cancel{W = \frac{d}{dt}|c^{(1)}+c^{(2)}|^2 = \frac{1}{\hbar}\left(\frac{e^2\hbar}{2mV\epsilon_0}\right)^2 (n_{q\beta'}+1)\,n_{q\beta} \left(\frac{\omega_q'}{\omega_q}\right) \left|\cdots\right|^2 \times \frac{1}{2\pi c^3} \left(\frac{d\Omega}{2\pi}\right)}$$

$$\sum_{\{f'\}} |c^{(1)}+c^{(2)}|^2 = \underbrace{\left(\int d\Omega' \; \frac{1}{\hbar^2}\left(\frac{e^2\hbar}{2mV\epsilon_0}\right)^2 \frac{(n_{q'\beta'}+1)\,n_{q\beta}}{\omega_q\,\omega_{q'}} \left|--\right|^2 2\pi\hbar \, \frac{V}{(2\pi)^3}\, \frac{E_{q'}^2}{\hbar c^2}\times\frac{1}{\hbar c}\right)}\, t$$

TRANSITION RATE $W = \frac{d}{dt}|c^{(1)}+c^{(2)}|^2$, $E_{q'} = \hbar\omega_{q'}$

$$\Rightarrow W = \int d\Omega' \left(\frac{e^2}{2m\epsilon_0}\right)^2 \frac{(n_{q'\beta'}+1)\,n_{q\beta}}{(2\pi)^2 c^3} \left|--\right|^2 \times \frac{\omega_{q'}}{\omega_q}\times\frac{1}{V}$$

$$\boxed{W = \int d\Omega' \left(\frac{e^2}{4\pi\epsilon_0}\times\frac{1}{mc^2}\right)^2 (n_{q'\beta'}+1)\,n_{q\beta}\left(\frac{\omega_{q'}}{\omega_q}\right)\left|--\right|^2 \times \frac{c}{V}}$$

TOTAL CROSS SECTION $\sigma = \dfrac{\text{TRANSITION RATE SUMMED OVER ALL FINAL STATES}}{\text{FLUX OF INCOMING PHOTONS}}$

$$= \frac{W}{n_{q\beta}\times\frac{c}{V}}$$

KRAMERS-HEISENBERG FORMULA

$$\Rightarrow \boxed{\sigma = \int d\Omega' \; r_0^2 \, (n_{q'\beta'}+1)\, \frac{\omega_{q'}}{\omega_q}\left| N_{elec}\,\delta_{fi}\,\hat{e}_\beta\cdot\hat{e}^*_{\beta'} - \frac{1}{m}\sum_j \left(\frac{\langle - \rangle\langle - \rangle}{E_j+\hbar\omega_{q'}-E_i} + \frac{\langle - \rangle\langle - \rangle}{E_j-\hbar\omega_q-E_i}\right)\right|^2}$$

TOTAL CROSS SECTION IS PROPORTIONAL TO $\int d\Omega' \, r_0^2$ KRAMERS-HEISENBERG FORMULA

WHERE $r_0 = \dfrac{e^2}{4\pi\epsilon_0}\times\dfrac{1}{mc^2}$ ≡ CLASSICAL RADIUS OF THE ELECTRON

$\simeq 2{,}82\ 10^{-15}\ m$

CHARGING ENERGY OF A SPHERE $= \dfrac{e^2}{4\pi\epsilon_0 r_0} = mc^2$

RAYLEIGH SCATTERING

$\boxed{\text{ELASTIC SCATTERING}}$ → SPECIALLY RELEVANT WHEN THE INCOMING PHOTONS DO NOT HAVE ENOUGH ENERGY TO EXCITE THE ATOM

$$\Rightarrow \omega_q = \omega_{q'}$$
$$E_i = E_f$$

CONSIDER A SINGLE PHOTON : $\boxed{n_{q\beta} = 1 \ , \ n_{q'\beta'} = 0}$

AND A SINGLE ELECTRON $\boxed{N_{ELEC} = 1}$

$\boxed{\text{NEW TRICK}}$: $\left[\hat{e}_\beta \cdot \vec{r} \, , \, \hat{e}^*_{\beta'} \cdot \vec{P} \right] = i\hbar \ \hat{e}_\beta \cdot \hat{e}^*_{\beta'}$

$$\Rightarrow \hat{e}_\beta \cdot \hat{e}^*_{\beta'} = \frac{1}{i\hbar} \langle i | \left[\hat{e}_\beta \cdot \vec{r} , \hat{e}^*_{\beta'} \cdot \vec{P} \right] | i \rangle$$

$$= \sum_j \frac{1}{i\hbar} \left(\langle i | \hat{e}_\beta \cdot \vec{r} | j \rangle \langle j | \hat{e}^*_{\beta'} \cdot \vec{P} | i \rangle - \langle i | \hat{e}^*_{\beta'} \cdot \vec{P} | j \rangle \langle j | \hat{e}_\beta \cdot \vec{r} | i \rangle \right)$$

