PERTURBATION THEORY

BASIC CONCEPTS:

LET $$I(g) = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2 - g x^4} = \frac{e^{\frac{1}{32g}}\, \mathrm{BesselK}\left(\frac{1}{4}, \frac{1}{32g}\right)}{\sqrt{8g}}$$

$\mathrm{BesselK}(n, z) \equiv K_n(z) \equiv$ MODIFIED BESSEL FUNCTION OF $2^{\underline{ND}}$ KIND.

EXPANDING FOR $g = 0$ $$e^{-g x^4} = 1 - g x^4 + \cdots = \sum_{n=0}^{\infty} \frac{(-g)^n x^{4n}}{n!}$$

$$\Rightarrow I(g) = \sum_{n=0}^{\infty} g^n I_n$$

WHERE $$I_n = \frac{(-1)^n}{n!} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}}\, x^{4n}\, e^{-\frac{1}{2}x^2}$$

$$= \frac{(-1)^n\, 2^{2n-1} \left(1 + (-1)^{4n}\right) \Gamma(2n + 1/2)}{n!\,\sqrt{\pi}} = \frac{(-1)^n (4n-1)!!}{n!}$$

FOR $n \in \mathbb{N}^*$

ALTERNATIVE WAY:

$$I_n = \frac{(-1)^n}{n!}\langle x^{4n}\rangle = \frac{(-1)^n}{n!}\langle \underbrace{x \cdot x \cdots x}_{4n \text{ TIMES}} \rangle = \frac{(-1)^n}{n!} \sum_{\substack{\text{ALL POSSIBLE} \\ \text{PAIRING OF } 4n \\ \text{ELEMENTS}}} \langle x^2 \rangle$$

$$= \frac{(-1)^n}{n!} (4n-1)!!\, \langle x^2 \rangle$$

BUT $$\langle x^2 \rangle = \int \frac{dx}{\sqrt{2\pi}}\, x^2 e^{-\frac{1}{2}x^2} = 1$$

$$\Rightarrow I(g) = \sum_{n=0}^{\infty} I_n g^n \equiv \text{SERIES}$$

DOES THE SERIES CONVERGE?

$$I_n g^n = \frac{(4n-1)!!}{n!}(-g)^n \xrightarrow[\text{STIRLING'S}]{} \left(-\frac{gn}{e}\right)^n$$

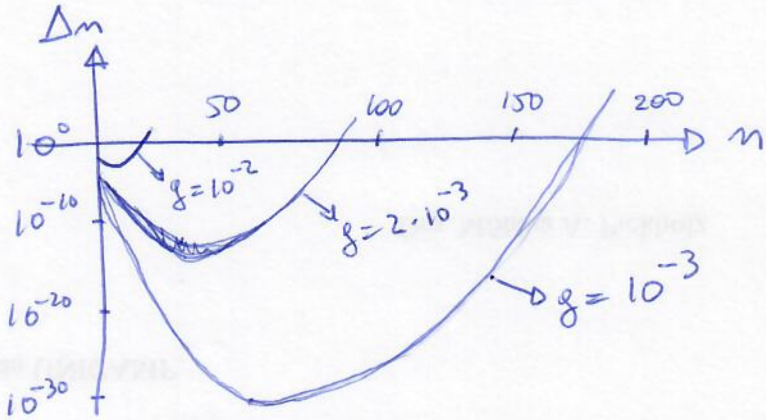
STIRLING'S APPROX. $n! \approx n^n e^{-n}$ FOR $n \gg 1$

$\Rightarrow$ PERTURBATIVE TERMS BECOME MORE AND MORE IMPORTANT AFTER ORDER $n^* = e/g$

- REASON: COMBINATORIAL PROLIFERATION OF <u>PAIRINGS</u>

CONCLUSION: THIS "PERTURBATION" SERIES HAS ILL-CONVERGENCY ISSUES

LET $\Delta_n = \left| I(g) - \sum_{k=0}^{n} I_k g^k \right|$



WHY IT IS ILL-CONVERGENT? $\begin{cases} g > 0 \rightarrow I(g) \text{ IS FINITE} \\ g < 0 \rightarrow I(g) \rightarrow \infty \end{cases}$

THE <u>RADIUS OF CONVERGENCY</u> IS $R = 0$ <u>AROUND $g = 0$</u>

<u>LESSON</u>: A GOOD STARTING POINT IS IMPORTANT.

HOWEVER, IS THE PERTURBATIVE APPROACH DOOMED TO FAIL WHEN $\underline{R=0}$?

- AS SHOWN IN THE GRAPH FOR Δm, IT IS ONLY FOR LARGE ORDER AND LARGE g.

LESSON: EVEN THOUGH $R=0$, THE PERTURBATIVE APPROACH IS GOOD FOR SMALL g.

* IN GENERAL, WE DO NOT KNOW HOW SMALL MUST g BE AND WHAT IS THE OPTIMAL PERTURBATIVE ORDER (HARD TO KNOW)

THE ϕ^4- THEORY

- SIMPLEST INTERACTING FIELD THEORY:

$$Z = \int \mathcal{D}[\phi]\, e^{-S[\phi]} , \qquad S[\phi] = \int d\vec{x} \left(\frac{1}{2}(\partial\phi)^2 + \frac{1}{2} r \phi^2 + g\phi^4 \right)$$

$\equiv$ GINZBURG-LANDAU FREE ENERGY FUNCTIONAL

$\phi \equiv \phi(\vec{x})$ IS A SCALAR BOSONIC FIELD

- $\frac{1}{2} r \phi^2 \rightarrow$ MASS TERM $\rightarrow$ IN MEAN-FIELD LEVEL, PHASE TRANSITION WHEN $r=0$

- $\frac{1}{2}(\partial\phi)^2 = \sum_{i=1}^{d} \left(\frac{\partial \phi}{\partial x_i}\right)^2 \rightarrow$ FLUCTUATING TERM WHICH COSTS ENERGY

  i.e., FLUCTUATIONS ARE PENALIZED ENERGY-WISE

  OF COURSE, FLUCTUATIONS ARE IMPORTANT FOR ENTROPIC PURPOSES

- $g\phi^4 \rightarrow$ INTERACTION TERM

- MOTIVATION:
  - ISING MODEL OR UNIAXIAL MAGNETS
  - FLUIDS
  - SYSTEM WITH SIMPLE ORDER PARAMETERS
  - ALLOWS US FOR GENERALIZATION

FORMAL DERIVATION FROM THE ISING MODEL: $H = \sum_{ij} S_i C_{ij} S_j - \sum_i H_i S_i$

$$Z = \sum_{\{S_i\}} e^{-\beta H} = \sum_{\{S_i\}} e^{\sum_{ij} S_i K_{ij} S_j + \sum_i h_i S_i} , \quad \boxed{S_i = \pm 1} \quad \begin{cases} K_{ij} = \beta C_{ij} \\ h_i = \beta H_i \end{cases}$$

$K_{ij} = K_{ji}$

- HUBBARD-STRATONOVICH TRANSFORMATION

$$* \int_{-\infty}^{\infty} \frac{d\phi_1}{2\sqrt{\pi}} \cdots \frac{d\phi_N}{2\sqrt{\pi}}\, e^{-\frac{1}{4}\vec{\phi}^T \cdot \underset{\substack{\text{SEMI-DEFINITE} \\ \text{POSITIVE MATRIX}}}{\mathbb{M}} \cdot \vec{\phi} + \vec{v}^T \cdot \vec{\phi}} = \frac{1}{\sqrt{\det \mathbb{M}}}\, e^{\vec{v}^T \cdot \mathbb{M}^{-1} \cdot \vec{v}} , \quad \forall \vec{v}$$

(5)

$$\Rightarrow Z = \sum_{\{S_i\}} e^{\vec{S}^T \cdot \mathbb{K} \cdot \vec{S} + \vec{h}^T \cdot \vec{S}} \quad , \quad \vec{S} = (S_1, S_2, \dots, S_N)$$

$$= \sum_{\{S_i\}} \sqrt{\operatorname{DET} \mathbb{K}^{-1}} \int_{-\infty}^{\infty} \frac{d\phi_1}{2\sqrt{\pi}} \cdots \frac{d\phi_N}{2\sqrt{\pi}} \, e^{-\frac{1}{4}\vec{\phi}^T \cdot \mathbb{K}^{-1} \cdot \vec{\phi} + \underbrace{\vec{S}^T \cdot \vec{\phi} + \vec{h}^T \cdot \vec{S}}_{(\vec{\phi}^T + \vec{h}^T) \cdot \vec{S}}}$$

- THE SPINS DO NOT INTERACT <u>DIRECTLY</u> ANYMORE

SINCE $S_i = \pm 1$, THE SUM OVER THE SPINS ARE EASY

$$Z = \sqrt{\operatorname{DET} \mathbb{K}^{-1}} \int_{-\infty}^{\infty} \frac{d\phi_1}{2\sqrt{\pi}} \cdots \frac{d\phi_N}{2\sqrt{\pi}} \, e^{-\frac{1}{4}\vec{\phi}^T \mathbb{K}^{-1} \vec{\phi}} \prod_{i=1}^{N} \underbrace{e^{\phi_i + h_i} + e^{-\phi_i - h_i}}_{2\cosh(\phi_i + h_i)}$$

CANCEL

$$\Rightarrow \boxed{Z = \int \mathcal{D}[\phi] \, e^{-\frac{1}{4}\phi \cdot \mathbb{K} \cdot \phi + \sum_{i=1}^{N} \ln(\cosh(\phi_i + h_i))}}$$

THE FACTOR $\prod_i 2 = 2^N \rightarrow$ ABSORBED IN $\mathcal{D}[\phi]$

$$\int \mathcal{D}[\phi] = \int_{-\infty}^{\infty} \frac{d\phi_1}{\sqrt{\pi}} \cdots \int_{-\infty}^{\infty} \frac{d\phi_N}{\sqrt{\pi}}$$

