BOSONS ON A LATTICE

BOSE-HUBBARD MODEL:

$$H = -t \sum_{\langle i,j \rangle} \left(b_i^{+} b_j + h.c. \right) - \mu \sum_i n_i + \sum_i \epsilon_i n_i + \frac{1}{2} U \sum_i n_i (n_i - 1)$$

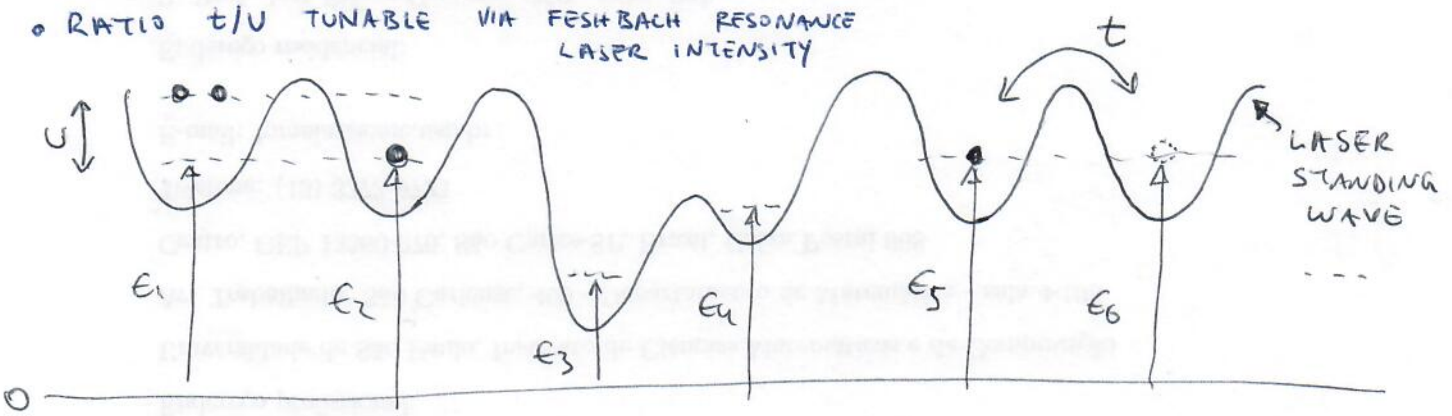
(HOPPING / KINETIC ENERGY) (CHEMICAL POTENTIAL CONTROLS THE DENSITY) (EXTERNAL/ OR RANDOM POTENTIAL) (INTERACTION $U > 0 \rightarrow$ REPULSIVE)

↓

EACH PAIR OF BOSONS ON SITE i PAYS INTERACTION ENERGY U

$n_i = b_i^{+} b_i \rightarrow$ LOCAL OCCUPATION NUMBER OPERATOR

- REALIZED IN OPTICAL LATTICES OF ULTRACOLD ATOMS
- RATIO t/U TUNABLE VIA FESHBACH RESONANCE, LASER INTENSITY



HERE, WE WILL CONSIDER THAT $\epsilon_i = \epsilon \Rightarrow -\mu + \epsilon \rightarrow -\mu$

- KINETIC x INTERACTIONS : LIMITING CASES

① $t = 0 \rightarrow$ NO KINETIC ENERGY (ATOMIC LIMIT) $\rightarrow$ LOCAL HAMILTONIAN: $H = \sum_i H_i$

$$H_i = -\mu n_i + \frac{1}{2} U n_i (n_i - 1)$$

EIGEN STATES : $|\psi\rangle = \bigotimes_i |\psi_i\rangle$

↓

PARTICLE NUMBER EIGENSTATES $\equiv |n_i\rangle$

$$\hat{n}_i |\psi\rangle = \hat{b}_i^{+} \hat{b}_i \bigotimes_i |n_i\rangle$$

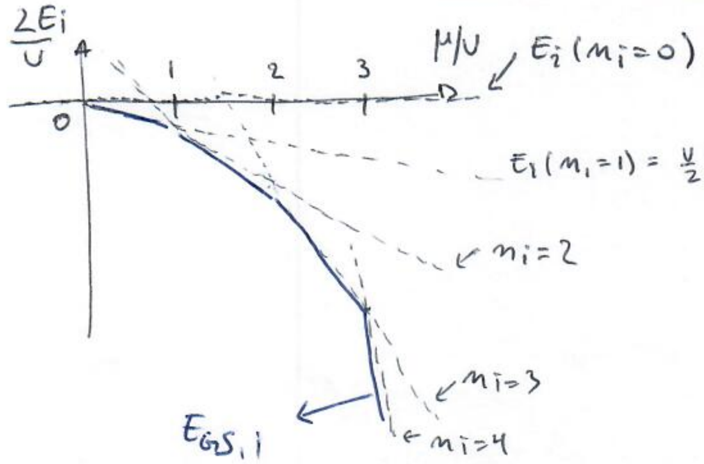
$$= \hat{b}_i^{+} \hat{b}_i |n_1, n_2, \ldots, n_i, \ldots\rangle$$

$$= n_i |\psi\rangle$$

GROUND STATE: $E = \sum_i E_i \,, \quad E_i = \frac{U}{2}\left(n_i^2 - \left(\frac{2\mu}{U} + 1 \right) n_i \right)$ ②

MINIMIZE E_i $\rightarrow$ FIND THE OPTIMAL $n_i \equiv n_{OPT}$

$$\frac{\partial E_i}{\partial n_i} = 0 \Rightarrow \boxed{n_{OPT} = \frac{1}{2} + \frac{\mu}{U}} \rightarrow \text{BUT } n_i \in \mathbb{N}$$



- $\frac{\mu}{U} < 0 \rightarrow n_{OPT} = 0$

$0 < \frac{\mu}{U} < 1 \rightarrow n_{OPT} = 1$

$1 < \frac{\mu}{U} < 2 \rightarrow n_{OPT} = 2$

$\vdots$

$$\boxed{n-1 < \frac{\mu}{U} < n \rightarrow n_{OPT} = n}$$

② $U=0 \rightarrow$ NO INTERACTIONS $\rightarrow H = -t \sum_{\langle i,j \rangle} (b_i^\dagger b_j + h.c.) - \mu \sum_i b_i^\dagger b_i$

FOURIER: $b_j = \frac{1}{\sqrt{N_{SITES}}} \sum_{\vec{k}} b_{\vec{k}}\, e^{i\vec{k}\cdot\vec{R}_j}$

$$\sum_{\langle i,j \rangle} b_i^\dagger b_j = \frac{1}{2} \sum_i \sum_{\{\vec{a}\}} b_i^\dagger b_j = \frac{1}{2} \sum_i \sum_{j \in (NN)_i} b_i^\dagger b_j = \frac{1}{2} \sum_i \sum_{j \in NN_i} \frac{1}{N_{SITES}} \sum_{\vec{k},\vec{q}} b_{\vec{k}}^\dagger b_{\vec{q}}\, e^{-i(\vec{k}\cdot\vec{R}_i - \vec{q}\cdot\vec{R}_j)}$$

ALL N_{SITES}

VECTORS CONNECTING NEAREST-NEIGHBORS

HERE, WE WILL ASSUME A BRAVAIS LATTICE

NOW, USE THAT $\vec{R}_j = \vec{R}_i + \vec{a}_j$

$$\sum_i \frac{e^{-i(\vec{k}-\vec{q})\cdot\vec{R}_i}}{N_{SITES}} = \delta_{\vec{k},\vec{q}}$$

$$-t \sum_{\langle i,j \rangle} (b_i^\dagger b_j + h.c.) = \sum_{\vec{k}} b_k^\dagger b_k \underbrace{\sum_{j \in NN_i} (-t) e^{i\vec{q}\cdot\vec{a}_j}}_{\epsilon_k} = \sum_{\vec{k}} \epsilon_k\, b_k^\dagger b_k$$

FOR A d-DIMENSIONAL HYPERCUBIC LATTICE, THEN $(\vec{k} = (k_1, k_2, \ldots, k_d))$

$$\frac{\epsilon_k}{-2t} = \cos k_1 a + \cos k_2 a + \cdots + \cos k_d a$$

GROUND STATE: ALL PARTICLES IN THE $\vec{k}=0$ STATE

$$|GS\rangle = \frac{(b^+_{k=0})^{N_{BOSONS}}}{\sqrt{N_{BOSONS}!}}|0\rangle = \frac{\left(\frac{1}{\sqrt{N_{SITES}}}\sum_i b^+_i\right)^{N_{BOSONS}}}{\sqrt{N_{BOSONS}!}}|0\rangle$$

$N_{BOSON} \gg 1$

$$\approx \prod_i e^{\sqrt{\rho}\, b^+_i}|0\rangle, \qquad \rho = \frac{N_{BOSONS}}{N_{SITES}} \equiv \text{DENSITY}$$