$\boxed{\text{OLD TRICK}}$: $\langle i | \vec{r} | j \rangle = \dfrac{1}{E_i - E_j} \langle i | [H_0, \vec{r}] | j \rangle$

$$\text{BUT} \quad [H_0, \vec{r}] = \frac{1}{2m} [P^2, \vec{r}] = -\frac{i\hbar}{m} \vec{P}$$

$$\Rightarrow \langle i | \hat{e}_\beta \cdot \vec{r} | j \rangle = -\frac{i\hbar}{m} \frac{\langle i | \hat{e}_\beta \cdot \vec{P} | j \rangle}{E_i - E_j}$$

FINALLY,

$$\hat{e}_\beta \cdot \hat{e}^*_{\beta'} = -\frac{1}{m} \left(\sum_j \frac{\langle i | \hat{e}_\beta \cdot \vec{P} | j \rangle \langle j | \hat{e}^*_{\beta'} \cdot \vec{P} | i \rangle}{E_i - E_j} - \frac{\langle i | \hat{e}^*_{\beta'} \cdot \vec{P} | j \rangle \langle j | \hat{e}_\beta \cdot \vec{P} | i \rangle}{E_j - E_i} \right)$$

27.9

$$\left[\hat{e}_\beta \cdot \vec{r}, \hat{e}^*_{\beta'} \cdot \vec{p}\right] = \cancel{(\hat{e}_\beta \cdot \vec{r})(\hat{e}^*_{\beta'} \cdot \vec{p})}$$

$$= e^x_\beta e^{*x}_{\beta'} \left[x, p^x\right] + e^y_\beta e^{*y}_{\beta'} \left[y, p^y\right] + \cdots$$

$$= \left(e^x_\beta e^{*x}_\beta + e^y_\beta e^{*y}_{\beta'} + \cdots\right) i\hbar$$

$$= i\hbar \, \hat{e}_\beta \cdot \hat{e}^*_{\beta'}$$

THE TOTAL CROSS SECTION THEN BECOMES

$$\sigma = r_0^2 \int d\Omega' \left|\frac{1}{m}\right|^2 \left| \sum_j \langle i| e_\beta \cdot \vec{p}\, |j\rangle\langle j|\hat{e}^*_{\beta'} \cdot \vec{p}\, |i\rangle \left(\frac{1}{E_i - E_j} + \frac{1}{E_j - E_i + \hbar\omega_f} \right) \right.$$

$$\left. + \sum_j \langle i|\hat{e}^*_{\beta'} \cdot \vec{p}\, |j\rangle\langle j|\hat{e}_\beta \cdot \vec{p}\, |i\rangle \left(\frac{1}{E_i - E_j} + \frac{1}{E_j - E_i \mp \hbar\omega_f} \right) \right|^2$$

~~THE SUM IN $|j\rangle$ WILL BE ZERO~~

~~THE~~ THE $\hat{e}_\beta \cdot \langle i|\vec{p}\,|j\rangle$ WILL SELECT $|j\rangle$ STATES

USUALLY $\langle i|\vec{p}\,|i\rangle = 0 \Rightarrow |j\rangle$ STATE IS AN EXCITED STATE OF THE MATTER

SINCE THE INCOMING PHOTON IS LOW-ENERGETIC $\Rightarrow \hbar\omega_f \ll E_j - E_i$

$$\Rightarrow \frac{1}{E_j - E_i \pm \hbar\omega_f} \approx \frac{1}{E_j - E_i}\left(1 \mp \frac{\hbar\omega_f}{E_j - E_i}\right) \Rightarrow \frac{1}{E_i - E_j} + \frac{1}{E_j - E_i \mp \hbar\omega} \approx \pm \frac{\hbar\omega_f}{(E_j - E_i)^2}$$

$$\Rightarrow \sigma \approx \frac{r_0^2}{m^2} \int d\Omega' \left| \sum_j \frac{\langle i|\hat{e}_\beta \cdot \vec{p}\,|j\rangle\langle j|\hat{e}^*_{\beta'} \cdot \vec{p}\,|i\rangle}{(E_j - E_i)^2} (-\hbar\omega_f) + \frac{\langle i|\hat{e}^*_{\beta'} \cdot \vec{p}\,|j\rangle\langle j|\hat{e}_\beta \cdot \vec{p}\,|i\rangle}{(E_j - E_i)^2} (+\hbar\omega_f) \right|^2$$

HOWEVER, THIS IS ZERO!