- FOR CONVENIENCE $\phi_i + h_i \rightarrow \phi_i$

$$\Rightarrow \vec{\phi}^T \mathbb{K}^{-1} \vec{\phi} \rightarrow (\vec{\phi}^T - \vec{h}^T) \cdot \mathbb{K}^{-1} \cdot (\vec{\phi} - \vec{h}) = \vec{\phi}^T \mathbb{K}^{-1} \vec{\phi} - \vec{h}^T \cdot \mathbb{K}^{-1} \cdot \vec{\phi} - \vec{\phi}^T \cdot \mathbb{K}^{-1} \cdot \vec{h} + \vec{h}^T \cdot \mathbb{K}^{-1} \cdot \vec{h}$$

$$= \sum_{ij} \phi_i (\mathbb{K}^{-1})_{ij} \phi_j - 2 \sum_{ij} h_i (\mathbb{K}^{-1})_{ij} \phi_j + \vec{h}^T \cdot \mathbb{K}^{-1} \cdot \vec{h}$$

$$\Rightarrow \cosh(\phi_i + h_i) \Rightarrow \cosh \phi_i$$

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- ALSO $\frac{1}{2}\sum_j (\mathbb{K}^{-1})_{ij}\phi_j \to \phi_j$

$$\Rightarrow \frac{1}{2}\mathbb{K}^{-1}\cdot\vec{\phi} \to \vec{\phi}$$

$$\Rightarrow \vec{\phi}^T\cdot\mathbb{K}^{-1}\cdot\vec{\phi} = 4\,\frac{\vec{\phi}^T\cdot\mathbb{K}^{-1}}{2}\,\mathbb{K}\,\frac{\mathbb{K}^{-1}\cdot\vec{\phi}}{2} \to 4\,\vec{\phi}^T\cdot\mathbb{K}\cdot\vec{\phi}$$

$$\Rightarrow -2\,\vec{h}^T\cdot\mathbb{K}^{-1}\cdot\vec{\phi} \to -4\,\vec{h}^T\cdot\vec{\phi}$$

$$\Rightarrow \cosh\phi_i \to \cosh\left(2\sum_j \mathbb{K}_{ij}\phi_j\right)$$

FINALLY, ABSORBING THE TERM $e^{\vec{h}^T\mathbb{K}\vec{h}}$ IN $\mathcal{D}[\phi]$

$$\boxed{Z = \int \mathcal{D}[\phi]\, e^{-\vec{\phi}^T\mathbb{K}\vec{\phi} + \vec{h}^T\cdot\vec{\phi} + \sum_{i=1}^{N}\ln\left(\cosh\left(2\sum_j \mathbb{K}_{ij}\phi_j\right)\right)}}$$

NOTICE THE SIMILARITY $Z = \sum_{\{S_i\}} e^{\vec{S}\,\mathbb{K}\,\vec{S} + \vec{h}\cdot\vec{S}}$

- $\langle S_j\rangle = \frac{1}{Z}\sum_{\{S_i\}} S_j\, e^{\vec{S}\mathbb{K}\vec{S} + \vec{h}\cdot\vec{S}} = \frac{1}{Z}\frac{\partial Z}{\partial h_j}$

- $\langle \phi_j\rangle = \frac{1}{Z}\int\mathcal{D}[\phi]\,\phi_j\, e^{-\vec{\phi}\mathbb{K}\vec{\phi} + \vec{h}\cdot\vec{\phi} + \sum_i \ln\left(\cosh\left(2\sum_j \mathbb{K}_{ij}\phi_j\right)\right)} = \frac{1}{Z}\frac{\partial Z}{\partial h_j}$

$$\Rightarrow \boxed{\langle S_j\rangle = \langle \phi_j\rangle}$$

$$S[\phi] = \vec{\phi}\,\mathbb{K}\,\vec{\phi} - \vec{h}\cdot\vec{\phi} - \sum_i \ln\left(\cosh\left(2\sum_j \mathbb{K}_{ij}\phi_j\right)\right)$$

FOR SYSTEMS WITH TRANSLATIONAL SYMMETRY (7)

- $$\phi_j = \frac{1}{\sqrt{N}} \sum_{\vec{k}} e^{-i\vec{k}\cdot\vec{r}_j} \phi(\vec{k})$$

- $$h_j = \sum_{\vec{k}} \frac{e^{-i\vec{k}\cdot\vec{r}_j}}{\sqrt{N}} h(\vec{k})$$

- $$\mathbb{K}_{ij} = \frac{1}{N} \sum_{\vec{k}} e^{-i\vec{k}\cdot(\vec{r}_i-\vec{r}_j)} \mathbb{K}(\vec{k})$$

$$\Rightarrow \sum_{ij} \phi_i \mathbb{K}_{ij} \phi_j = \sum_{k_1k_2k_3} \phi(k_1)\mathbb{K}(k_2)\phi(k_3) \sum_{ij} \frac{e^{-i\vec{r}_i\cdot(\vec{k}_1+\vec{k}_2)}}{N} \frac{e^{-i\vec{r}_j\cdot(\vec{k}_3-\vec{k}_2)}}{N}$$

$$= \sum_{\vec{k}} \phi(\vec{k})\mathbb{K}(\vec{k})\phi(-\vec{k})$$

$$\Rightarrow \sum_i h_i \phi_i = \sum_{\vec{k}_1} \sum_{\vec{k}_2} h(\vec{k}_1)\phi(\vec{k}_2) \sum_j \frac{e^{-i\vec{r}_j\cdot(\vec{k}_1+\vec{k}_2)}}{N} = h(-\vec{k})\phi(\vec{k})$$

ALSO $$\ln(\cosh(2x)) \simeq 2x^2 - \frac{4}{3}x^4 + O(x^6)$$

$$\Rightarrow \sum_i \sum_{jl} \mathbb{K}_{ij}\phi_j \mathbb{K}_{il}\phi_l = \sum_{\substack{k_1k_2\\k_3k_4}} \mathbb{K}(k_1)\phi(k_2)\mathbb{K}(k_3)\phi(k_4) \frac{1}{N^3}\sum_{ijl} e^{-i\vec{r}_j(\vec{k}_2-\vec{k}_1)} e^{-r_i(\vec{k}_1+\vec{k}_3)} * e^{-r_l(\vec{k}_4-\vec{k}_3)}$$

$$= \sum_{\vec{k}} \phi(\vec{k})\mathbb{K}(\vec{k})\mathbb{K}(-\vec{k})\phi(\vec{k})$$

$$\Rightarrow \sum_i \sum_{\substack{jl\\mn}} \mathbb{K}_{ij}\phi_j \mathbb{K}_{il}\phi_l \mathbb{K}_{im}\phi_m \mathbb{K}_{in}\phi_n = \sum_{\substack{k_1k_2\\k_3k_4\\k_5k_6\\k_7k_8}} \mathbb{K}_1\phi_2\mathbb{K}_3\phi_4\mathbb{K}_5\phi_6\mathbb{K}_7\phi_8 \sum_{\substack{ijl\\mn}} \frac{1}{N^6} e^{-i\vec{r}_i\cdot(k_1+k_3+k_5+k_7)} * e^{-i\vec{r}_j(\vec{k}_2-\vec{k}_1)} e^{-i\vec{r}_l(\vec{k}_4-\vec{k}_3)} e^{-i\vec{r}_m(\vec{k}_6-\vec{k}_5)} e^{-i\vec{r}_n\cdot(\vec{k}_8-\vec{k}_7)}$$

$$= \frac{1}{N} \sum_{\substack{k_1k_2\\k_3k_4}} \mathbb{K}(\vec{k}_1)\mathbb{K}(\vec{k}_2)\mathbb{K}(\vec{k}_3)\mathbb{K}(\vec{k}_4)\phi(\vec{k}_1)\phi(\vec{k}_2)\phi(\vec{k}_3)\phi(\vec{k}_4) * \delta_{\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4,0}$$

EXPANDING $\mathbb{K}(\vec{k}) = \mathbb{K}(0) + \frac{1}{2}\mathbb{K}''(0)\, k^2$

BECAUSE $\mathbb{K}_{ij} = \mathbb{K}_{ji} \Rightarrow$ EVEN FUNCTION

$$\Rightarrow S[\phi] \cong \sum_{\vec{k}} \phi(\vec{k})\left(\mathbb{K}(0) + \frac{1}{2}\mathbb{K}''(0)k^2 - 2\left[\mathbb{K}^2(0) + \mathbb{K}(0)\mathbb{K}''(0)k^2\right]\right)\phi(-\vec{k}) - h(\vec{k})\phi(-\vec{k})$$

$$+ \frac{\mathbb{K}^4(0)}{N}\sum_{\vec{k}_1 \cdots \vec{k}_4} \phi(\vec{k}_1)\phi(\vec{k}_2)\phi(\vec{k}_3)\phi(\vec{k}_4)\,\delta_{\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4,\,0}$$

SWITCHING BACK TO SPACE REPRESENTATION

$$S[\phi] = \int d\vec{x}\,\left\{ c_1\phi^2 + c_2|\nabla\phi|^2 + c_3\, h(\vec{x})\phi(\vec{x}) + c_4\phi^4(\vec{x}) \right\}$$

WHERE
$$\begin{cases} c_1 = \mathbb{K}(0)\left(1 - 2\mathbb{K}(0)\right) \\ c_2 = \frac{1}{2}\mathbb{K}''(0)\left(1 - 4\mathbb{K}(0)\right) \\ c_3 = 1 \\ c_4 = \frac{4}{3}\mathbb{K}^4(0) \end{cases}$$

MAKING $\sqrt{2c_2}\,\phi \to \phi$

$$\boxed{S[\phi] = \int d\vec{x}\,\left\{ \frac{1}{2}|\nabla\phi|^2 + \frac{1}{2}r\phi^2 + \frac{1}{4}g\phi^4 - \alpha h(x)\phi(x) \right\}}$$

where $r = \frac{c_1}{c_2} = \frac{2\mathbb{K}(0)(1-2\mathbb{K}(0))}{\mathbb{K}''(0)(1-4\mathbb{K}(0))}$, $g = \frac{c_4}{c_2^2} = \frac{16\mathbb{K}^4(0)}{3\mathbb{K}''^2(0)(1-4\mathbb{K}(0))^2}$