= APPROXIMATELY A PRODUCT OF ONSITE COHERENT STATES

$$\Rightarrow b_i|GS\rangle \approx \sqrt{\rho}\,|GS\rangle \Rightarrow \boxed{\langle GS|b_i|GS\rangle = \sqrt{\rho}}$$

THIS IS THE SIGNATURE OF A SUPERFLUID STATE

ALTERNATIVE DERIVATION:

$$b_i b^+_{k=0} = \frac{1}{\sqrt{N_S}} b_i\left(b^+_1 + b^+_2 + \cdots + b^+_i + \cdots\right) = \frac{1}{\sqrt{N_S}}\left(b^+_1 b_i + b^+_2 b_i + \cdots + \underbrace{b_i b^+_i}_{1 + b^+_i b_i} + \cdots\right)$$

$$= \frac{1}{\sqrt{N_S}} + b^+_{k=0} b_i$$

$$b_i b^{+2}_{k=0} = \left(\frac{1}{\sqrt{N}} + b^+_{k=0} b_i\right) b^+_{k=0} = \frac{b^+_{k=0}}{\sqrt{N_S}} + b^+_{k=0}\left(\frac{1}{\sqrt{N_S}} + b^+_{k=0} b_i\right)$$

$$= \frac{2}{\sqrt{N_S}} b^+_{k=0} + b^{+2}_{k=0} b_i$$

$\vdots$

$$b_i b^{+N_B}_{k=0} = \frac{N_B}{\sqrt{N_S}}\left(b^+_{k=0}\right)^{N_B-1} + \left(b^+_{k=0}\right)^{N_B} b_i$$

$$\Rightarrow b_i|GS\rangle = \frac{b_i\left(b^+_{k=0}\right)^{N_B}}{\sqrt{N_B!}}|0\rangle = \left[\sqrt{\frac{N_B}{N_S}}\frac{\left(b^+_{k=0}\right)^{N_B-1}}{\sqrt{(N_B-1)!}} + \frac{\left(b^+_{k=0}\right)^{N_B}}{\sqrt{N_B}} b_i\right]|0\rangle$$

$$\Rightarrow \quad b_i |GS\rangle = \sqrt{\rho}\, \frac{(b^+_{k=0})^{N_B-1}}{\sqrt{(N_B-1)!}} |0\rangle \qquad (4)$$

$$\simeq \sqrt{\rho}\, |GS\rangle \quad \text{FOR} \quad N_B \gg 1$$

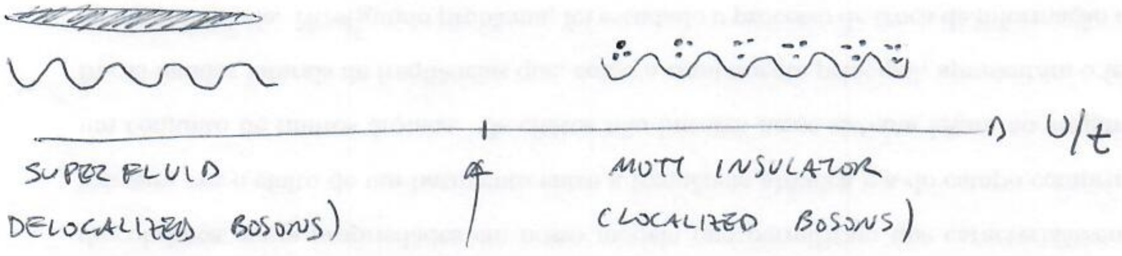
PARTICLE NUMBER: $\langle n_i \rangle = \langle GS| b_i^+ b_i |GS\rangle = \rho$

$$\langle n_i^2 \rangle = \rho \langle GS| b_i b_i^+ |GS\rangle = \rho(1+\rho)$$

$$\Rightarrow \Delta n_i^2 = \langle n_i^2 \rangle - \langle n_i \rangle^2 = \rho = \langle n_i \rangle$$

IN CONTRAST, THE $t=0$ (ATOMIC LIMIT) CASE $\Delta n_i^2 = 0$ AND $\langle b_i \rangle = 0$

SUMMARY SO FAR:



SEE EXPERIMENTS IN GREINER ET AL., NATURE 415, 39 (2002)

MEAN-FIELD TREATMENT BY METZNER + VOLLHART PRB 39, 4462 (1989)

$$H^{(MF)} = \sum_i h_i = \sum_i -\mu \hat{n}_i + \frac{1}{2} U \hat{n}_i(\hat{n}_i+1) - t z (\psi^* \hat{b}_i + \psi \hat{b}_i^+) + t \frac{z}{2} \psi^* \psi$$

THIS APPROXIMATION FAVORS THE ATOMIC LIMIT

• HERE $\hat{b}_i = \psi + \delta \hat{b}_i \Rightarrow \sum_{\langle i,j\rangle} \hat{b}_i^+ \hat{b}_j + h.c. = \sum_{\langle i,j\rangle} \psi^*\psi + \psi^* \delta\hat{b}_j + \psi \delta\hat{b}_i^+ + \underbrace{\delta\hat{b}_i^+ \delta\hat{b}_j}_{\text{NEGLECT}} + h.c.$

z = COORDINATION NUMBER

$$= \sum_{\langle i,j\rangle} \psi^*\psi + \psi^*(\hat{b}_j - \psi) + \psi(\psi^* - \hat{b}_i^+) + h.c.$$

$$= \frac{z}{2}\left(\sum_i \psi^* \hat{b}_i + \hat{b}_i^+ \psi - \psi^*\psi\right) + h.c.$$

NOTICE $H^{(MF)}$ IS ALMOST DIAGONAL IN THE PARTICLE BASIS

$$h_i = \underbrace{-\mu \hat{n}_i + \frac{1}{2} U \hat{n}_i(\hat{n}_i - 1) + \frac{tz}{2}|\psi|^2}_{\text{MOTT TERM}} - \underbrace{\frac{tz}{2}\left(\psi^* \hat{b}_i + \psi \hat{b}_i^+\right)}_{\substack{\text{SUPERFLUID TERM} \\ \text{(DOES NOT CONSERVE} \\ \text{PARTICLE NUMBER)}}}$$

START ANALYSIS FROM THE MOTT INSULATOR PHASE $\Rightarrow t \ll 1$ (PERTURBATIVE PARAMETER)

ZERO ORDER: $E_n^{(0)} = -\mu n + \frac{1}{2} U n(n-1)$

1st ORDER: $E_n^{(1)} = \langle n | \frac{tz}{2}|\psi|^2 - \frac{tz}{2}(\psi^*\hat{b} + \psi\hat{b}^+) | n \rangle = \frac{tz}{2}|\psi|^2$

LANDAU IDEA: ψ IS SUCH THAT MINIMIZES E

$\Rightarrow |\psi|^2 = 0 \Rightarrow \psi = 0 \Rightarrow$ ALWAYS MOTT INSULATOR

$\Rightarrow$ NEED FURTHER ORDERS IN PERT. THEORY

2ND ORDER:
$$E_n^{(2)} = \sum_{n' \neq n} \frac{\langle n | \frac{tz}{2}(\psi^*\hat{b} + \psi\hat{b}^+) | n' \rangle \langle n' | \frac{tz}{2}(\psi^*\hat{b} + \psi\hat{b}^+) | n \rangle}{E_n^{(0)} - E_{n'}^{(0)}}$$

$$= \left(\frac{tz}{2}\right)^2 \left[\sum_{n \neq n'} \frac{\langle n | \psi^* \hat{b} | n' \rangle \langle n' | \psi \hat{b}^+ | n \rangle}{E_n^{(0)} - E_{n'}^{(0)}} + \frac{\langle n | \psi \hat{b}^+ | n' \rangle \langle n' | \psi \hat{b} | n \rangle}{E_n^{(0)} - E_{n'}^{(0)}} \right]$$

$$= \left(\frac{tz}{2}\right)^2 \left\{ \frac{(n+1)|\psi|^2}{E_n^{(0)} - E_{n+1}^{(0)}} + \frac{n|\psi|^2}{E_n^{(0)} - E_{n-1}^{(0)}} \right\}$$

$$= \left(\frac{tz}{2}\right)^2 |\psi|^2 \left\{ \frac{n+1}{\mu - Un} + \frac{n}{-\mu + U(n-1)} \right\}$$

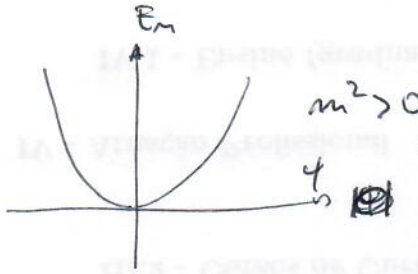
$$= \left(\frac{tz}{2}\right)^2 |\psi|^2 \frac{+\mu + U}{U^2\left(n - \frac{\mu}{U}\right)\left(n - 1 - \frac{\mu}{U}\right)}$$