SUM RULES

$$\sum_j \frac{\langle i| e_\beta \cdot \vec{p}\,|j\rangle\langle j|\hat{e}^*_{\beta'} \cdot \vec{p}\,|i\rangle}{(E_j - E_i)^2} - \frac{\langle i|\hat{e}^*_{\beta'} \cdot \vec{p}\,|j\rangle\langle j|\hat{e}_\beta \cdot \vec{p}\,|i\rangle}{(E_j - E_i)^2}$$

$$= \sum_j \left(\frac{m}{i\hbar}\right)^2 \frac{\langle i|[\hat{e}_\beta \cdot \vec{r}, H_0]|j\rangle\langle j|[\hat{e}^*_{\beta'} \cdot \vec{r}, H_0]|i\rangle - \langle i|[\hat{e}^*_{\beta'} \cdot \vec{r}, H_0]|j\rangle\langle j|[\hat{e}_\beta \cdot \vec{r}, H_0]|i\rangle}{(E_j - E_i)^2}$$

$$= -\left(\frac{m}{i\hbar}\right)^2 \sum_j \langle i|\hat{e}_\beta \cdot \vec{r}\,|j\rangle\langle j|\hat{e}^*_{\beta'} \cdot \vec{r}\,|i\rangle - \langle i|\hat{e}^*_{\beta'} \cdot \vec{r}\,|j\rangle\langle j|\hat{e}_\beta \cdot \vec{r}\,|i\rangle$$

$$= -\left(\frac{m}{i\hbar}\right)^2 \langle i|\,[\hat{e}_\beta \cdot \vec{r}, \hat{e}^*_{\beta'} \cdot \vec{r}\,]\,|i\rangle = 0$$

THEREFORE, WE NEED TO GO TO FURTHER ORDER (29)
IN THE EXPANSION

$$-\frac{1}{E_j-E_i}+\frac{1}{E_j-E_i\mp\hbar\omega_f}\approx\pm\frac{\hbar\omega_f}{(E_j-E_i)^2}+\frac{(\hbar\omega_f)^2}{(E_j-E_i)^3}\mp\frac{(\hbar\omega_f)^3}{(E_j-E_i)^4}+\cdots$$

THE SURVIVING TERM IS THUS,

$$\sigma\approx\frac{r_0^2}{m^2}\int d\Omega'\left|\sum_j\left(\frac{\langle i|\hat{e}_p\cdot\vec{p}|j\rangle\langle j|\hat{e}_{p'}^*\cdot\vec{p}|i\rangle+h.c}{(E_j-E_i)^3}\right)\right|^2*(\hbar\omega_f)^4$$

$$=\frac{r_0^2m^2}{\hbar^4}\int d\Omega'\left|\sum_j\frac{\langle i|\hat{e}_p\cdot\vec{r}|j\rangle\langle j|\hat{e}_{p'}^*\cdot\vec{r}|i\rangle}{E_j-E_i}+h.c.\right|^2*(\hbar\omega_f)^4$$

FINALLY, THE PROB OF SCATTERING IS PROPORTIONAL TO $\underline{\omega^4}$

RAYLEIGH'S LAW

VISIBLE LIGHT OBEYS THE APPROXIMATION $\hbar\omega \ll \overbrace{E_j-E_i}^{\text{ULTRAVIOLET }(2p\to1s)}$

FOR ORDINARY ATOMS ("GASES")
IN THE ATMOSPHERE

ORDERS OF MAGNITUDE

HYDROGEN:

$$\sigma\sim\frac{r_0^2m^2}{\hbar^4}\int d\Omega'\left|\frac{a_0^2}{\underbrace{\tfrac{3}{4}}_{\text{NEGLECT}}13.6\text{ eV}}\right|^2*(\hbar\omega)^4$$

$$\sim r_0^2m^2\omega^4\left(\frac{a_0^2}{13.6\text{eV}}\right)^2$$

$\Rightarrow$ BLUE SKY
RED SUNSET
LUNAR ECLIPSE - MOON IS RED

RED ($\lambda=7000\,\text{Å}$), $\hbar\omega=0.28\,\text{eV}$ $\Rightarrow$ $\sigma_R=9\,10^{-3}\,\text{cm}^2=0.9\,\text{mm}^2$