• MEAN-FIELD CRITICAL TEMPERATURE: $c_1 = r = 0 \Rightarrow \mathbb{K}(0) = \boxed{\beta^* C(0) = \frac{1}{2}}$

$\alpha = \frac{c_3}{\sqrt{2c_2}} = \frac{1}{\sqrt{\mathbb{K}''(0)(1-4\mathbb{K}(0))}}$

DIMENSIONAL ANALYSYS

$$[d\vec{x}\,|\nabla\phi|^2] = 1 \Rightarrow \begin{cases} [\phi^2] = (\text{LENGTH})^{2-d} \\ [r] = (\text{LENGTH})^{-2} \\ [g] = (\text{LENGTH})^{d-4} \end{cases}$$

$$[\alpha h] = (\text{LENGTH})^{-1-\frac{d}{2}}$$

EXPANSION IS GOOD FOR $|\phi| << 1$

AND FOR SMALL K → LONG WAVE LENGTHS

n-POINT CORRELATION FUNCTION

$$C_n(x_1 \dots x_n) \equiv \langle \phi(\vec{x}_1) \dots \phi(\vec{x}_n) \rangle$$

THE CASE $n=2$ IS SPECIAL

$$C_2(\vec{x}_1, \vec{x}_2) \equiv \mathcal{G}(\vec{x}_1 - \vec{x}_2) \equiv$$ PROPAGATOR OF THE THEORY

GREEN'S FUNCTION

RESOLVENT

TRANSLATION INVARIANCE

EXAMPLE: FREE THEORY

$$\mathcal{G}_0(\vec{x}) = C_2(\vec{x}+\vec{x}_1, \vec{x}_1) = \langle \phi(0)\phi(\vec{x}) \rangle_0 = \frac{\int \mathcal{D}[\phi]\, \phi(0)\phi(\vec{x})\, e^{-S_0}}{\int \mathcal{D}[\phi]\, e^{-S_0}}$$

WITH $$S_0[\phi] = \frac{1}{2} \sum_{\vec{k}} \phi(\vec{k})(r + k^2)\phi(-\vec{k})$$ $\boxed{\text{WITH } r > 0}$

$$\Rightarrow \tilde{\mathcal{G}}_0(\vec{k}) = \int d\vec{x}\, e^{i\vec{k}\cdot\vec{x}}\, \mathcal{G}_0(x) = \Big\langle \sum_{k_1 k_2} \phi(\vec{k}_1)\phi(\vec{k}_2) \int_N d\vec{x}\, e^{i\vec{x}\cdot(\vec{k}-\vec{k}_2)} \Big\rangle_0$$

$$= \sum_{\vec{k}_1} \langle \phi(\vec{k}_1)\phi(\vec{k}) \rangle_0$$

$$= \sum_{\vec{k}_1} \frac{\delta_{\vec{k}_1, -\vec{k}}}{r + k^2} \quad \text{(GAUSSIAN INTEGRATION)}$$

$$\Rightarrow \boxed{\tilde{\mathcal{G}}_0(\vec{k}) = \frac{1}{r + k^2}}$$

INVERSE OF THE FREE PROPAGATOR

NOTICE THAT $$S_0[\phi] = \frac{1}{2} \sum_{\vec{k}} \phi(\vec{k})\, \tilde{\mathcal{G}}_0^{-1}(\vec{k})\, \phi(-\vec{k})$$

THEREFORE, $\mathcal{G}_0(\vec{x},\vec{x}') \equiv \mathcal{G}_0(\vec{x}-\vec{x}')$ IS THE GREEN'S FUNCTION OF THE DIFFERENTIAL EQUATION

$$\left(-\nabla_{\vec{x}}^2 + r\right)\mathcal{G}_0(\vec{x},\vec{x}') = \delta(\vec{x}-\vec{x}')$$

$$\Rightarrow \mathcal{G}_0(\vec{x}) = \frac{1}{V}\sum_{\vec{k}} e^{-i\vec{k}\cdot\vec{x}}\,\tilde{\mathcal{G}}_0(\vec{k}) \underset{V\to\infty}{=} \int \frac{d\vec{k}}{(2\pi)^d}\,\frac{e^{-i\vec{k}\cdot\vec{x}}}{r+k^2}$$

$\underline{d=1}$: $\mathcal{G}_0(\vec{x}) = \dfrac{e^{-\sqrt{r}x}}{2\sqrt{r}}$

$\underline{d=2}$: $\mathcal{G}_0(\vec{x}) = \dfrac{K_0(\sqrt{r}x)}{2\pi} \simeq \begin{cases} \dfrac{-\ln\left(\frac{1}{2}\sqrt{r}x\right)}{2\pi}, & \sqrt{r}x \ll 1 \\ \dfrac{e^{-\sqrt{r}x}}{2\sqrt{2\pi\sqrt{r}x}}, & \sqrt{r}x \gg 1 \end{cases}$

$\underline{d=3}$: $\mathcal{G}_0(\vec{x}) = \dfrac{e^{-\sqrt{r}x}}{4\pi x}$

$\underline{\text{CORRELATION LENGTH}}$: $\xi \equiv \dfrac{1}{\sqrt{r}} = r^{-\nu}$, $\nu = \frac{1}{2}$ MEAN-FIELD CORRELATION LENGTH

IT COULD BE OBTAINED BY DIMENSIONAL ANALYSIS $\rightarrow \mathcal{G}_0 \sim e^{-x/\xi}$, $[\xi] = $ LENGTH

SINCE $\begin{cases} [\phi] = L^{1-\frac{d}{2}} \\ [r] = L^{-2} \end{cases}$

$\rightarrow$ THEN, ONLY LENGTH SCALE IN THE FREE THEORY IS $\dfrac{1}{\sqrt{r}}$

COMMENTS:

$$G_0(x \to 0) = \int \frac{d\vec{k}}{(2\pi)^d} \frac{1}{r + k^2}$$

- ULTRAVIOLET DIVERGENCY $(k \to \infty)$: $\int^{\infty} k^{d-3}\, dk \to \infty$ FOR $d \geq 2$

* RECALL THAT ~~...~~ IN A LATTICE SYSTEM $k \leq \frac{1}{a}$, a = LATTICE PARAMETER

* FOR SYSTEMS WITHOUT A LATTICE, RECALL THAT THE ϕ^4-THEORY WAS DERIVED FOR SMALL k (LONG WAVELENGTHS), THERE MUST BE A MICROSCOPIC LENGTH SCALE WHERE IT DOES NOT APPLY

$\to$ IN ANY CASE $\to$ k_{MAX} CUTOFF SOLVES THE PROBLEM

QUESTION: AFTER REGULARIZING THE ~~...~~ UV DIVERGENCY WITH A CUTOFF, IS OUR PERT. THEORY SENSITIVE TO IT?

FOR THE PHYSICS OF LONG WAVELENGTHS, IT SHOULD BE INSENSITIVE AS SHOWN BY RENORMALIZATION-GROUP METHODS

- INFRARED DIVERGENCY (LONG WAVELENGTHS) : $\int \frac{k^{d-1}\, dk}{r + k^2}$ $(k \to 0)$

FOR $r \to 0$ $\to$ $\int_0^{k_{max}} k^{d-3}\, dk \to \infty$ FOR $d \leq 2$

- THE ϕ^4-THEORY MUST BE RENORMALIZED TO CURE THIS DIVERGENCY

FOR $r > 0$ $\to$ NO DIVERGENCY IN THE IR LIMIT

PERTURBATION THEORY FOR THE ϕ^4-THEORY

~~(NOT GENERAL (NOT ONLY FOR THE ϕ^4-THEORY))~~

$$S_{INT}[\phi] = g \int d\vec{x}\, \phi^4(\vec{x}) \longrightarrow \text{INTERACTION OR VERTEX}$$

$$S_0[\phi] = \int d\vec{x} \left(\frac{r}{2}\phi^2 + \frac{1}{2}|\nabla\phi|^2 \right)$$

$$\langle O \rangle = \frac{\int D[\phi]\, e^{-S_0}\, e^{-S_{INT}}\, O}{\int D[\phi]\, e^{-S_0}\, e^{-S_{INT}}} = \frac{\langle e^{-S_{INT}}\, O \rangle_0}{\langle e^{-S_{INT}} \rangle_0}$$

$$= \frac{\sum_{n=0}^{\infty} \frac{1}{n!}(-g)^n \left\langle O \left(\int d\vec{x}\, \phi^4(x) \right)^n \right\rangle_0}{\sum_{n=0}^{\infty} \frac{(-g)^n}{n!} \left\langle \left(\int d\vec{x}\, \phi^4(\vec{x}) \right)^n \right\rangle_0} = \sum_{n=0}^{\infty} O_n$$