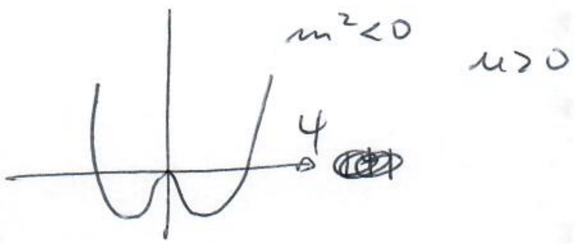
$$\Rightarrow E_n = E_n^{(0)} + m^2 |\psi|^2 + u |\psi|^4$$

(JUST AS IN A ϕ^4-THEORY)

$u |\psi|^4$ → APPEARS IN 4th ORDER OF PERT. THEORY

$$m^2 = \left(\frac{tz}{U}\right) + \left(\frac{tz}{U}\right)^2 \frac{1}{U} \frac{1+\mu/U}{\left(n-\frac{\mu}{U}\right)\left(n-1-\frac{\mu}{U}\right)}$$



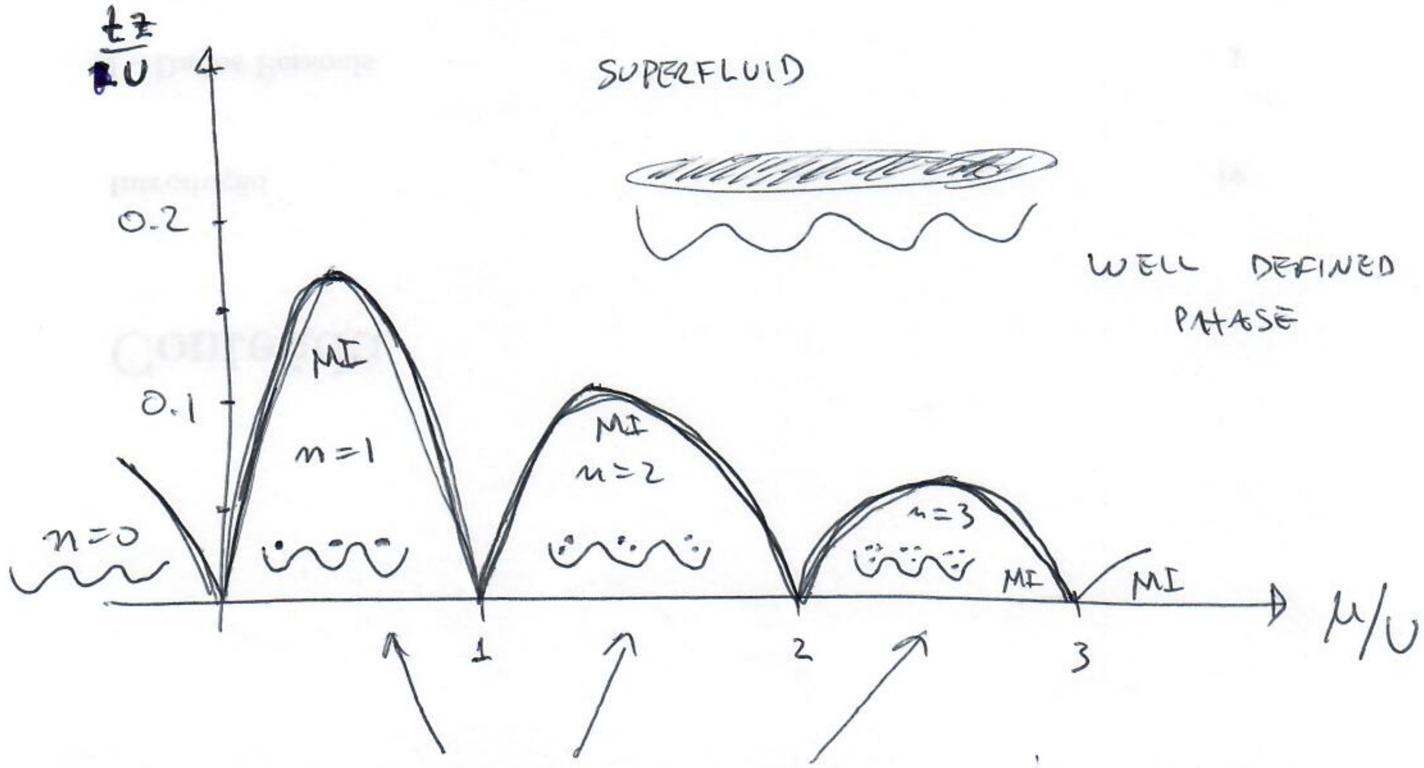


TRANSITION AT $m^2=0$

$$\Rightarrow 0 = 1 + \left(\frac{tz}{2U}\right)_c \left(\frac{1+\mu/U}{\left(n-\frac{\mu}{U}\right)\left(n-\frac{\mu}{U}-1\right)}\right)$$

$$\Rightarrow \boxed{\left(\frac{tz}{2U}\right)_c = \frac{\left(n-\frac{\mu}{U}\right)\left(1+\frac{\mu}{U}-n\right)}{1+\mu/U}}$$

MEAN-FIELD TRANSITION LINE



MOTT LOBES → WELL DEFINED PARTICLE NUMBER

RECALL THAT WE CAN CHANGE ~~THE~~ OPERATORS TO

$$\begin{cases} \hat{b}_i = \sqrt{\hat{N}_i}\, e^{+i\hat{\phi}_i} \\ \hat{b}_i^+ = e^{-i\hat{\phi}_i}\sqrt{\hat{N}_i} \end{cases}$$

FROM $[\hat{b}, \hat{b}^+] = 1 \Rightarrow \hat{N} - e^{-i\hat{\phi}}\hat{N}e^{+i\hat{\phi}} = 1$

$$\Rightarrow [e^{+i\hat{\phi}}, \hat{N}] = e^{+i\hat{\phi}}$$

$$\Rightarrow [+i\hat{\phi}, \hat{N}] = 1 \quad \Rightarrow [\hat{\phi}, \hat{N}] = -i$$

$\hat{\phi}$ AND $\hat{N}$ ARE CONJUGATE VARIABLES

MOTT PHASE → EIGENSTATES ARE PARTICLE NUMBER STATES

PROOF: $\hat{N} = \sum_i \hat{n}_i$, KINETIC PERTURBATION → $\hat{H}_{KIN} = -t \sum_{(i,j)} \hat{b}_i^+ \hat{b}_j$

$$\Rightarrow [\hat{N}, \hat{H}_{KIN}] = -t\left[\sum_i b_i^+ b_i , \sum_{(j,l)} b_j^+ b_l\right] = 0$$

- SWITCHING ON H_{KIN} ADIABATICALLY, THE GS REMAIN EIGENSTATE OF NUMBER OPERATOR

* ACTUALLY, THIS IS "ALWAYS" TRUE BECAUSE EACH TERM OF $\hat{H}$ CONSERVES N.
THIS IS NOT TRUE ONLY WHEN THERE IS A PHASE TRANSITION (WHICH HAPPENS ONLY IN THE THERMODYNAMIC LIMIT)

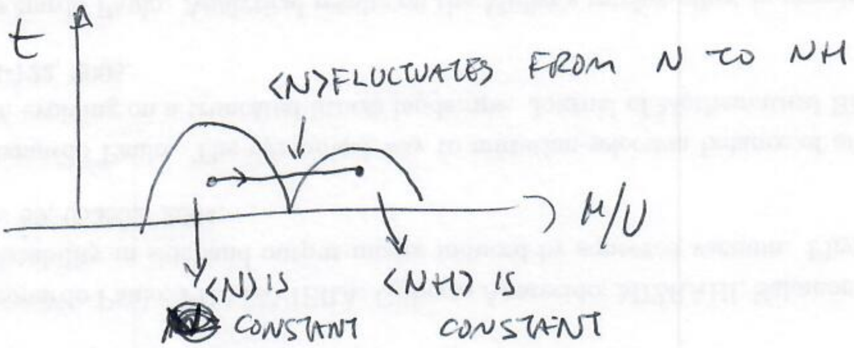
$\Rightarrow \langle \hat{N} \rangle$ IS CONSTANT IN EACH MOTT LOBE

$\frac{\partial}{\partial \mu}\langle \hat{N} \rangle = 0 \rightarrow$ INCOMPRESSIBLE STATE

THUS, INSTEAD OF ALLOWING FOR ~~TOTAL~~ PARTICLE FLUCTUATIONS WHEN GOING FROM $n-1 < \frac{\mu}{U} < n$ TO

$n < \frac{\mu}{U} < n+1$

THE SYSTEM CHOOSES TO BE INCOMPRESSIBLE INSIDE THE MOTT LOBES, THEN UNDERGOES TO A PHASE TRANSITION INTO THE SUPERFLUID PHASE, WHERE THE PARTICLE NUMBER FLUCTUATES, AND ONLY THEN COMES BACK TO THE MOTT INSULATOR WITH ONE MORE PARTICLE PER SITE



HOW ABOUT THE EXCITATION SPECTRUM?