BLUE ($\lambda=4000\,\text{Å}$), $\sigma_B=\left(\frac{7}{4}\right)^4$ TIMES BIGGER THAN σ_R

$\simeq$
9.38

(30)

THOMPSON SCATTERING

(ESSENTIALLY)

- FREE ELECTRON
- $\omega_{\vec{q}} = \omega_{\vec{q}'}$

γ-RAYS, FOR INSTANCE

NEGLECT THE 2ND ORDER TERMS BECAUSE OF THE DENOMINATOR

$\Rightarrow$ KRAMERS-HEISENBERG FORMULA FOR $\boxed{\hbar\omega_{\vec{q}} \gg \text{BINDING ENERGY}}$

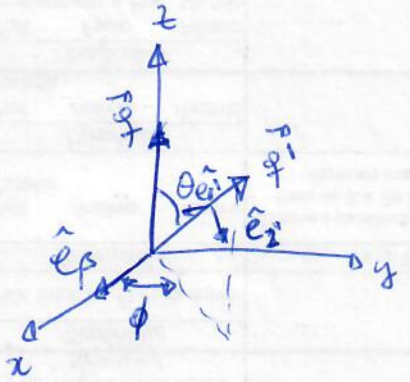
$$\boxed{\frac{d\sigma}{d\Omega} = r_0^2 \left| \hat{e}_\rho \cdot \hat{e}^*_{\rho'} \right|^2}$$

(CHOOSE) CONSIDER $\vec{q} = q\,\hat{z}$

$\hat{e}_1 = \hat{x}\ ,\ \hat{e}_2 = \hat{y}$

$$\hat{e}_{1'} = \left(\sin\phi\ ,\ -\cos\phi\ ,\ 0\right)$$

$$\hat{e}_{2'} = \left(\cos\theta\cos\phi\ ,\ \sin\phi\cos\theta\ ,\ -\sin\theta\right)$$



$$\Rightarrow \left.\frac{d\sigma}{d\Omega}\right|_{\rho=1} = r_0^2 \begin{cases} \sin^2\phi\ , & \rho'=1 \\ \cos^2\theta\cos^2\phi\ , & \rho'=2 \end{cases}$$

$$\left.\frac{d\sigma}{d\Omega}\right|_{\rho=2} = r_0^2 \begin{cases} \cos^2\phi\ , & \rho'=1 \\ \sin^2\phi\cos^2\theta\ , & \rho'=2 \end{cases}$$

INCOMING LIGHT IS UNPOLARIZED $\to$ OUTGOING LIGHT POLARIZATION IS NOT MEASURED

$$\Rightarrow \sigma_{TOT} = \int d\Omega\ r_0^2 \left[\underbrace{\frac{1}{2}\left(\sin^2\phi + \cos^2\theta\cos^2\phi\right)}_{\text{INTEGRATE THE OUTPUT}} + \underbrace{\frac{1}{2}\left(\cos^2\phi + \sin^2\phi\cos^2\theta\right)}_{\text{INTEGRATED OUTPUT}} \right]$$

$$= \int d\Omega\ \frac{r_0^2}{2}\left(1 + \cos^2\theta\right) = \frac{1}{2} r_0^2 \left(4\pi + 2\pi \times \frac{2}{3}\right) = \frac{8\pi}{3} r_0^2 = 6.65 \times 10^{-25}\ \text{cm}^2$$

FREE ELECTRON

$$H_0 = \frac{p^2}{2m} + \sum_{\vec{k},\alpha} \hbar\omega_k N_{\vec{k}\alpha}$$

$$H_{INT} = \frac{1}{2m} e^2 A^2$$

DIPOLE APPROXIMATION $\rightarrow e^{ik\cdot r} \simeq 1$

$$c^{(1)} = \frac{1}{i\hbar}\int dt' e^{i\omega_{FI}t'} \frac{e^2}{2m} \langle F | \frac{1}{V} \frac{\hbar}{2\epsilon_0} \times \frac{1}{\sqrt{\omega_q\omega_{q'}}} (a^+a + a^+a) | I \rangle$$

AS BEFORE

$$W = r_0^2 \int d\Omega' \frac{\omega_{q'}}{\omega_q} \left| N_{ELECT} \times \delta_{fi}\, \hat{e}_\beta \cdot \hat{e}_{\beta'} \right|^2 (n_{q'\beta'}+1)$$

$$\rightarrow r_0^2 \int d\Omega' \left| \hat{e}_\beta \cdot \hat{e}^*_{\beta'} \right|^2$$