THIS STRUCTURE IS GENERAL. NOT ONLY FOR THE ϕ^4-THEORY

- EXAMPLE: $O = G(x, x') = G(x - x') = \langle \phi(x)\phi(x') \rangle$

$$\Rightarrow G = \frac{G_0 - g \langle \phi(\vec{x})\phi(\vec{x}') \int d\vec{y}\, \phi^4(\vec{y}) \rangle + O(g^2)}{1 - g \langle \int d\vec{y}\, \phi^4(\vec{y}) \rangle_0 + O(g^2)}$$

$$= G_0 - g \left\{ \langle \phi(\vec{x})\phi(\vec{x}') \int d\vec{y}\, \phi^4(\vec{y}) \rangle - G_0 \langle \int d\vec{y}\, \phi^4(y) \rangle_0 \right\} + O(g^2)$$

$$= G_0 + G_1 + O(g^2)$$

WITH $G_1 = -g \int d\vec{y} \left\{ \langle \phi(x)\phi^4(y)\phi(y) \rangle_0 - \langle \phi(x)\phi(x') \rangle_0 \langle \phi^4(y) \rangle_0 \right\}$

$$\langle \phi(x)\phi(x') \rangle_0 = G_0(x - x')$$

NOW, WE USE WICK'S THEOREM

$$\bullet \; \langle \phi(x)\,\phi^4(y)\,\phi(x')\rangle_0 = \langle \phi(x)\,\phi(y)\,\phi(y)\,\phi(y)\,\phi(y)\,\phi(x')\rangle_0$$

$$= 3\,\langle \phi(x)\,\phi(x')\rangle_0 \left(\langle \phi(y)\,\phi(y)\rangle_0\right)^2$$

$$+\, 12\,\langle \phi(x)\,\phi(y)\rangle_0\,\langle \phi(x')\,\phi(y)\rangle_0\,\langle \phi(y)\,\phi(y)\rangle_0$$

$$= 3\,\mathcal{G}_0(x-x')\,\mathcal{G}_0^2(0) + 12\,\mathcal{G}_0(x-y)\,\mathcal{G}_0(y-x')\,\mathcal{G}_0(0)$$

$$\bullet \; \langle \phi^4(y)\rangle_0 = 3\left(\langle \phi(y)\,\phi(y)\rangle_0\right)^2 = 3\,\mathcal{G}_0^2(0)$$

THUS, $$\mathcal{G}_1 = -\cancel{\tfrac{1}{24}}\,g \int d\bar{y} \left\{ \underline{3\,\mathcal{G}_0(x-x')\,\mathcal{G}_0^2(0)} + 12\,\mathcal{G}_0(x-y)\,\mathcal{G}_0(y-x')\,\mathcal{G}_0(0) \; \underline{- 3\,\mathcal{G}_0(x-x')\,\mathcal{G}_0^2(0)} \right\}$$

CANCELATION

$$\Rightarrow \boxed{\mathcal{G}_1 = -\cancel{\tfrac{1}{24}}\,g \int d\bar{y}\; 12\,\mathcal{G}_0(x-y)\,\mathcal{G}_0(0)\,\mathcal{G}_0(y-x')}$$

HOW ABOUT 2^{ND} ORDER CORRECTION $\mathcal{G}_2$?

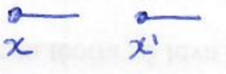
WE NEED A CLEVER WAY TO KEEP TRACK THE CONTRACTIONS OF THE WICK'S THEOREM

$\Rightarrow$ DIAGRAMATICS

ASSOCIATE $\phi(x) \longrightarrow$ 

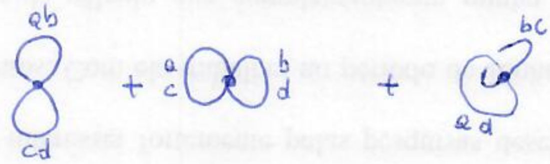
THUS $\phi^4(y) = \phi(y)\phi(y)\phi(y)\phi(y) \longrightarrow$ 

CONTRACTIONS

$G_0(x-x') \equiv \langle \phi(x)\,\phi(x') \rangle_0 = \langle$  $\rangle_0 \equiv$ 

IN THIS GRAPHIC REPRESENTATION, (FREE PROPAGATOR)
CONTRACTION IS THUS SIMPLY
GLUING THE LINES

HOW ABOUT $\langle \phi^4(y) \rangle_0$?

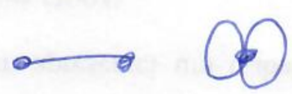
$\langle \phi^4(y) \rangle_0 = \langle$  $\rangle_0 =$ 

$= 3$ 

$\Rightarrow \langle \phi(x)\phi(x') \rangle_0 \, \langle \phi^4(y) \rangle_0 = 3$  $\equiv$ DISCONNECTED DIAGRAM

HOW ABOUT $\langle \phi(x)\,\phi(x')\,\phi^4(y) \rangle_0$?

$\langle \phi(x)\,\phi^4(y)\,\phi(x') \rangle_0 = \langle$  $\rangle_0$

12  $+ 3$ 

CONNECTED DIAGRAM

DISCONNECTED DIAGRAM

THUS,

$$\mathcal{Y}_1 = -g\int d\vec{y}\,\Big\{ \langle \overset{x}{\bullet} - \underset{y}{\times} - \overset{x'}{\bullet} \rangle_0 - \langle \overset{x}{\bullet} - \overset{x'}{\bullet} \rangle_0 \langle \underset{y}{\times} \rangle_0 \Big\}$$

$$= -g\int d\vec{y}\,\Big\{ 12$$  $$\Big\}$$

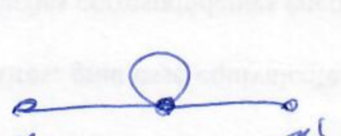
WE NOW REDEFINE THIS TO

$$\mathcal{G}_1(x-x') = 12$$ 

 $$\equiv G_0(x-y) = G_0(y-x)$$

 $$\equiv G_0(0)$$

EACH INTERNAL VERTEX $\underset{y}{\bullet} \equiv -g\int d\vec{y}$

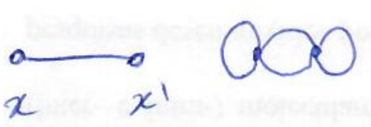
THEREFORE,  $$\equiv -g\int d\vec{y}\; G_0(x-y)\,G_0(y-x')\,G_0(0)$$

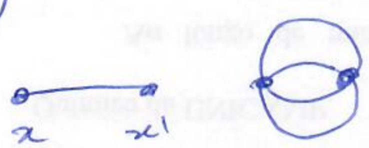
HOW ABOUT 2ND ORDER?

$$\frac{1}{2!}(-g)^2 \langle \phi(x)\,\phi(x') \Big(\int dy\, \phi^4(y)\Big)^2 \rangle_0 \to \langle \overset{x}{\bullet} - \underset{y}{\times}\ \underset{y'}{\times} - \overset{x'}{\bullet} \rangle_0$$

HAVE TO COUNT ALL POSSIBLE CONTRACTIONS (GLUING THE LEGS)

1.  → 3×3 → 9 TIMES

2.  → CONNECT ONLY 2 LEGS, CONNECT ONLY 2 LEGS: $\binom{4}{2}\times 4\times 3 = 72$ TIMES

3.  → $4\times 3\times 2 = 24$ TIMES

4.  → $12\times 3\times 2 = 72$

5.  → $12\times 12\times 2 = 288$

6.  → $\binom{4}{2}\times 12\times 2\times 2 = 288$

   EXCHANGE $x \leftrightarrow x'$

   EXCHANGE $y \leftrightarrow y'$

7.  → $4\times 3\times 2\times (2\times 4) = 192$

- DIAGRAMS 1, 2, 3 AND 4 → DISCONNECTED DIAGRAMS ALSO KNOWN AS <u>VACUUM</u> <u>POLARIZATION GRAPHS</u>

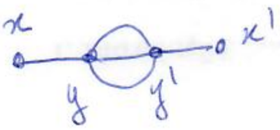
- DIAGRAMS 5, 6 AND 7 → CONNECTED DIAGRAMS

• WE HAVE ALSO TO COMPUTE THE DIAGRAMS OF $O(g^2)$ COMING FROM THE DENOMINATOR $\dfrac{1}{\sum_n \frac{(-g)^n}{n!} \langle (\mathcal{X})^n \rangle_0}$

THE LINKED CLUSTER THEOREM SAYS THEY WILL CANCEL OUT THE DISCONNECTED DIAGRAMS AND ONLY THE CONNECTED ONES WILL REMAIN

$\Rightarrow G_2 = 192$  $+ \; 288$  $+ \; 288$ 

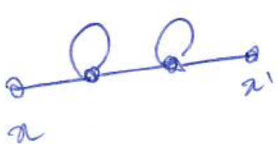
HOW DO WE TRANSLATE TO INTEGRALS ?



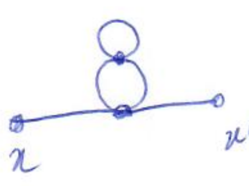
$n = 2$ INTERNAL VERTICES $\Rightarrow \dfrac{(-g)^n}{n!}$

INTEGRATE OVER y AND y'

$$\Rightarrow \frac{(-g)^2}{2!} \int dy\, dy' \; G_0(x-y)\, G_0^3(y-y')\, G_0(x'-y')$$



$$= \frac{(-g)^2}{2!} G_0^2(0) \int dy\, dy' \; G_0(x-y)\, G_0(y'-y)\, G_0(x'-y')$$



$$= \frac{(-g)^2}{2!} G_0(0) \int dy\, dy' \; G_0(x-y)\, G_0^2(y-y')\, G_0(x'-y)$$

LINKED CLUSTER THEOREM : PROOF

$$\langle O \rangle = \frac{\text{NUMERATOR}}{\text{DENOMINATOR}} \quad , \quad \text{NUMERATOR} = \sum_{n=0}^{\infty} \frac{(-g)^n}{n!} \langle O (X)^n \rangle_0 \qquad X = \int dy\, \phi^4(y)$$

$$\text{DENOMINATOR} = \sum_{n=0}^{\infty} \frac{(-g)^n}{n!} \langle (X)^n \rangle_0$$