HARD ON HAMILTONIAN FRAMEWORK

EASY ON THE PATH-INTEGRAL FRAMEWORK

$$Z = \mathrm{tr}\, e^{-\beta(H-\mu N)} = \int \mathcal{D}[b,\bar b]\, e^{-S_{BH}[b,\bar b]}$$

$$S_{BH} = S_{LOCAL} + S_{KIN}$$

$$S_{LOCAL} = \int_0^\beta d\tau \sum_i \left\{ \bar b_i (\partial_\tau - \mu) b_i + \frac{1}{2} U \bar b_i^2 b_i^2 \right\}$$

↓ THIS IS BECAUSE $\hat n_i(\hat n_i - 1) = b_i^+ b_i^+ b_i b_i$ WHICH IS NORMAL ORDERED

$$S_{KIN} = \int_0^\beta d\tau \left(-t \sum_{(i,j)} \bar b_i b_j \right) = - \vec{\bar b}\, T\, \vec b$$

$+ \bar b_j b_i$

HOW DO WE PROCEED? → **HUBBARD–STRATONOVICH TRANSFORMATION** 9

USE THE IDENTITY $\int D[\psi,\bar\psi]\, e^{-(\vec{\bar\psi}-\vec{\bar b})\,\mathbb{T}\,(\vec\psi-\vec b)} = (\det \mathbb{T})^{-1}$

MAKING $D[\psi,\bar\psi] \to D[\psi,\bar\psi](\det \mathbb{T})^{-1}$

$$\Rightarrow \int D[\psi,\bar\psi]\, e^{-(\vec{\bar\psi}-\vec{\bar b})\,\mathbb{T}\,(\vec\psi-\vec b)} = 1$$

$$\Rightarrow Z = \int D[b,\bar b]\, D[\psi,\bar\psi]\, e^{-S_{loc} + \vec{\bar b}\,\mathbb{T}\,\vec b - (\vec{\bar\psi}-\vec{\bar b})\,\mathbb{T}\,(\vec\psi-\vec b)}$$

$$= \int D[b,\bar b]\, D[\psi,\bar\psi]\, e^{-S_{loc} - \int_0^\beta d\tau \left(t \sum_{\langle i,j\rangle} \bar\psi_i \psi_j + \bar\psi_j\psi_i\right) + \int_0^\beta d\tau \left[t \sum_{\langle i,j\rangle} \left(\bar\psi_i b_j + \bar b_i \psi_j\right)\right]}$$

NOW, WE INTEGRATE OVER THE BOSONS $b, \bar b$

$$\Rightarrow Z = \int D[\psi,\bar\psi]\, e^{-\int_0^\beta d\tau \sum_{i,j} t_{ij}\bar\psi_i\psi_j} * \underbrace{\int D[b,\bar b]\, e^{-S_{loc}}}_{Z_{LOC}} \underbrace{\left(\frac{\int D[b,\bar b]\, e^{-S_{loc} + \int_0^\beta d\tau \cdots}}{\int D[b,\bar b]\, e^{-S_{loc}}}\right)}_{\left\langle e^{\int_0^\beta d\tau \sum_{i,j} t_{ij}(\bar\psi_i b_j + \bar b_i \psi_j)}\right\rangle_{LOC}}$$

USE S_{LOC} AS REFERENCE
⇒ GOOD DESCRIPTION OF THE MOTT PHASE

$$\Rightarrow Z = Z_{LOC} * \int D[\psi,\bar\psi]\, e^{-S_{eff}[\psi,\bar\psi]}$$

$$S_{eff} = \int_0^\beta d\tau \left(\sum_{i,j} t_{i,j}\bar\psi_i\psi_j\right) - \ln \left\langle e^{\int_0^\beta d\tau \sum_{i,j} t_{ij}(\bar\psi_i b_j + \bar b_i \psi_j)} \right\rangle_{LOC}$$

NOW WE EXPAND THE LOG (CUMULANT EXPANSION)

$$S_{eff} = \int_0^\beta d\tau \sum_{i,j} t_{i,j}\bar\psi_i\psi_j - \left(\langle x\rangle_{loc} + \frac{1}{2}\left(\langle x^2\rangle_{loc} - \langle x\rangle^2_{loc}\right) + \cdots\right)$$

WHERE $x = \int_0^\beta d\tau \sum_{i,j} t_{ij}(\bar\psi_i b_j + \bar b_i \psi_j)$

SINCE S_{loc} IS QUADRATIC ON $b, \bar b$ THEN $\langle b_j\rangle = \langle \bar b_j\rangle = 0$.

THEN, IN SECOND ORDER, WE ONLY HAVE $\langle x^2\rangle_{loc}$

$$\langle x^2\rangle_{LOC} = \left\langle \iint_0^\beta d\tau\, d\tau' \sum_{ij}\sum_{i'j'} t_{i,j}\, t_{i',j'} \left(\bar{\psi}_i b_j + \bar{b}_i \psi_j\right)\left(\bar{\psi}_{j'} b_{j'} + \bar{b}_{i'} \psi_{j'}\right) \right\rangle_{LOC}$$

$$= \iint_0^\beta d\tau\, d\tau' \sum_{ij}\sum_{i'j'} t_{ij}\, t_{i'j'} \Big[\bar{\psi}_i(\tau)\psi_{j'}(\tau') \langle T\, b_j(\tau)\, \bar{b}_{i'}(\tau')\rangle_{LOC}$$

$$+ \psi_j(\tau)\bar{\psi}_{i'}(\tau') \langle T\, \bar{b}_i(\tau)\, b_{j'}(\tau')\rangle_{LOC} \Big]$$

WHERE $T(\cdots)$ IS THE TIME ORDERING OPERATOR

NOW,
OUR TASK IS COMPUTING THE GREEN'S FUNCTION

$$\langle T\, b_j(\tau)\, \bar{b}_{i'}(\tau')\rangle_{LOC} \quad \text{AND} \quad \langle T\, \bar{b}_i(\tau)\, b_{j'}(\tau')\rangle_{LOC}$$

SINCE S_{LOC} IS A 1-PARTICLE ACTION, THEY CAN BE SOLVED EXACTLY.

BEFORE COMPUTING THEM, LETS FURTHER SIMPLIFY USING THE FACT THAT $\langle b_\alpha \bar{b}_\beta\rangle \propto \delta_{\alpha\beta}$

- IN THE 1st SUM, MAKE $\boxed{\begin{array}{l} i \to i \\ j \to j \\ i' \to j \\ j' \to k \end{array}}$, IN THE 2ND $\boxed{\begin{array}{l} i' \to i \\ j \to k \\ i \to j \\ j' \to j \end{array}}$

THEN

$$\langle x^2\rangle_{LOC} = \iint_0^\beta d\tau\, d\tau' \sum_{i,j,k} t_{ij}\, t_{jk} \Big\{ \bar{\psi}_i(\tau)\psi_k(\tau') \langle T\, b_j(\tau)\, \bar{b}_j(\tau')\rangle_{LOC}$$

$$+ \psi_k(\tau)\bar{\psi}_i(\tau') \langle T\, \bar{b}_j(\tau)\, b_j(\tau')\rangle_{LOC} \Big\}$$

EXCHANGING $\tau \leftrightarrow \tau'$ IN THE 2ND SUM

$$\langle x^2\rangle_{LOC} = \iint_0^\beta d\tau\, d\tau' \sum_{ijk} t_{ij}\, t_{jk}\, \bar{\psi}_i(\tau)\psi_k(\tau')\; 2\langle T\, b_j(\tau)\, \bar{b}_j(\tau')\rangle_{LOC}$$

- RECALL THAT $T\big(A(\tau)B(\tau')\big) = \begin{cases} A(\tau)B(\tau'), & \tau > \tau' \\ B(\tau')A(\tau), & \tau < \tau' \end{cases}$

FOR BOSONIC VARIABLES

WE CONTINUE FROM NOW ON FOCUSING ON THE T=0 CASE ($\beta \to \infty$).