CLASSICAL REVIEW ON SCATTERING OF LIGHT BY ATOMIC ELECTRON

$$m\ddot{\vec{r}} = \underbrace{-m\omega_0^2\vec{r}}_{\text{APROXIMATE THE BOUNDED } e \text{ BY HARMONIC OSCILLATOR}} + e\vec{E}_0 e^{i\omega t}$$

(NEGLECT MAGNETIC FIELD → CONTRIBUTION OF ORDER $\frac{v}{c}$)

$$\Rightarrow \ddot{\vec{r}}(t) = -\frac{e}{m}\left(\frac{\omega^2}{\omega_0^2-\omega^2}\right)\vec{E}_0 e^{i\omega t}$$

$$\vec{r} = \frac{e}{m}\left(\frac{1}{\omega_0^2-\omega^2}\right)\vec{E}_0 e^{i\omega t}$$

FAR FIELD IRADIATE POWER

$$\boxed{\frac{dP}{d\Omega} = \frac{\omega^4 p_0^2}{32\pi^2\epsilon_0 c^3}\sin^2\theta}, \quad \vec{p}_0 \equiv \text{ELECTRIC DIPOLE} = -e\vec{r}$$

$$\text{INCIDENT FLUX} = c \times \frac{\epsilon_0}{2}E_0^2$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{dP/d\Omega}{\text{INCIDENT FLUX}} = \frac{\omega^4 p_0^2}{32\pi^2\epsilon_0 c^3}\sin^2\theta \times \frac{2}{c\,\epsilon_0 E_0^2}$$

$$= \frac{\omega^4}{16\pi^2\epsilon_0^2 c^4}\times\left(\frac{e^2}{m}\right)^2\times\frac{\cancel{E_0^2}}{(\omega_0^2-\omega^2)^2}\times\frac{1}{\cancel{E_0^2}}\times\sin^2\theta$$

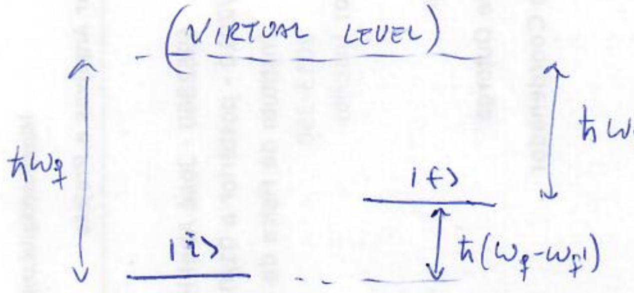
$$\Rightarrow \sigma_{TOT} = \underbrace{\frac{1}{16\pi^2\epsilon_0^2 c^4}\left(\frac{e^2}{m}\right)^2}_{r_0^2}\times\frac{\omega^4}{(\omega_0^2-\omega^2)^2}\times\frac{8\pi}{3} = \frac{8\pi r_0^2}{3}\,\frac{\omega^4}{(\omega_0^2-\omega^2)^2}$$

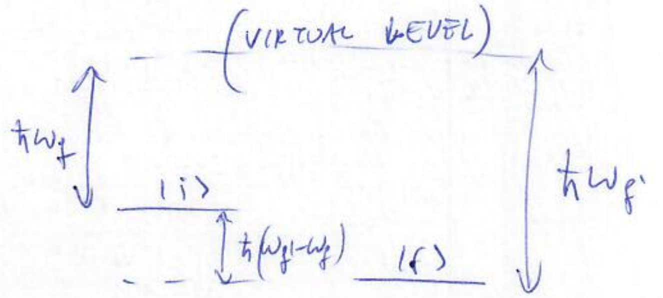
FOR $\omega \ll \omega_0 \Rightarrow \sigma_{TOT} \sim \left(\frac{\omega}{\omega_0}\right)^4$ RAYLEIGH

FOR $\omega \gg \omega_0 \Rightarrow \sigma_{TOT} \sim \text{CONST} = \frac{8\pi}{3}r_0^2$ THOMPSON

RAMAN EFFECT

- INELASTIC SCATTERING (2ND ORDER PERT. THEORY)