LET'S APPLY WICK'S THEOREM FOR THE NUMERATOR

$$\langle O \overbrace{X\, X \cdots X}^{n \text{ TIMES}} \rangle_0 = \langle O \overbrace{X \cdots X}^{n-1 \text{ TIMES}} \rangle_0^{\text{CONNECTED}} * \langle X \rangle_0 * n$$

$$+ \langle O (X)^{n-2} \rangle_0^{\text{CONNECTED}} * \langle X X \rangle_0 * \binom{n}{2}$$

$$+ \cdots$$

$$= \sum_{p=0}^{n} \langle O (X)^{n-p} \rangle_0^{\text{CONNECTED}} \langle (X)^p \rangle_0 * \binom{n}{p}$$

$$\Rightarrow \text{NUMERATOR} = \sum_{n=0}^{\infty} \sum_{p=0}^{n} (-g)^n \frac{1}{(n-p)!} \frac{1}{p!} \langle O (X)^{n-p} \rangle_0^{\text{CONN}} \langle (X)^p \rangle_0$$

$$= \sum_{n=0}^{\infty} a_{n,p} = a_{0,0} + a_{1,0} + a_{1,1} + a_{2,0} + a_{2,1} + a_{2,2} + \cdots$$

$$\sum_{\alpha=0}^{\infty} a_{\alpha+2,\alpha} \qquad \sum_{\alpha=0}^{\infty} a_{\alpha+1,\alpha} \qquad \sum_{\alpha=0}^{\infty} a_{\alpha,\alpha}$$

$$= \sum_{\beta=0}^{\infty} \sum_{\alpha=0}^{\infty} a_{\alpha+\beta,\alpha} \qquad \text{WHERE} \quad \begin{cases} \alpha+\beta = n \\ \alpha = p \end{cases}$$

$$= \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} \frac{(-g)^{\alpha+\beta}}{\beta!\, \alpha!} \langle O (X)^{\beta} \rangle_0^{\text{CONN}} \langle (X)^{\alpha} \rangle_0$$

$$= \left(\sum_{n=0}^{\infty} \frac{(-g)^n}{n!} \langle O (X)^n \rangle_0^{\text{CONN}} \right) \underbrace{\left(\sum_{n=0}^{\infty} \frac{(-g)^n}{n!} \langle (X)^n \rangle_0 \right)}_{\text{DENOMINATOR}}$$

THEREFORE, ONLY THE CONNECTED DIAGRAMS CONTRIBUTE.

FINALLY, $\langle \mathcal{O} \rangle = \sum_{n\geq 0}^{\infty} \mathcal{O}_n = \sum_{n\geq 0}^{\infty} \langle \mathcal{O}(\times)^n \rangle_0^{\text{CONNECTED}}$

FOR THE CASE $\mathcal{O} = \phi(x)\,\phi(x')$

$$\Rightarrow \boxed{\langle \phi(x)\,\phi(x') \rangle = G(x-x') = \sum_{n=0}^{\infty} \left\langle \overset{x}{\bullet}\!-\!(\times)^n\!-\!\overset{x'}{\bullet} \right\rangle_0^{\text{CONNECTED}}}$$

- ALTHOUGH WE GOT RID OF THE DISCONNECTED DIAGRAMS, THE CONNECTED ONES STILL AMOUNTS A GREAT DEAL OF CALCULATIONS.

~~THERE ARE~~

- ENTHUSIASTS WRITE COMPUTER PROGRAMS FOR CALCULATING THE COMBINATORIES IN HIGHER ORDERS.

- FINALLY, NOTICE THIS THEOREM IS MORE GENERAL SINCE WE DID NOT ESPECIFIED THE VERTEX $\times$. AS LONG AS THE WICK'S THEOREM CAN BE USED, THEN THE THEOREM FOLLOWS.

PERTURBATION THEORY FOR THE JELLIUM MODEL

RECALL THE JELLIUM MODEL

$$H = \sum_{\vec{k},\sigma} c^{+}_{\vec{k}\sigma}\left(\frac{k^2}{2m}\right)c_{\vec{k}\sigma} + \frac{e^2}{2V}\sum_{\sigma_1\sigma_2}\sum_{\vec{k}_1\vec{k}_2}\sum_{\vec{q}\neq 0}\frac{4\pi}{q^2}\, c^{+}_{\vec{k}_1+\vec{q},\sigma_1}\, c^{+}_{\vec{k}_2-\vec{q},\sigma_2}\, c_{\vec{k}_2\sigma_2}\, c_{\vec{k}_1\sigma_1}$$

RECALL $\vec{q}\neq 0$ DUE TO CHARGE NEUTRALITY

WHICH WE DENOTE AS

$$H = \sum_{\vec{k},\sigma} c^{+}_{\vec{k}\sigma}\frac{k^2}{2m}c_{\vec{k}\sigma} + \frac{1}{2V}\sum_{\substack{\sigma_1\sigma_2\\ \vec{k}_1\vec{k}_2}}\sum_{\vec{q}\neq 0} c^{+}_{\vec{k}_1+\vec{q},\sigma_1}\, c^{+}_{\vec{k}_2-\vec{q},\sigma_2}\, \tilde{v}(\vec{q})\, c_{\vec{k}_2\sigma_2}\, c_{\vec{k}_1\sigma_1}$$

THE PARTITION FUNCTION IS THEN

$$Z = \int \mathcal{D}[\bar{\psi}(\tau),\psi(\tau)]\; e^{-S_E[\bar{\psi},\psi]} = \operatorname{tr} e^{-\beta(H-\mu N)}$$

WHERE $S_E = \int_0^\beta d\tau\; \bar{\psi}\cdot\left(\frac{\partial}{\partial\tau}-\mu\right)\psi + H[\bar{\psi},\psi]$

EXCHANGING TO THE MATSUBARA FREQUENCIES

$$\Rightarrow S_E = \sum_{\vec{k},\sigma,m} \bar{\psi}_{\vec{k},m,\sigma}\left(-i\omega_m + \frac{k^2}{2m} - \mu\right)\psi_{\vec{k},m,\sigma}$$

$$+ \frac{1}{2\beta V}\sum_{\substack{\vec{k}_1\vec{k}_2\\ \sigma_1\sigma_2}}\sum_{\substack{m_1 m_2\\ \vec{q}\\ m}} \bar{\psi}_{\vec{k}_1+\vec{q},m_1+m,\sigma_1}\, \bar{\psi}_{\vec{k}_2-\vec{q},m_2-m,\sigma_2}\, \tilde{v}(\vec{q})\, \psi_{\vec{k}_2,m_2,\sigma_2}\, \psi_{\vec{k}_1,m_1,\sigma_1}$$

NOTATION: "4-MOMENTUM" $p = (\vec{k},\omega_m)$

$$\Rightarrow S_E \equiv S = \sum_{p,\sigma} \bar{\psi}_{p,\sigma}\left(-i\omega_n + \frac{k^2}{2m} - \mu\right)\psi_{p,\sigma}$$

$$+ \frac{T}{2V} \sum_{\substack{p_1 p_2 q \\ \sigma_1 \sigma_2}} \bar{\psi}_{p_1+q,\sigma_1} \bar{\psi}_{p_2-q,\sigma_2} \tilde{v}(q) \psi_{p_1,\sigma_1} \psi_{p_2,\sigma_2}$$

$\boxed{\text{NOTE: } \sum_q = \sum_{\vec{q}\,\omega_q} \text{ AND } \tilde{v}(q) = \tilde{v}(\vec{q})}$

FOCUS OF THE FREE ENERGY IN ORDER TO PERFORM ~~THE~~ PERTURBATION THEORY

* OBS: IN MOST BOOKS, AUTHORS FOCUS ON THE GREEN'S FUNCTION

ZERO ORDER (NON-INTERACTING CASE $\tilde{v} = 0$)

$$F_0 = -k_BT \ln Z = -k_BT \ln\left(\prod_{\vec{k},n,\sigma}\left(\beta\left(\frac{k^2}{2m} - \mu - i\omega_n\right)\right)^{-\zeta}\right)$$

(SUM OVER THE MATSUBARA FREQUENCIES) $= \zeta k_BT \sum_{\vec{k},\sigma} \ln\left(1 - \zeta e^{-\beta\left(\frac{k^2}{2m} - \mu\right)}\right) =$

$(k_B = 1)$ $$= -T \sum_{\vec{k},\sigma} \ln\left(1 + e^{-\beta\left(\frac{k^2}{2m} - \mu\right)}\right)$$

$$\xrightarrow{T=0} -\lim_{T\to 0} T \sum_{\frac{k^2}{2m} < \mu} \sum_{\sigma} \beta\left(\frac{k^2}{2m} - \mu\right)$$

$$= \sum_{\sigma} \sum_{\frac{k^2}{2m} < \mu} \frac{k^2}{2m} - \mu$$

$=$ GROUND STATE ENERGY OF THE FERMI SEA WITH RESPECT TO THE FERMI ENERGY μ

(CHEMICAL POTENTIAL = FERMI ENERGY WHEN $T=0$) $T \to 0$

$$\sum_\sigma \sum_{k<k_F} \frac{k^2}{2m} = 2\frac{V}{(2\pi)^3}\int d\vec{k}\,\frac{k^2}{2m} = \frac{2V}{(2\pi)^3}\times\frac{4\pi}{2m}\frac{k_F^5}{5}$$ (22)

RECALL THAT $k_F = \sqrt{2m\mu}$

$$\sum_\sigma \sum_{k<k_F} (-\mu) = \frac{2V}{(2\pi)^3}(-\mu)\frac{4}{3}\pi k_F^3$$

$$= (-\mu)\sum_{\sigma, k<k_F}(1) = -\mu * \text{No. OF OCCUPIED STATES} = -\mu N$$

$$\Rightarrow F_0 = \frac{2V}{(2\pi)^3}4\pi\left[\frac{k_F^5}{10m} - \frac{\mu k_F^3}{3}\right] = \frac{2V}{(2\pi)^3}4\pi\left[\frac{-2}{15}\mu k_F^3\right]$$

$$= \frac{2V}{(2\pi)^3}\frac{4}{3}\pi k_F^3\left(-\frac{2}{5}\mu\right) = \boxed{-\frac{2}{5}N\mu = F_0}$$