FOR $\tau > \tau'$, $\langle b_j(\tau) \bar{b}_j(\tau') \rangle_{loc} \overset{T=0}{=} \langle GS | b_j(\tau) b_j^+(\tau) | GS \rangle$

$|GS\rangle = |n\rangle$, WHERE n MINIMIZES

$$E_n = -\mu n + \frac{1}{2} U n(n-1)$$

(SEE PAGE 2)

$$\Rightarrow \langle n | e^{\tau H_{loc}} b_j e^{(-\tau+\tau')H_{loc}} b_j^+ e^{-\tau' H_{loc}} | n \rangle$$

$$= (n+1)\, e^{(E_{n+1} - E_n)(\tau' - \tau)}$$

FOR $\tau < \tau'$, $\langle \bar{b}_j(\tau') b_j(\tau) \rangle_{loc} = n\, e^{(E_{n-1} - E_n)(\tau - \tau')}$

THUS,

$$\langle x^2 \rangle_{loc} = 2 \sum_{ijk} t_{ij} t_{jk} \int_0^\beta d\tau\, \bar{\psi}_i(\tau) \left\{ \int_0^\tau d\tau'\, \psi_k(\tau')\, (n+1)\, e^{(E_{n+1} - E_n)(\tau' - \tau)} + \int_\tau^\beta d\tau'\, \psi_k(\tau')\, n\, e^{(E_{n-1} - E_n)(\tau - \tau')} \right\}$$

FOURIER $\to$ $\psi(\tau) = \frac{1}{\sqrt{\beta}} \sum_m \psi_m e^{-i\omega_m \tau}$; $\psi_m = \frac{1}{\sqrt{\beta}} \int_0^\beta d\tau\, \psi(\tau) e^{i\omega_m \tau}$

$$\to \sum_{m_1, m_2} \frac{\bar{\psi}_{i,m_1} \psi_{k,m_2}}{\beta} \underbrace{\int_0^\beta d\tau\, e^{(i\omega_1 - E_{n+1} + E_n)\tau} \int_0^\tau d\tau'\, e^{(-i\omega_2 + E_{n+1} - E_n)\tau'}}_{\frac{e^{\beta(E_n - E_{n+1})} - 1}{(E_n - E_{n+1} + i\omega_1)(E_n - E_{n+1} + i\omega_2)} + \frac{\beta \delta_{\omega_1, \omega_2}}{E_{n+1} - E_n - i\omega_2}}$$

$$\to \sum_{m_1 m_2} \frac{\bar{\psi}_{i,m_1} \psi_{k,m_2}}{\beta} \underbrace{\int_0^\beta d\tau\, e^{(i\omega_1 + E_{n-1} - E_n)\tau} \int_\tau^\beta d\tau'\, e^{(-i\omega_2 - E_{n-1} + E_n)\tau'}}_{\frac{1 - e^{\beta(E_n - E_{n-1})}}{(E_n - E_{n-1} - i\omega_1)(-E_n + E_{n-1} + i\omega_2)} + \frac{\beta \delta_{\omega_1, \omega_2}}{E_{n-1} - E_n + i\omega_2}}$$

SINCE $|GS\rangle = |m\rangle \Rightarrow e^{\beta(E_n - E_{m\pm1})} \xrightarrow{T=0} 0$, (12)

ALSO, AS $\beta \to \infty$, ONLY THE 2ND TERM SURVIVES IN EACH INTEGRAL.

FINALLY,

$$\langle x^2 \rangle_{Loc} = 2 \sum_{i,j,k} t_{ij} t_{jk} \sum_m \psi_{i,m} \psi_{k,m} \left\{ \underbrace{\frac{m+1}{E_{m+1} - E_m - i\omega_m}}_{Um - \mu - i\omega_m} + \underbrace{\frac{m}{E_{m-1} - E_m + i\omega_m}}_{\mu - U(m-1) + i\omega_m} \right\}$$

THUS

$$S_{EFF} \simeq \sum_m \left\{ \sum_{i,j} t_{ij} \psi_{i,m} \psi_{j,m} - \sum_{i,j,k} t_{ij} t_{jk} \psi_{i,m} \psi_{k,m} \left\{ \frac{m+1}{\cdots} + \frac{m}{\cdots} \right\} \right\}$$

NOW, WE FOURIER TRANSFORM IN SPACE

$$\psi_{j,m} = \frac{1}{\sqrt{N}} \sum_{\vec{k}} e^{-i\vec{k}\cdot\vec{r}_j} \psi_{\vec{k},m}$$

$$(+) t_{j,l} = \frac{1}{N} \sum_{\vec{k}} e^{-i\vec{k}\cdot(\vec{r}_j - \vec{r}_l)} \epsilon_{\vec{k}}$$

NOTICE A SIGN DIFFERENCE FROM THE DEFINITION OF PAGE 2

$$\Rightarrow \sum_{ij} t_{ij} \bar{\psi}_{im} \psi_{jm} = \frac{+1}{N^2} \sum_{\vec{k}_1 \vec{k}_2 \vec{q}} \bar{\psi}_{\vec{k}_1 m} \psi_{\vec{k}_2 m} \epsilon_{\vec{q}} \underbrace{\sum_{i,j} e^{i\vec{k}_1\cdot\vec{r}_i - i\vec{q}\cdot(\vec{r}_i - \vec{r}_j) - i\vec{k}_2\cdot\vec{r}_j}}_{N^2 \delta_{\vec{k}_1,\vec{q}} \delta_{\vec{k}_2,\vec{q}}}$$

$$= +\sum_{\vec{q}} \epsilon_{\vec{q}} \bar{\psi}_{\vec{q},m} \psi_{\vec{q},m}$$

$$\Rightarrow \sum_{i,j,l} t_{ij} t_{jl} \bar{\psi}_{i,m} \psi_{l,m} = \frac{1}{N^3} \sum_{\substack{\vec{k}_1 \vec{k}_2 \\ \vec{k}_3 \vec{q}}} \epsilon_{\vec{k}_1} \epsilon_{\vec{k}_2} \bar{\psi}_{\vec{k}_3 m} \psi_{\vec{q} m} \underbrace{\sum_{i,j,l} e^{i\vec{r}_i\cdot(\vec{k}_3 - \vec{k}_1)} e^{i\vec{r}_j\cdot(\vec{k}_1 - \vec{k}_2)} e^{i\vec{r}_l\cdot(\vec{k}_2 - \vec{q})}}_{N^3 \delta_{\vec{k}_1,\vec{k}_2} \delta_{\vec{k}_1,\vec{k}_3} \delta_{\vec{k}_2,\vec{q}}}$$

$$= \sum_{\vec{q}} \epsilon_{\vec{q}}^2 \bar{\psi}_{\vec{q},m} \psi_{\vec{q},m}$$

FINALLY, (13)

$$\boxed{S_{EFF} = \sum_{\vec{q}, m} \bar{\psi}_{\vec{q}, m}\, \mathcal{G}^{-1}(\vec{q}, i\omega_m)\, \psi_{\vec{q}, m}}$$

WHERE

$$\mathcal{G}^{-1}(\vec{q}, i\omega) \simeq \epsilon_{\vec{q}} - \epsilon_{\vec{q}}^2 \left\{ \frac{n+1}{Un - \mu - i\omega_m} + \frac{n}{\mu - U(n-1) + i\omega_m} \right\}$$

FOR HYPERCUBIC LATTICE, $\epsilon_{\vec{q}} = +2t \sum_{j=1}^{d} \cos(q_j a)$ → NOTICE DIFFERENT SIGN FROM PAGE 2

COMPARE WITH THE ϕ^4 THEORY (EVIDENTLY, THE $\bar{\psi}^2 \psi^2$ TERM APPEARS ONLY IN 4th ORDER IN CUMULANT EXPANSION)

NONETHELESS, $\mathcal{G}^{-1} \simeq +m^2 + q^2 + (i\omega_m)^2$

↑ FOR THE ϕ^4 THEORY

THE PHASE TRANSITION HAPPENS WHEN $m = 0$

$$\Rightarrow \mathcal{G}^{-1}(\vec{q} = 0, \omega_m = 0) = 0$$

$$\Rightarrow |\epsilon_0| \left(\frac{n+1}{Un - \mu} + \frac{n}{\mu - U(n-1)} \right) = +1 \quad , \text{ AS } \epsilon_0 = +tz$$

$$\boxed{\left(\frac{tz}{U} \right)_c = \frac{(n - \mu/U)(-n + 1 + \mu/U)}{1 + \mu/U}}$$

COMPARE WITH PAGE 6

NOW WE GO FURTHER AND OBTAIN THE SPECTRUM FROM $\mathcal{G}^{-1}(\vec{q}, \omega_m \to -i\omega) = 0$