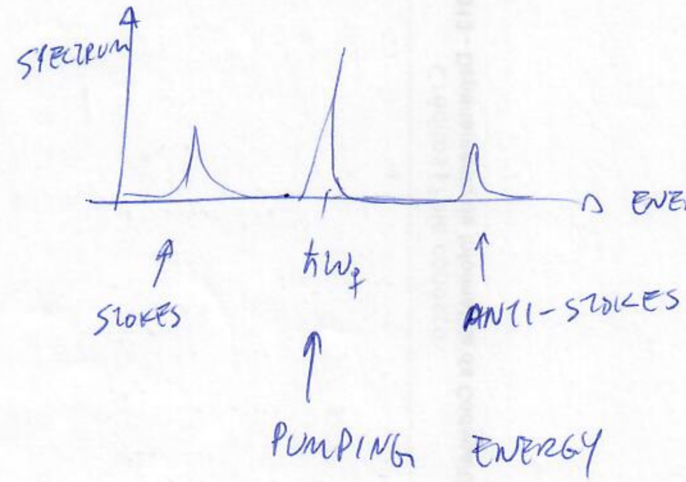


SCATTERED PHOTON IS LESS ENERGETIC

STOKE'S LINE

SCATTERED PHOTON IS MORE ENERGETIC

ANTI-STOKE'S LINE



$$\sigma = r_0^2 (n+1)\frac{\omega'}{\omega}\int d\Omega' \Big| \cancel{\delta_{fi}} - \frac{1}{m}\sum_l \left(\frac{\langle\ldots\rangle\langle\ldots\rangle}{E_l - E_i + \hbar\omega'} + \frac{\langle\ldots\rangle\langle\ldots\rangle}{E_l - E_i \mp \hbar\omega} \right)\Big|^2$$

- RAMAN EFFECT IS NOT FLUORESCENCE

IN FLUORESCENCE THE PHOTON IS COMPLETELY ABSORBED AND THE MATTER GOES TO A REAL EXCITED STATE. NOT A "VIRTUAL" ONE

(THE RESULT OF BOTH ARE THE SAME: A PHOTON OF DIFFERENT FREQUENCY IS EMITTED)

RESONANT SCATTERING

IN THE KRAMERS - HEISENBERG FORMULA, DIVERGENCIES APPEAR

WHEN $\begin{cases} E_j - E_i = \hbar\omega_f & (E_j > E_i) \rightarrow \text{USUALLY WHEN } |i\rangle \text{ IS G.S.} \\ E_i - E_j = \hbar\omega_{f'} & (E_i > E_j) \rightarrow \text{WHEN } |i\rangle \text{ IS EXCITED STA} \end{cases}$

THUS, WHEN RESONANCE APPEARS, $c^{(2)} \rightarrow \infty$.

OF COURSE, THIS CANNOT BE.

HOW DO WE SOLVE THIS PROBLEM?

- ONE POSSIBLE WAY IS TO INCORPORATE THE SPONTANEOUS DECAY OF THE TARGET STATE (IN THIS CASE, $|j\rangle$) SINCE ONLY THE GROUND STATE IS THE TRUE STABLE ONE.

- THE SPONTANEOUS DECAY IS MODELED BY SOME SORT OF DISSIPATION.

REVIEW IN CLASSICAL MECHANICS:

THOMPSON'S MODEL: $m\ddot{\vec{r}} = -m\omega_0^2\vec{r} + e\vec{E}_0 e^{i\omega t} - \underbrace{m\gamma\dot{\vec{r}}}_{\text{DISSIPATION}}$

$\Rightarrow \ddot{r} + \gamma\dot{r} + \omega_0^2 r = eE_0 e^{i\omega t}$

$\Rightarrow r(t) = \dfrac{-eE_0}{m} \dfrac{1}{\sqrt{(\omega_0^2-\omega^2)^2 + (\gamma\omega)^2}} \cos(\omega t + \phi)$ FOR $t \rightarrow \infty$

$\Rightarrow \dfrac{d\sigma}{d\Omega} = \dfrac{dP/d\Omega}{\text{INCIDENT FLUX}} = \omega^4 \Big(\underbrace{\dfrac{e^2E_0}{m} \times \dfrac{1}{\sqrt{\ }}}_{P_0}\Big)^2 \times \dfrac{1}{32\pi^2\epsilon_0 c^3} \times \sin^2\theta \times \left(\dfrac{2}{c\epsilon_0 E_0^2}\right)$

$$\Rightarrow \boxed{\sigma_{TOT} = \frac{8\pi r_0^2}{3} \frac{\omega^4}{(\omega_0^2-\omega^2)^2 + (\gamma\omega)^2}}$$

IN QUANTUM MECHANICS

1st ORDER: $C_{FI}^{(1)} = \frac{1}{i\hbar}\int_{t_0}^{t} dt' \, e^{i\omega_{FI}t'} \langle F | \cancel{H_{int}} | I \rangle$