RECALL THE DIMENSIONLESS DENSITY PARAMETER

$$r_s = 2\frac{\text{MEAN DISTANCE BETWEEN } e\text{'s}}{\text{BOHR RADIUS}} = \frac{\text{MEAN COULOMB ENERGY}}{\text{KINETIC ENERGY}}$$

$$= \frac{e^2/r_0}{\frac{(\Delta P)^2}{2m}} = \frac{e^2/r_0}{\frac{\hbar^2/r_0^2}{2m}} = \frac{2e^2 m}{\hbar^2}\times r_0 \quad , \quad \text{SINCE } a_0 = \frac{\hbar^2}{me^2}$$

$$= 2\frac{r_0}{a_0} \rightarrow \boxed{r_s = \frac{r_0}{a_0}}$$

IGNORE THIS FACTOR

IN TERMS OF THE RYDBERG ENERGY : $E_{RY} = \frac{me^4}{2\hbar^2} = 13.6\,eV$

$$\Rightarrow \frac{F_0}{E_{RY}} \sim \frac{N\mu}{\frac{me^4}{2\hbar^2}} \sim \frac{N}{\frac{me^4}{\hbar^2}}\times\frac{1}{r_0^2 m} = N\frac{a_0^2}{r_0^2} \qquad \text{SINCE } \frac{2V}{(2\pi)^3}\frac{4}{3}\pi k_F^3 = N$$

$$V = \frac{4}{3}\pi r_0^3 N$$

$$\Rightarrow r_0^3 (m\mu)^{3/2} \sim 1$$

$$\boxed{\frac{F_0}{E_{RY}} \sim \frac{N}{r_s^2}}$$

$\rightarrow$ JUST FOR GETTING A FEELING ON THE ORDER OF MAGNITUDE

FREE ENERGY SERIES

$$F = -T \ln Z = -T \ln\left(\int D[\bar\psi,\psi]\, e^{-S_0 - S_{INT}}\right)$$

$$= -T \ln\left[\int e^{-S_0} - \int e^{-S_0} S_{INT} + \frac{1}{2!}\int e^{-S_0} S_{INT}^2 - \cdots\right]$$

$$= -T \ln\left\{\int e^{-S_0}\left(1 - \frac{\int e^{-S_0} S_{INT}}{\int e^{-S_0}} + \frac{\int e^{-S_0} S_{INT}^2}{2!\int e^{-S_0}} - \cdots\right)\right\}$$

$$= -T \ln\left(\int e^{-S_0}\right) - T \ln\left(1 - \underbrace{\frac{\int e^{-S_0} S_{INT}}{\int e^{-S_0}}}_{\langle S_{INT}\rangle_0} + \underbrace{\frac{\int e^{-S_0} S_{INT}^2}{\int e^{-S_0}}}_{\frac{1}{2!}\langle S_{INT}^2\rangle_0} - \cdots\right)$$

$$= F_0 - T\left\{ -\langle S_{INT}\rangle_0 + \frac{1}{2!}\left(\langle S_{INT}^2\rangle_0 - \langle S_{INT}\rangle_0^2\right) - \frac{1}{3!}\left(\langle S_{INT}^3\rangle_0 - 3\langle S_{INT}^2\rangle_0\langle S_{INT}\rangle_0 + 2\langle S_{INT}\rangle_0^3\right) + \frac{1}{4!}\left(\langle S_{INT}^4\rangle_0 - 4\langle S_{INT}^3\rangle_0\langle S_{INT}\rangle_0 - 3\langle S_{INT}^2\rangle_0^2 + 12\langle S_{INT}^2\rangle_0\langle S_{INT}\rangle_0^2 - 6\langle S_{INT}\rangle_0^4\right) \cdots \right\}$$

→ JUST LIKE A CUMULANT EXPANSION

1st ORDER CORRECTION

$$F_1 = T\langle S_{INT}\rangle_0 = \frac{T^2}{2V}\sum_{\substack{p_1 p_2 q\\ \sigma_1\sigma_2}} \tilde v(q)\left\langle \bar\psi_{p_1+q,\sigma_1}\bar\psi_{p_2-q,\sigma_2}\psi_{p_2\sigma_2}\psi_{p_1\sigma_1}\right\rangle_0$$

USING WICK'S THEOREM, WE CONTRACT

$$F_1 = \frac{T^2}{2V}\sum_{\substack{p_1 p_2 q\\ \sigma_1\sigma_2}} \tilde v(q)\left\{\langle\bar\psi_{p_1+q,\sigma_1}\psi_{p_1,\sigma_1}\rangle_0\langle\bar\psi_{p_2-q,\sigma_2}\psi_{p_2,\sigma_2}\rangle_0 - \langle\bar\psi_{p_1+q,\sigma_1}\psi_{p_2,\sigma_2}\rangle_0\langle\bar\psi_{p_2-q,\sigma_2}\psi_{p_1\sigma_1}\rangle_0\right\}$$

- NOTICE THE MINUS SIGN IN THE 2ND TERM BECAUSE THESE ARE GRASSMANN VARIABLES

THESE MEAN VALUES ARE RELATED TO THE GREEN'S FUNCTION

$$\boxed{\begin{aligned} \mathcal{G}_0(P_1, P_2) &\equiv \langle \psi_{P_1\sigma_1} \bar{\psi}_{P_2\sigma_2} \rangle_0 \\ &= \frac{\delta_{\vec{k}_1,\vec{k}_2} * \delta_{m_1,m_2} * \delta_{\sigma_1,\sigma_2}}{\frac{k_1^2}{2m} - \mu - i\omega_{m_1}} \end{aligned}}$$

$$= \delta_{\vec{k}_1\vec{k}_2}\, \delta_{m_1 m_2}\, \delta_{\sigma_1\sigma_2} * \mathcal{G}_0(P_1)$$

WHERE WE ARE SHORTENING THE NOTATION $\mathcal{G}_0(P_1, \sigma_1) \to \mathcal{G}_0(P_1)$

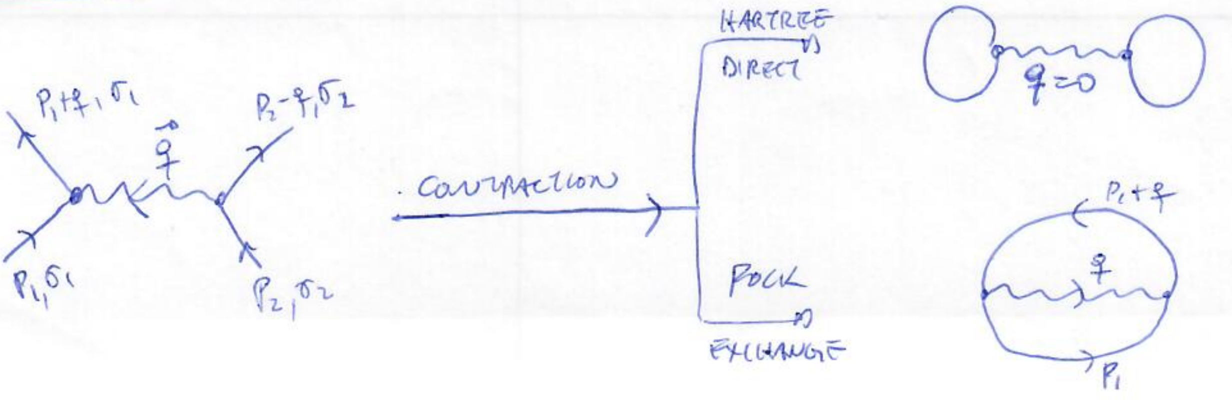
THEREFORE,

$$F_1 = \frac{T^2}{2V} \sum_{\substack{P_1 P_2 q \\ \sigma_1 \sigma_2}} \Big\{ \tilde{v}(q)\, \mathcal{G}_0(P_1)\, \delta_{\vec{q},0}\, \mathcal{G}_0(P_2)\, \delta_{\vec{q},0} - \tilde{v}(q)\, \mathcal{G}_0(P_1+q)\, \mathcal{G}_0(P_1)\, \delta_{P_2, P_1+q}\, \delta_{\sigma_1\sigma_2} \Big\}$$

1ST TERM $\to$ CALLED HARTREE (OR DIRECT) CONTRIBUTION

2ND TERM $\to$ " FOCK (OR EXCHANGE) CONTRIBUTION

CORRESPONDING FEYNMAN DIAGRAMS



THE HARTREE CONTRIBUTION IS NULL BECAUSE
$\tilde{v}(q=0)=0$ DUE TO CHARGE NEUTRALITY

THUS, $F_1 = -\frac{T^2}{2V} \sum_{p p'} \sum_{\sigma} \tilde{v}(p-p')\, \mathcal{G}_0(p)\, \mathcal{G}_0(p')$

$$= -\frac{T^2}{2V} \sum_{\vec{p}\,\vec{p}'} \sum_{m, m'} \sum_{\sigma} \tilde{v}(\vec{p}-\vec{p}')\, \frac{1}{\frac{p^2}{2m}-\mu-i\omega_m} \times \frac{1}{\frac{p'^2}{2m}-\mu-i\omega_{m'}}$$

RECALL THAT $\langle n_j \rangle = \frac{1}{S\beta} \sum_m \frac{1}{\epsilon_j - \mu - i\omega_m}$ (SEE PAGE 6 NOTES: PARTITION FUNCTION)

THUS, $F_1 = -\frac{T^2}{V} (S\beta)^2 \sum_{\vec{p}, \vec{p}'} \tilde{v}(\vec{p}-\vec{p}') \langle n_{\vec{p}} \rangle \langle n_{\vec{p}'} \rangle$

WHERE $\langle n_{\vec{p}} \rangle = \frac{1}{e^{\beta(\frac{p^2}{2m}-\mu)}+1}$ $\equiv$ FERMI-DIRAC DISTRIBUTION

FOR $T \to 0 \Rightarrow \langle n_{\vec{p}} \rangle = \Theta(k_F - p)$

$$\Rightarrow F_1 = -\frac{1}{V} \sum_{p, p' < k_F} \frac{4\pi e^2}{|\vec{p}-\vec{p}'|^2} = -\frac{1}{V}\left(\frac{V}{(2\pi)^3}\right)^2 4\pi e^2 \int d\vec{p}\, d\vec{p}'\, \frac{1}{p^2 - 2\vec{p}\cdot\vec{p}' + p'^2}$$