RECALL THE ANALYTIC CONTINUATION

$$\Rightarrow \quad 1 = |\epsilon_{\vec{q}}| \left(\frac{-(m+1)}{\omega + \mu - Um} + \frac{m}{\omega + \mu - U(m-1)} \right)$$

$$\Rightarrow -1 = |\epsilon_{\vec{q}}| \left(\frac{\omega + \mu + U}{(\omega+\mu)^2 - U(2m-1)(\omega+\mu) + U^2 m(m-1)} \right)$$

$$(\omega+\mu)^2 - \left(U(2m-1) - |\epsilon_{\vec{q}}| \right)(\omega+\mu) + U\left(Um(m-1) + |\epsilon_{\vec{q}}| \right) = 0$$

$$\boxed{\omega_\pm = -\mu + \frac{U}{2}(2m-1) - \frac{|\epsilon_{\vec{q}}|}{2} \pm \frac{1}{2}\sqrt{\epsilon_{\vec{q}}^2 - 2U(2m+1)|\epsilon_{\vec{q}}| + U^2}}$$

$\omega^+ \rightarrow$ PARTICLE DISPERSION > 0

$\omega^- \rightarrow$ HOLE " < 0

} EXCITATION SPECTRUM (NOT THE TRUE SPECTRUM (EIGENENERGY))

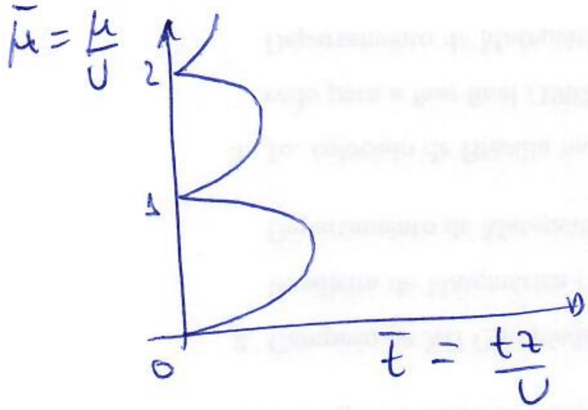
TRUE SPECTRUM IS SIMPLY $\omega_{\vec{q}}^{\pm} = \pm \omega^{\pm} \rightarrow$ 2 BRANCHES

BOTH ARE GAPPED, $\omega_{\vec{q}}^{\pm} \propto U \rightarrow$ MOTT INSULATOR

$$\boxed{\omega_{\vec{q}}^{\pm} = \pm\left(-\mu + \frac{U}{2}(2m-1) - \frac{\epsilon_{\vec{q}}}{2} \right) + \frac{1}{2}\sqrt{\epsilon_{\vec{q}}^2 - 2U(2m+1)\epsilon_{\vec{q}} + U^2}}$$

EXCITATION AT THE TRANSITION BORDER

RECALL THAT $\left(\frac{t z}{U}\right)_c^* \left(\frac{n+1}{n - \frac{\mu_c}{U}} + \frac{n}{\frac{\mu_c}{U} - (n-1)} \right) = 1$



$\bar{\mu} = \frac{\mu}{U}$

$\bar{t} = \frac{tz}{U}$

$$\Rightarrow \boxed{\bar{t}\left(\frac{n+1}{n-\bar{\mu}_c} + \frac{n}{\bar{\mu}_c - n + 1} \right) = 1}$$

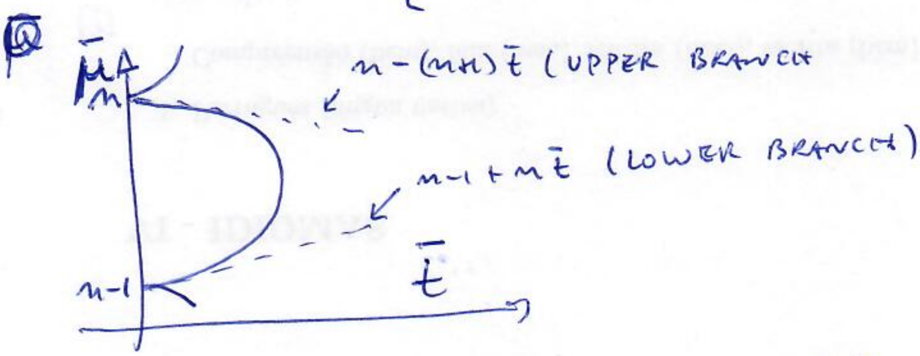
LETS START WITH $\bar{t} \ll 1$

$\Rightarrow \bar{\mu}_c \approx$ INTEGER

FOR A BETTER APPROXIMATION $\Rightarrow$ EITHER $\bar{t}\,\frac{n+1}{n-\bar{\mu}_c} \simeq 1$

OR $\bar{t}\,\frac{n}{\bar{\mu}_c - n + 1} \approx 1$

THUS $\bar{\mu}_c \approx \begin{cases} n - (n+1)\bar{t} & \rightarrow \text{UPPER} \\ n - 1 + n\bar{t} & \rightarrow \text{LOWER BRANCH} \end{cases}$



$$W_q^{\pm} = U \left\{ \pm\left(-\bar{\mu}_c + n - \frac{1}{2} - \frac{1}{2}\bar{\epsilon}_q \right) + \frac{1}{2}\sqrt{1 - 2(2n+1)\bar{\epsilon}_q + \bar{\epsilon}_q^2} \right\}$$

WHERE $\bar{\epsilon}_q = \frac{\epsilon_q}{U} = \frac{2\bar{t}}{z} \sum_{j=1}^{d} \cos(q_j a)$

FOR SMALL $q \Rightarrow \bar{\epsilon}_q \approx \frac{2\bar{t}}{z}\left(\frac{z}{2} - \frac{1}{2} q^2 a^2 \right)$

$$= \bar{t}\,(1 - \bar{q}^2), \quad \bar{q}^2 = \frac{q^2 a^2}{z}$$

FOR THE UPPER BRANCH, $\mu_c \approx n-(n+1)\bar{t}$

$$\Rightarrow \omega_q^{\pm} \approx U\left\{ \pm\left(-\frac{1}{2} + (n+1)\bar{t} - \frac{\bar{\epsilon}_q}{2} \right) + \frac{1}{2}\sqrt{1-2(2n+1)\bar{\epsilon}_q + \bar{\epsilon}_q^{\,2}} \right\}$$

FOR SMALL q

$$\Rightarrow \omega_q^{\pm} \approx U\left\{ \pm\left(-\frac{1}{2} + (n+1)\bar{t} - \frac{\bar{\epsilon}_q}{2} \right) + \frac{1}{2} - \left(n+\frac{1}{2}\right)\bar{\epsilon}_q \right\}$$

$$= U\left\{ \begin{array}{l} -\frac{1}{2}+\frac{1}{2} + (n+1)(\bar{t}-\bar{\epsilon}_q) \\ +\frac{1}{2}+\frac{1}{2} - (n+1)\bar{t} - n\,\bar{\epsilon}_q \end{array} \right. \begin{array}{l} \rightarrow \text{PARTICLE} \\ \rightarrow \text{HOLE} \end{array}$$

$$\Rightarrow \boxed{\omega_q^{+} \approx U(n+1)\left(\bar{t} - \bar{t}(1-\bar{q}^2)\right) \approx (n+1)\,t\,(qa)^2}$$

NO GAP, QUADRATIC DISPERSION

$$\boxed{\omega_q^{-} \approx U \rightarrow \text{GAPPED MODE}}$$

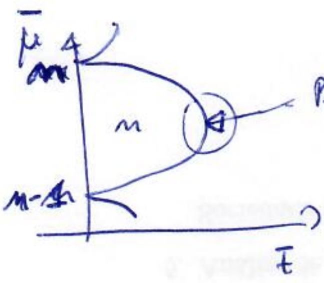
FOR THE LOWER BRANCH, $\mu_c \approx n-1+n\bar{t}$

AND SMALL q

$$\omega_q^{\pm} \approx U\left\{ \pm\left(+\frac{1}{2} - n\bar{t} - \frac{\bar{\epsilon}_q}{2} \right) + \frac{1}{2}\left(1-(2n+1)\bar{\epsilon}_q\right) \right\}$$

$$\Rightarrow \left\{ \begin{array}{l} \omega_q^{+} \approx U \rightarrow \text{GAPPED MODE} \\ \omega_q^{-} \approx U n(\bar{t}-\bar{\epsilon}_q) = n\,t\,(qa)^2 \rightarrow \text{QUADRATIC DISPERSION} \end{array} \right.$$

- HOW ABOUT THE EXCITATIONS AT THE TIP OF THE LOBES?
  → PARTICLES / HOLES ARE NOT WELL DEFINED SINCE $\bar{t}$ IS NOT SMALL
  → EXTRA SYMMETRY → PARTICLE-HOLE SYMMETRY



PARTICLE-HOLE SYMMETRIC POINT $\rightarrow \dfrac{\partial \bar{t}}{\partial \bar{\mu}} = 0$