$\Rightarrow \dot{C}^{(1)} = \frac{1}{i\hbar} \cancel{H}_{FI} e^{i\omega_{FI}t}$

MODIFY TO $\dot{C}_{FI}^{(1)} = \frac{1}{i\hbar} V_{FI} e^{i\omega_{FI}t} - \frac{1}{2}\frac{C_{FI}^{(1)}}{\tau_F}$

τ_F = LIFE TIME OF STATE $|F\rangle$

THE TERM $-\frac{1}{2\tau_F} C_{FI}^{(1)}$ CORRESPONDS TO THE EXPONENTIAL DECAY OF THE AMPLITUDE C_{FI} (SEE LIST OF EXERCISE No. 1)

$\dot{C}_{FI}^{(1)} = -\frac{1}{2}\frac{C_{FI}^{(1)}}{\tau_F} \rightarrow C_{FI}^{(1)} = C_{FI}^{(1)}(0)\, e^{-t/(2\tau_F)}$

SOLVING THE WHOLE EQUATION

$C_{FI\,\text{HOMOGENEOUS}} = A\, e^{-t/(2\tau_F)}$

$C_{FI\,\text{PARTICULAR}} = \cancel{\frac{V_{FI}}{i\hbar(i\omega_{FI})}} B\, e^{i\omega_{FI}t} \Rightarrow i\omega_{FI}B = \frac{V_{FI}}{i\hbar} - \frac{B}{2\tau_F} \Rightarrow B = \frac{V_{FI}}{i\hbar(i\omega_{FI}+\frac{1}{2\tau_F})}$

INITIAL CONDITION: $C_{FI}(0) = 0 \Rightarrow A = -\frac{V_{FI}}{i\hbar(i\omega_{FI}+\frac{1}{2\tau_F})} = \frac{V_{FI}}{\hbar\omega_{FI} - \frac{i\hbar}{2\tau_F}}$

$\Rightarrow C_{FI}^{(1)} = \frac{V_{FI}\left(e^{-\frac{t}{2\tau_F}} - e^{i\omega_{FI}t}\right)}{\hbar\omega_{FI} - \frac{i\hbar}{2\tau_F}}$

$= \frac{V_{FI}}{E_F - E_I - \frac{i\Gamma_F}{2}} \left(e^{-\frac{t}{2\tau_F}} - e^{i\omega_{FI}t}\right)$, $\Gamma_F = \frac{\hbar}{\tau_F}$

= LARGURA DE LINHA

IN THE DENOMINATOR $E_F - E_I \rightarrow E_F - E_I - \frac{i}{2}\Gamma_F$

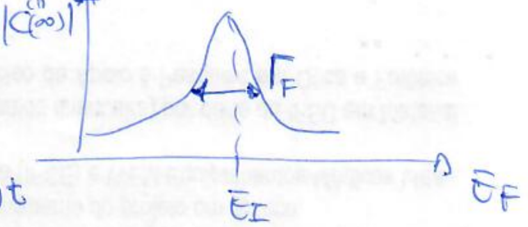
~~IT IS LIKE THE ENERGY HAD ACQUIRED AN IMAGINARY PART IN THE INFERIOR COMPLEX PLANE~~

IT IS LIKE THE ENERGY WAS SHIFTED DOWNWARD IN THE COMPLEX PLANE

$\Rightarrow$ THE NEGATIVE IMAGINARY PART OF THE ENERGY IS THE LINE WIDTH

FOR $t \gg \tau_F \Rightarrow C^{(1)} \rightarrow \dfrac{V_{FI}}{E_F - E_I - \frac{i}{2}\Gamma_F}\left(-e^{i\omega_{FI}t}\right)$

$$\Rightarrow \left|C^{(1)}(\infty)\right|^2 = \frac{|V_{FI}|^2}{(E_F - E_I)^2 + \Gamma_F^2/4}$$



*NOTE: IF $t \ll \tau_F \Rightarrow |C^{(1)}|^2 \sim |V_{FI}|^2 \frac{2\pi}{\hbar}\delta(E_F - E_I)\, t$

LET US RETURN TO THE SCATTERING PROBLEM

$$C^{(2)}_{FI} = \left(\frac{1}{i\hbar}\right)^2 \sum_L \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' e^{i\omega_{FL}t'} V_{FL}\, e^{i\omega_{LI}t''} V_{LI}$$