INTEGRATING OVER $\vec{p}'$:

$$I = \int \frac{d\vec{p}'}{p^2 - 2\vec{p}\cdot\vec{p}' + p'^2} = 2\pi \int_0^{\pi} d\theta \int_0^{k_F} dp'\, \frac{p'^2 \sin\theta}{p^2 + p'^2 - 2pp'\cos\theta} = 2\pi \int_0^{k_F} dp'\, \frac{p'^2}{2pp'} \ln\left(p^2 + p'^2 - 2pp'\cos\theta\right)\Big|_0^{\pi}$$

$$= \frac{\pi}{p} \int_0^{k_F} dp'\, p' \ln\frac{(p+p')^2}{(p-p')^2} = \frac{\pi p}{\cancel{p}} \int_0^{\frac{k_F}{p}} dx\, x \ln\left(\frac{1+x}{1-x}\right)^2$$

$$\left(\frac{x-1}{2}\right)\left(4 + (x+1)\ln\left(\frac{1+x}{1-x}\right)^2\right)\Big|_0^{\frac{k_F}{p}}$$

$$= \frac{2}{p/k_F} + \left(\frac{1}{(p/k_F)^2} - 1\right) \ln\left|\frac{1+\frac{1}{p/k_F}}{1-1/p/k_F}\right|$$

$$\Rightarrow I = \pi k_F \left[2 + \left(\frac{k_F}{p} - \frac{p}{k_F} \right) \ln \left| \frac{1 + p/k_F}{1 - p/k_F} \right| \right]$$

$$= 4\pi k_F \, \mathcal{F}(p/k_F) \,, \quad \text{WHERE}$$

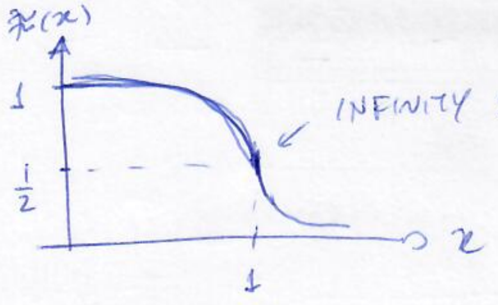
$$\boxed{\mathcal{F}(x) = \frac{1}{2} - \frac{1}{4}\left(x - \frac{1}{x} \right) \ln \left| \frac{1+x}{1-x} \right| = \text{LINDHARD'S FUNCTION}}$$

THUS, $F_1 = -\frac{V}{(2\pi)^6} 4\pi e^2 \int d\vec{p} \; 4\pi k_F \mathcal{F}(p/k_F)$

$$= \frac{-V}{(2\pi)^6} (4\pi)^3 k_F e^2 \underbrace{\int_0^{k_F} p^2 \mathcal{F}(p/k_F) \, dp}_{k_F^3 \underbrace{\int_0^1 x^2 \mathcal{F}(x) \, dx}_{= \frac{1}{4}}}$$

$$\boxed{F_1 = -\frac{V (4\pi)^3}{4 (2\pi)^6} e^2 k_F^4 = -4\pi V \frac{e^2}{(2\pi)^4} k_F^4}$$

CURIOSITY:



INFINITY SLOPE (BUT ONLY LOG-INFINITY)

NOTICE ALSO THAT $F_1 \sim V e^2 k_F^4 \sim N a_0^3 e^2 (m\mu)^2$

$$\sim N a_0^3 e^2 \frac{1}{r_0^4}$$

$$\boxed{\frac{F_1}{E_{Ry}} \sim \frac{N a_0^3 e^2}{\frac{m e^4}{\hbar^2} r_0^4} \sim N \frac{a_0}{r_0} \sim \frac{N}{r_s}}$$

THESE CALCULATIONS ARE THE SAME AS WE DID (27)
IN THE BEGINNING OF THE COURSE FOR THE JELLIUM MODEL. IF WE PLUG BACK THE PREFACTORS

$$\Rightarrow \quad E_{HF} = \underbrace{F_0 + F_1}_{\text{AT } T=0} \equiv \text{GROUND-STATE ENERGY IN THE HARTREE-FOCK APPROXIMATION}$$

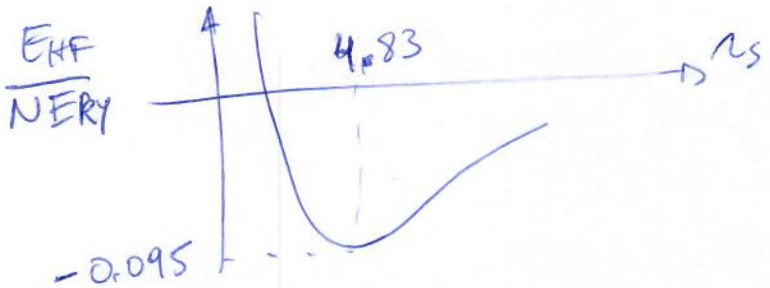
$$\Rightarrow \frac{E_{HF}}{E_{RY}} \cong N\left(\frac{2.210}{r_s^2} - \frac{0.916}{r_s} \right) \rightarrow$$

NOW MEASURED WITH RESPECT TO THE MINIMUM KINETIC ENERGY LEVEL $K=0$

$\Rightarrow F_0 \Rightarrow F_0 + N\mu = \frac{3}{5} N\mu$

DUE TO ANTI-COMMUTATION (PAULI PRINCIPLE) $\boxed{\text{IMPLIES IN AN ENERGY GAIN}}$

AT THAT TIME OF THE COURSE,
WE MINIMIZED E_{HF} WITH RESPECT TO r_s



COMPARING WITH METALLIC SODIUM

$r_s \simeq 3.96$

$\frac{E_{GS}}{NE_{RY}} \simeq -0.065$

WE THEN CONCLUDE THAT THIS SIMPLE THEORY IS VERY GOOD.

HOWEVER, NOT ALL IS GOOD, AND WE SHALL NOW SEE WHY. BUT BEFORE THAT, LET US TELL YOU ~~THAT~~ THIS APPROXIMATION IS OKAY. THE REASON WHY IT WILL GIVE NON PHYSICAL RESULT (IN A MOMENT) IS BECAUSE NEW PHYSICS MUST BE TAKEN INTO ACCOUNT WHICH IS NOT IN THIS LEVEL OF APPROXIMATION
WE WILL NEED HIGHER ORDERS IN PERT. THEORY.

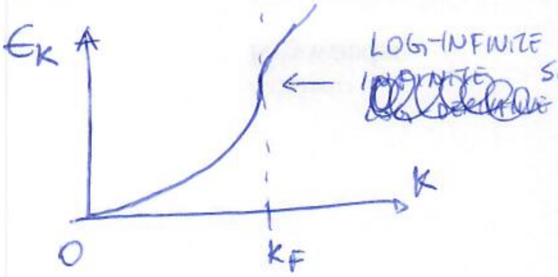
AT $T=0$, THE FREE ENERGY IS SIMPLY THE GROUND STATE ENERGY. THUS,

$$E_{HF} = \underbrace{F_0 + F_1}_{\text{AT } T=0} = 2\sum_{p<k_F}\left[\underbrace{\frac{p^2}{2m} - \mu}_{\text{UNPERTURBED EIGEN ENERGIES}} \; \underbrace{- \frac{1}{2V}\sum_{q<k_F}\frac{4\pi e^2}{|\vec{p}-\vec{q}|^2}}_{\text{1st ORDER CORRECTION TO THE EIGEN ENERGIES DUE TO INTERACTIONS}}\right]$$

$$= 2\sum_{p<k_F}\left\{\frac{p^2}{2m} - \mu - \frac{1}{2V}\frac{V e^2}{(2\pi)^3}(4\pi)^2 k_F\, \mathcal{F}(p/k_F)\right\}$$

$$= 2\sum_{p<k_F}\left\{\underbrace{\frac{p^2}{2m} - \frac{e^2}{\pi}k_F\,\mathcal{F}(p/k_F)}_{\epsilon_p = \epsilon_p^{(0)} + \epsilon_p^{(1)}} - \mu\right\}$$

$$\boxed{\text{DEFINITION: } \epsilon_{\vec{p},\sigma} \equiv \epsilon^{(0)}_{\vec{p},\sigma} + \text{SELF-ENERGY} = \epsilon^{(0)}_{\vec{p},\sigma} + \Sigma_{\vec{p},\sigma} \quad (\text{MORE ON THE SELF-ENERGY LATER})}$$



$\longrightarrow$ CONSEQUENCES TO THE DENSITY OF STATES

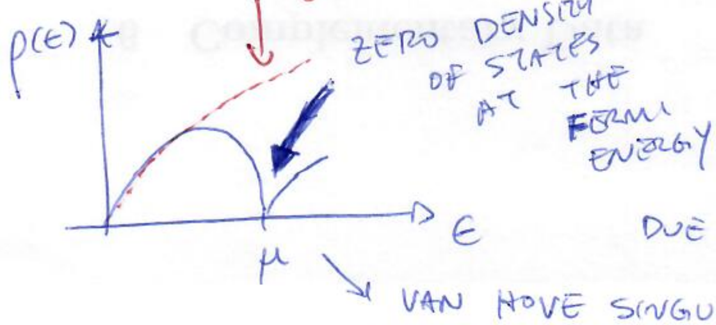
$$\rho(\epsilon) = \sum_{\vec{k},\sigma}\delta(\epsilon - \epsilon_{\vec{k},\sigma})$$

$$= \frac{V}{\pi^2}\int_0^\infty \delta(\epsilon_{\vec{k},\sigma} - \epsilon)\, k^2\, dk$$

$$= \frac{V}{\pi^2}\frac{k_\epsilon^2}{\left|\frac{d\epsilon_k}{dk}\right|_{k=k_\epsilon}}$$

WHERE k_ϵ SOLVES $\epsilon_{\vec{k}_\epsilon,\sigma} = \epsilon$

THEREFORE,



DUE TO THE INFINITY SLOPE OF $\mathcal{F}$

VAN HOVE SINGULARITY

THIS SINGULARITY IS BECAUSE OF THE LONG-RANGE CHARACTER OF THE COULOMB POTENTIAL $\rightarrow \frac{1}{r} \rightarrow \frac{1}{q^2}$ AND THE SINGULARITY APPEARS WHEN INTEGRATING NEAR $q=0$

- WHAT IS THE PROBLEM WITH THIS VANISHING DENSITY OF STATES (DOS)?