DIFFERENTIATING CRITICAL LINE OF PAGE 15

$$\frac{\partial \bar{t}}{\partial \bar{\mu}}\left(\frac{m+1}{m-\bar{\mu}_c}+\frac{m}{\bar{\mu}_c-m+1}\right)+\bar{t}\left(\frac{m+1}{(m-\bar{\mu}_c)^2}-\frac{m}{(\bar{\mu}_c-m+1)^2}\right)=0$$

$$\Rightarrow \left(\bar{\mu}_c-(m-1)\right)^2(m+1)=\left(\bar{\mu}_c-m\right)^2 m \Rightarrow \left(\bar{\mu}_c^2-2\bar{\mu}_c(m-1)+(m-1)^2\right)(m+1)=\left(\bar{\mu}_c^2-2\bar{\mu}_c m+m^2\right)m$$

$$\Rightarrow \bar{\mu}_c^2+2\bar{\mu}_c+\underbrace{(m-1)^2(m+1)-m^3}_{1-m-m^2}=0$$

$$\Rightarrow \mu_c^*=\frac{-2+\sqrt{4-4(1-m-m^2)}}{2}=-1+\sqrt{m(m+1)}\xrightarrow{m\gg 1} m-\frac{1}{2}$$

$$\Rightarrow \bar{t}^*\left(\frac{m+1}{m-\bar{\mu}_c^*}+\frac{m}{\bar{\mu}_c^*-m+1}\right)=1 \Rightarrow \bar{t}^*=2m+1-2\sqrt{m(m+1)}=2m+1-2\left(\bar{\mu}_c^*+1\right)$$

NOTICE THAT $\omega^{\pm}_{q=0}=0 \rightarrow$ 2 GAPLESS MODES

THEREFORE, THE SMALL q SPECTRUM IS DOMINATED BY THE SQUARE ROOT PART

$$\Rightarrow \omega_q^+ \approx \omega_q^- \approx \frac{U}{2}\sqrt{2(2m+1)\bar{t}\bar{q}^2-2\bar{t}\bar{q}^2}=\sqrt{\bar{t}Um}\,q$$

2 MODES WITH LINEAR DISPERSION

HOW ABOUT THE EXCITATIONS IN THE SUPERFLUID PHASE?

IN ORDER TO ACCESS THE BROKEN SYMMETRY PHASE, WE NEED THE ψ^4 TERM IN ORDER TO ENSURE ENERGY STABILITY.

WE WILL NOT DO THIS HERE, INSTEAD, LET US CONSIDER THE PROBLEM IN A MORE GENERAL FRAMEWORK.

CONSIDER THE ϕ^4- THEORY

$$S = \int d\vec{x} \left\{ \frac{1}{2} r |\vec{\phi}|^2 + \frac{1}{2} |\nabla \vec{\phi}|^2 + \frac{1}{4} g |\vec{\phi}|^4 \right\}$$

WHERE $\vec{\phi} = (\phi_1, \phi_2, \ldots, \phi_N) \to$ N-COMPONENT VECTOR

- NOTICE S IS SYMMETRIC UNDER TRANSFORMATIONS OF THE $O(N)$ GROUP

- MEAN-FIELD SOLUTION: $\vec{\phi} = \vec{\phi}_{MF}$ = CONSTANT VECTOR

MINIMIZING $S \to \frac{\delta S}{\delta \vec{\phi}_{MF}} = 0 \Rightarrow |\vec{\phi}_{MF}| = \begin{cases} 0, & r \geq 0 \\ \sqrt{-\frac{r}{g}}, & r < 0 \end{cases}$

- INCLUDING FLUCTUATIONS: $\vec{\phi} = \vec{\phi}_{MF} + \vec{\psi}$

WITHOUT LOSS OF GENERALITY, $\vec{\phi}_{MF} = \phi_{MF} \hat{e}_N \to$ ALIGNED WITH THE LAST DIRECTION $\vec{N}$

$= (0, 0, \ldots, 0, \phi_{MF})$

IN OTHER WORDS, THE SYMMETRY WAS SPONTANEOUSLY BROKEN IN THE $\hat{e}_N$ DIRECTION

$$\to \frac{1}{2} r |\vec{\phi}|^2 = \frac{r}{2} |\vec{\phi}_{MF} + \vec{\psi}|^2 = \frac{r}{2} \left(\phi_{MF}^2 + 2 \phi_{MF} \psi_N + \psi^2 \right)$$

WHERE $\psi^2 = |\vec{\psi}|^2$, $\psi_N = \vec{\psi} \cdot \hat{e}_N$

$$\to \frac{1}{2} |\nabla \vec{\phi}|^2 = \frac{1}{2} |\nabla \vec{\psi}|^2 = \frac{1}{2} \sum_{l=1}^{N} |\nabla \psi_l|^2$$

$$\to \frac{1}{4} g |\vec{\phi}|^4 = \frac{1}{4} g \left(\phi_{MF}^2 + 2\phi_{MF}\psi_N + \psi^2 \right)^2$$

$$= \frac{1}{4} g \left(\phi_{MF}^4 + 4\phi_{MF}^3\psi_N + 2\phi_{MF}^2\psi^2 + 4\phi_{MF}^2\psi_N^2 + 4\phi_{MF}\psi_N\psi^2 + \psi^4 \right)$$

PLUGGING ALL OF THESE BACK IN THE ACTION

$$S = \int d\vec{x} \left\{ \frac{r\phi_{MF}^2}{2} + r\phi_{MF}\psi_N + \frac{r}{2}\psi^2 + \frac{1}{2}|\nabla\vec{\psi}|^2 + \frac{1}{4}g\phi_{MF}^4 + g\phi_{MF}^3\psi_N + \frac{1}{2}g\phi_{MF}^2\psi^2 + \phi_{MF}^2\psi_N^2 g + O(\psi^3,\psi^4) \right\}$$

$$= \int d\vec{x} \left\{ \frac{r}{2}\phi_{MF}^2 + \frac{1}{4}g\phi_{MF}^4 \right\} + \int d\vec{x} \left\{ (r\phi_{MF} + g\phi_{MF}^3)\psi_N \right\} + \int d\vec{x} \left\{ \frac{r}{2}\psi^2 + \frac{1}{2}|\nabla\vec{\psi}|^2 + \frac{1}{2}g\phi_{MF}^2\psi^2 + g\phi_{MF}^2\psi_N^2 \right\}$$

* THE 1ST TERM IS SIMPLY THE MEAN-FIELD ACTION S_{MF}
* THE 2ND TERM VANISHES BECAUSE $r\phi_{MF} + g\phi_{MF}^3 = \frac{\delta S}{\delta \vec{\phi}_{MF}} = 0$
* THE 3RD TERM CAN BE SIMPLIFIED BECAUSE

$\frac{r}{2} + \frac{1}{2}g\phi_{MF}^2 = 0 \Rightarrow$ MASS TERM FOR $|\vec{\psi}|^2 = 0$

FINALLY, $$S = S_{MF} + \int d\vec{x} \left\{ g\phi_{MF}^2\psi_N^2 + \frac{1}{2}|\nabla\vec{\psi}|^2 \right\}$$

IN THE ORDERED PHASE $(r<0)$, $\phi_{MF} \neq 0$. THEN

$$S = S_{MF} + \underbrace{\int d\vec{x} \left\{ |r|\psi_N^2 + \frac{1}{2}|\nabla\psi_N|^2 \right\}}_{\text{MASS TERM FOR THE } \psi_N \text{ MODE}} + \underbrace{\int d\vec{x}\, \frac{1}{2}\left\{ |\nabla\psi_1|^2 + |\nabla\psi_2|^2 + \dots + |\nabla\psi_{N-1}|^2 \right\}}_{\text{ALL OTHER MODES ARE MASSLESS}}$$

MASS TERM FOR THE ψ_N MODE

(HIGGS MODE)

ALL OTHER MODES ARE MASSLESS

(GOLDSTONE MODES)

SO FAR, WE HAVE NOT INCLUDED QUANTUM FLUCTUATIONS EXPLICITLY. WE NEED THAT FOR THE ACTUAL GOLDSTONE MODES.