BUT $\left(\frac{1}{i\hbar}\right)\int_{t_0}^{t'} dt''\, e^{i\omega_{LI}t''} V_{LI} = C^{(1)}_{LI}(t')$

$$\Rightarrow C^{(2)}_{FI} = \left(\frac{1}{i\hbar}\right)\sum_L \int_{t_0}^{t} dt' e^{i\omega_{FL}t'} V_{FL}\, C_{LI}(t')$$

$$\Rightarrow C^{(2)}_{FI} = \frac{-1}{i\hbar}\sum_L \int_{t_0}^{t} dt'\, e^{i\omega_{FL}t'} \frac{\langle F|V|L\rangle\langle L|V|I\rangle}{E_L - E_I - \frac{i}{2}\Gamma_I}\left(e^{i\omega_{LI}t'} - e^{-t'/2\tau_F}\right)$$

FOR $t' \gg \tau_F$ THEN

$$\boxed{C_{FI}^{(2)} = \frac{-1}{i\hbar}\sum_{L}\int_{t_0}^{t} dt'\, e^{i\omega_{FI}t'}\,\frac{\langle F|V|L\rangle\langle L|V|I\rangle}{E_L - E_I - \frac{i}{2}\Gamma_I}}$$

FOR OUR CASE OF INTEREST, IN THE ABSORBTION OF A RESONANT PHOTON, THE KRAMERS–HEISENBERG FORMULA BECOMES

$$\boxed{\sigma = r_0^2\,(M_{fp'}+1)\,\frac{\omega_{p'}}{\omega_p}\int d\Omega'\,\left|\frac{1}{m}\,\frac{\langle f|\hat{e}_{p'}^{*}\cdot\vec{p}\,|\ell\rangle\langle \ell|\,\hat{e}_p\cdot\vec{p}\,|i\rangle}{E_\ell - E_i - \hbar\omega_p - \frac{i}{2}\Gamma_\ell}\right|^2}$$

THE SUM IS DOMINATED BY THE RESONANT ~~TARGET~~ INTERMEDIATE STATE $|\ell\rangle$

THE OTHER TERM (EMISSION) IS NOT RESONANT, SO IT IS DROPPED OUT.

~~ALTERNATIVE WAY:~~

~~ON PAGE 24, WE FOUND THAT~~ $\cancel{C_{FI}^{(2)} = \frac{-1}{i\hbar}\sum_L \frac{\langle F|V|L\rangle\langle L|V|I\rangle}{E_L - E_I - i\epsilon}\int_{t_0}^{t} e^{i\omega_{FI}t'}}$

~~IT IS JUST LIKE WE HAVE JUST DERIVED.~~

~~TO COMPLETE, LETS USE THAT~~

$$\cancel{\frac{1}{E_L - E_I - i\epsilon} = \mathcal{P}\frac{1}{(E_L - E_I)} + i\pi\,\delta(E_L - E_I)}$$

NOTICE THAT

$$\frac{d\sigma}{d\Omega} = r_0^2 \left(n_{\vec{k}'\beta'}+1\right) \frac{\omega_{f'}}{\omega_f} \frac{1}{m^2} \frac{\left|\langle f|\hat{e}_{\beta'}^{*}\cdot\vec{p}|\ell\rangle\right|^2 \left|\langle \ell|\hat{e}_{\beta}\cdot\vec{p}|i\rangle\right|^2}{(E_\ell - E_i - \hbar\omega_f)^2 + \Gamma_\ell^2/4}$$

$$= \left(\underbrace{\text{ABSORPTION PROBABILITY}}_{\text{FLUX DENSITY}}\right) \times \left(\text{EMISSION } \frac{\text{RATE}}{d\Omega}\right)$$

PAGE 35 PAGE 18

$$= \left(\frac{e^2\hbar(n_{\vec{k}\beta})}{2\omega_f\epsilon_0 m^2 V} * \frac{|\langle \ell|\hat{e}_\beta\cdot\vec{p}|i\rangle|^2}{(E_\ell - E_i - \hbar\omega_f)^2 + \frac{\Gamma_\ell^2}{4}}\right) \times \left(\frac{2\pi}{\hbar}\frac{e^2\hbar(n_{\vec{k}'\beta'}+1)|\langle f|\hat{e}_{\beta'}^{*}\cdot\vec{p}|\ell\rangle|^2}{2m^2\epsilon_0 V\omega_f} * \frac{V\omega_{f'}^2}{(2\pi)^3\hbar c^3}\right)$$

$$n_{\vec{k}\beta} * \frac{c}{V}$$

RECALL $r_0^2 = \left(\frac{e^2}{4\pi\epsilon_0 mc^2}\right)^2$

THIS IS THE EXPECTED RESULT SINCE OUR APPROACH WAS TO DEFLATE THE EMISSION PART FROM THE ABSORPTION

THE RESONANT LEVEL BY EMISSION