THEN, WE GENERALIZE THE ϕ^4-THEORY TO

$$S = \int d\vec{x} \int d\tau \quad \bar{\phi}(\vec{x},\tau)\left(\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m} - \mu\right)\phi(\vec{x},\tau) + \frac{1}{2} g \,|\bar{\phi}\phi|^2$$

WHERE WE ARE CONSIDERING A COMPLEX FIELD $\phi(\vec{x},\tau) = \sqrt{\rho}\, e^{i\theta}$

- NOTICE THE SYMMETRY GROUP IS $U(1)$ (OR $O(2)$).
- NOTICE THE SIMILARITY WITH THE ACTION ON PAGE 8.
  THE DIFFERENCE IS THAT WE ARE NOT CONSIDERING THE LATTICE POTENTIAL. THIS SHOULDN'T BE A PROBLEM, SPECIALLY IN THE SUPERFLUID PHASE WHERE THE PARTICLES ARE DELOCALIZED.
- NOTICE ALSO THE SIMILARITY WITH THE ϕ^4 THEORY ON PAGE 18.
  THERE, $N=2$ (2 COMPONENTS VECTOR) AND μ PLAYS THE ROLE OF THE MASS TERM. IN ADDITION, WE HAVE THE IMAGINARY TIME DERIVATIVE DUE TO QUANTUM FLUCTUATIONS

WE NOW PROCEED WITH OUR DERIVATION OF THE EXCITATION SPECTRUM IN THE SUPERFLUID PHASE ($\mu > 0$)

* MEAN-FIELD SOLUTION $\rightarrow S_{MF} = S_{CLASSICAL} = V\beta\left(-\mu|\phi_{MF}|^2 + \frac{1}{2} g\,|\phi_{MF}|^4\right)$

$$\Rightarrow |\phi_{MF}|^2 = \rho_{MF} = \frac{\mu}{g}$$

* NOW, WE INCLUDE FLUCTUATIONS : $\phi = \phi_{MF} + \text{FLUCTUATIONS}$

CONVENIENTLY, LET US USE $\begin{cases} \phi(\vec{x},\tau) = \sqrt{\rho(\vec{x},\tau)}\, e^{i\theta(\vec{x},\tau)} \\ \bar{\phi}(\vec{x},\tau) = \sqrt{\rho(\vec{x},\tau)}\, e^{-i\theta(\vec{x},\tau)} \end{cases}$

$$\rightarrow \bar{\phi}\partial_\tau \phi = \sqrt{\rho}e^{-i\theta}\left(\partial_\tau \sqrt{\rho}e^{i\theta}\right) = i\rho\partial_\tau\theta + \frac{1}{2}\partial_\tau\rho$$

$$\rightarrow -\bar{\phi}\nabla^2\phi = +|\nabla\phi|^2 = \nabla\bar{\phi}\cdot\nabla\phi$$

THIS IS BECAUSE $\int d\vec{x}\left(-\bar{\phi}\frac{\partial^2}{\partial x^2}\phi\right) = \underbrace{-\bar{\phi}\frac{\partial}{\partial x}\phi\Big|_{-\infty}^{\infty}}_{=0} + \int d\vec{x}\,\left|\frac{\partial}{\partial x}\phi\right|^2$

$=0$ (WE USE BOUNDARY CONDITIONS ENSURING THIS RESULT)

$$\rightarrow -\bar{\phi}\nabla^2\phi = \left(\frac{e^{-i\theta}}{2\sqrt{\rho}}\nabla\rho - i\sqrt{\rho}e^{-i\theta}\nabla\theta\right)\cdot\left(\frac{\nabla\rho}{2\sqrt{\rho}} + i\sqrt{\rho}\nabla\theta\right)e^{i\theta}$$

$$= \frac{|\nabla\rho|^2}{4\rho} + \rho|\nabla\theta|^2$$

$$\rightarrow -\mu|\phi|^2 + \frac{1}{2}g|\phi|^4 = -\mu\rho + \frac{1}{2}g\rho^2$$

NOW, WE EXPAND FOR SMALL FLUCTUATIONS $\delta\rho$ AND θ

$$\begin{cases}\rho(\vec{x},\tau) = \rho_{MF} + \delta\rho(\vec{x},\tau) \\ \theta(\vec{x},\tau) = \theta_{MF} + \delta\theta(\vec{x},\tau) = \delta\theta(\vec{x},\tau) = \theta(x,\tau)\end{cases}$$ $\rightarrow$ WE HAVE SET $\theta_{MF}=0$

$$\rightarrow \bar{\phi}\partial_\tau\phi = \underbrace{i\rho_{MF}\partial_\tau\theta} + i\delta\rho\,\partial_\tau\theta + \frac{1}{2}\partial_\tau\delta\rho = i\delta\rho\,\partial_\tau\theta + \frac{1}{2}\partial_\tau\delta\rho$$

$\rightarrow$ VANISHES BECAUSE $\int d\vec{x}\int_0^\beta d\tau\, i\rho_{MF}\partial_\tau\theta = i\rho_{MF}\int d\vec{x}\,\underbrace{\theta(\vec{x},\beta) - \theta(\vec{x},0)}_{=0}$ (PERIODIC BOUNDARY CONDITIONS)

$$\rightarrow -\phi\nabla^2\phi \cong \frac{1}{4\rho_{MF}}|\nabla(\delta\rho)|^2 + \rho_{MF}|\nabla\theta|^2$$

$$\rightarrow -\mu\bar{\phi}\phi + \frac{1}{2}g|\bar{\phi}\phi|^2 = \underbrace{-\mu\rho_{MF} + \frac{1}{2}g\rho_{MF}}_{\text{GIVES RISE TO } S_{MF}} + \underbrace{(-\mu + g\rho_{MF})\delta\rho}_{\propto \left(\frac{\delta S}{\delta\rho}\right)\Big|_{\rho=\rho_{MF}} = 0} + \frac{1}{2}g(\delta\rho)^2$$

FINALLY,

$$S \approx S_{MF} + \int d\vec{x} \int_0^\beta d\tau \left\{ \frac{\rho_{MF}}{2m} |\nabla\theta|^2 + \frac{1}{2} g (\delta\rho)^2 + i\,\delta\rho\, \partial_\tau \theta \right\}$$

$$+ \int d\vec{x}\, d\tau \left\{ \frac{1}{8\rho_{MF} m} |\nabla \delta\rho|^2 + \frac{1}{2} \partial_\tau (\delta\rho) \right\}$$

- WE NEGLECT THE 3RD ACTION BECAUSE THE MASS TERM $\frac{1}{2} g (\delta\rho)^2$ PENALIZES FLUCTUATIONS OF THE $\delta\rho$ (HIGGS) MODE
- ALTHOUGH WE WANT THEM IN ORDER TO OBTAIN THE SPECTRUM OF THE HIGGS MODE, LET US FOCUS ON THE GOLDSTONE MODE WHICH IS MASSLESS.

$$\Rightarrow S \approx S_{MF} + \int d\vec{x}\, d\tau \left\{ \frac{\rho_{MF}}{2m} |\nabla\theta|^2 + \frac{1}{2} g \left[(\delta\rho)^2 + 2 \frac{i (\partial_\tau \theta)}{g} \delta\rho \right] \right\}$$

COMPLETING SQUARE AND PERFORMING THE GAUSSIAN INTEGRATION OVER THE $\delta\rho$, WE ARRIVE AT

$$S_{GOLDSTONE} = \int d\vec{x}\, d\tau \left\{ \frac{\rho_{MF}}{2m} |\nabla\theta|^2 + \frac{1}{2g} (\partial_\tau \theta)^2 \right\}$$

FOURIER TRANSFORMING: $\theta(\vec{x}, \tau) = \frac{1}{\sqrt{\beta V}} \sum_{\vec{k}, m} \tilde{\theta}_{\vec{k}, m}\, e^{i(\vec{k}\cdot\vec{x} - \omega_m \tau)}$

$$\Rightarrow S_{GOLDSTONE} = \sum_{\vec{k}, m} \left(\frac{\rho_{MF}}{2m} k^2 - \frac{1}{2g} \omega_m^2 \right) \tilde{\theta}_{k,m}^2$$

THIS IS EXACTLY THE ACTION OF AN HARMONIC OSCILLATOR WITH REAL FREQUENCY EQUAL TO

LINEAR DISPERSION $\rightarrow$ $\boxed{\omega_{\vec{k}} = c\, k}$ WITH SOUND VELOCITY $\boxed{c = \sqrt{\rho_{MF} \frac{g}{m}}}$

THIS IS THE GOLDSTONE MODE

COMPARE WITH THE RESULT OF THE BEGINNING OF THE COURSE

$c = \sqrt{\frac{N}{V} \frac{\tilde{v}_0}{m}}$

SUPER CURRENT: AS A FINAL REMARK, IT IS INTERESTING TO RELATE THE DENSITY CURRENT WITH THE ORDER PARAMETER PHASE

$$\vec{J}(\vec{x},\tau) = \frac{i}{2m}\left((\nabla \bar{\phi}) \phi - \bar{\phi} (\nabla \phi) \right) \simeq \rho_{MF} \frac{\nabla \theta(\vec{x},\tau)}{m}$$

$$\equiv \rho_{MF} \vec{v}_s$$

$\hookrightarrow$ SUPER FLUID VELOCITY $\Rightarrow \boxed{\vec{v}_s = \frac{\nabla \theta}{m}}$

SPECTRUM (SUMMARY):