  - NOT OBSERVED EXPERIMENTALLY
  - MANY PROPERTIES OF A METAL (AS CONDUCTIVITY) ARE PROPORTIONAL TO $\rho(E_F)$. THUS, WE EXPECT THAT OUR THEORY DOES NOT PREDICT ~~THAT~~ A VANISHING D.O.S. AT $E=\mu=E_F$.

PHYSICALLY, REPULSION IS VERY STRONG YIELDING FEW STATES AROUND E_F.

- ALL WE ARE DOING IS THEN WRONG?

  NO. WHAT WE NEED IS TO REALIZE THAT A SHORT-RANGE COULOMB INTERACTION WILL NOT PRODUCE THAT SINGULARITY AT $E=E_F$.

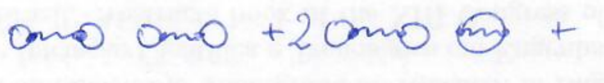
WE EXPECT THAT THE COULOMB POTENTIAL BE SCREENED BY THE POLARIZATION OF THE ELECTRON GAS ITSELF, BUT THIS PHENOMENA APPEARS IN HIGHER ORDER OF PERTURBATION THEORY. THEY WILL REMOVE THE VAN HOVE SINGULARITY FROM $E=E_F$.

2ND ORDER: THE PURPOSE IS TO GET FAMILIAR WITH SOME DIAGRAMS AND NOTICE WHAT PROCESSES ARE MORE IMPORTANT. (DIAGRAMS)

$$F_2 = \frac{-T}{2!}\left(\langle S_{INT}^2 \rangle_0 - \langle S_{INT} \rangle_0^2 \right)$$

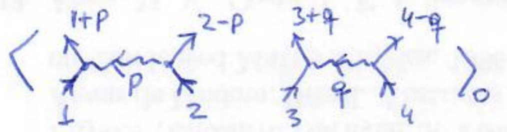
CONTRACTIONS: $\langle S_{INT} \rangle_0 =$

$\Rightarrow \langle S_{INT} \rangle_0^2 =$



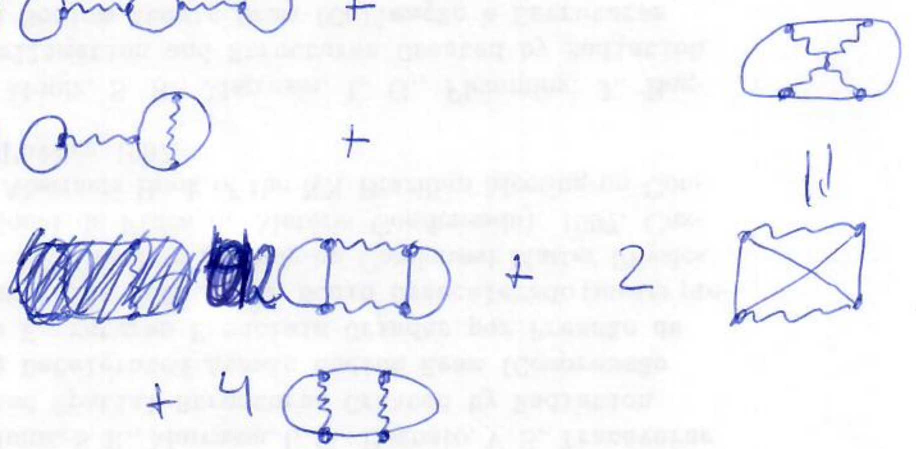
DISCONNECTED DIAGRAMS

$\langle S_{INT}^2 \rangle_0 =$



$= \langle S_{INT} \rangle_0^2 +$ CONNECTED DIAGRAMS

CONNECTED DIAGRAMS $=$



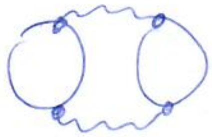
DISCONNECTED DIAGRAMS CANCELS OUT (IN ACCORDANCE WITH THE LINKED CLUSTER THEOREM)

$$\Rightarrow F_2 = -\frac{T}{2}\langle S_{INT}^2 \rangle_0^{\text{CONNECTED}}$$

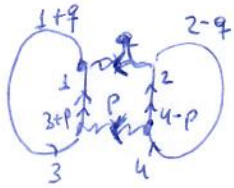
•

$= 0$ BECAUSE $\tilde{V}(q=0) = 0$ (CHARGE NEUTRALITY)

LET US WRITE DOWN THE CORRESPONDING VALUES (31)



$\rightarrow$



$$\rightarrow \left(\frac{T}{2V}\right)^2 \sum_{\substack{1,2,q\\3,4,p}} \tilde{v}(q)\tilde{v}(p)\, \mathcal{G}_0(1)\, \delta_{\sigma_1\sigma_3}\, \delta_{3+p,1} \,*$$

$$*\ \mathcal{G}_0(3)\, \delta_{1+q,3}\ *$$

WHICH IS RELATED TO

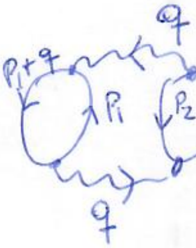
$$*\ \mathcal{G}_0(2)\, \mathcal{G}_0(4)\, \delta_{4-p,2}\, \delta_{4,2-q}\, \delta_{\sigma_2\sigma_4}$$

$$\left(\frac{T}{2V}\right)^2 \sum_{\substack{1,2,q\\3,4,p}} \tilde{v}(q)\tilde{v}(p) \left\langle \bar{\psi}_{1+q\,\sigma_1} \bar{\psi}_{2-q\,\sigma_2} \psi_{2\sigma_2} \psi_{1\sigma_1} \bar{\psi}_{3+p\,\sigma_3} \bar{\psi}_{4-p\,\sigma_4} \psi_{4\sigma_4} \psi_{3\sigma_3} \right\rangle_0$$

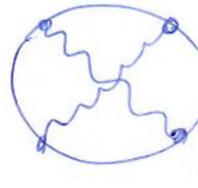
$$\rightarrow \left\langle \bar{\psi}_{1+q\,\sigma_1} \psi_{3\sigma_3} \right\rangle_0 \left\langle \bar{\psi}_{2-q\,\sigma_2} \psi_{4\sigma_4} \right\rangle_0 \left\langle \psi_{2\sigma_2} \bar{\psi}_{4-p\,\sigma_4} \right\rangle_0 \left\langle \psi_{1\sigma_1} \bar{\psi}_{3+p\,\sigma_3} \right\rangle_0$$

$$\boxed{\text{NOTICE THE EVEN No. OF PERMUTATIONS} \Rightarrow \text{NO MINUS SIGN}}$$

$\Rightarrow$



$$= \left(\frac{T}{2V}\right)^2 * \underset{\text{SPIN}}{4} \times \sum_{p_1 p_2 q} \tilde{v}(q)^2\, \mathcal{G}_0(p_1+q)\, \mathcal{G}_0(p_1)\, \mathcal{G}_0(p_2+q)\, \mathcal{G}_0(p_2)$$



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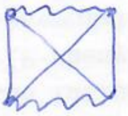

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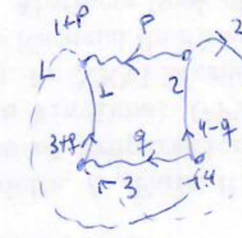
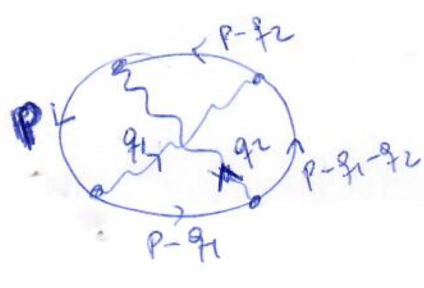


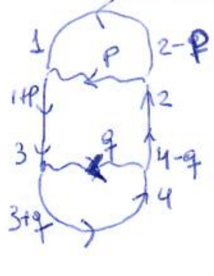
$$\rightarrow \left\langle \bar{\psi}_{1+p} \bar{\psi}_{2-p} \psi_2 \psi_1 \bar{\psi}_{3+q} \bar{\psi}_{4-q} \psi_4 \psi_3 \right\rangle_0$$

WHEN CONTRACTING, THERE WILL BE AN ODD NUMBER OF PERMUTATIONS $\Rightarrow$ NET MINUS SIGN



$$= \left(\frac{T}{2V}\right)^2 \times \underset{\text{SPIN}}{2} * \left(- \sum_{q_1 q_2 p} \tilde{v}(q_1)\tilde{v}(q_2)\, \mathcal{G}_0(p)\, \mathcal{G}_0(p-q_1)\, \mathcal{G}_0(p-q_1-q_2)\, * \right.$$

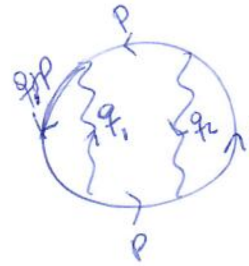
$$\mathcal{G}_0(p-q_2)$$

 $\to$ 

$$\to \left\langle \bar\psi_{1+p}\,\bar\psi_{2-p}\,\psi_2\,\psi_1\,\bar\psi_{3+q}\,\bar\psi_{4-q}\,\psi_4\,\psi_3 \right\rangle_0$$

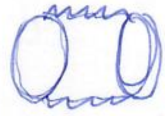
$$\Rightarrow \langle \bar\psi_{1+p}\psi_3\rangle_0 \langle \bar\psi_{2-p}\psi_1\rangle_0 \langle \bar\psi_{3+q}\psi_4\rangle_0 \langle \psi_2\bar\psi_{4-q}\rangle_0$$

EVEN No. OF PERMUTATIONS $\to$ NO MINUS SIGN



$$\to \left(\frac{T}{2V}\right)^2 \times \underset{\text{SPIN}}{2} \times \sum_{q_1 q_2 p} \tilde v^2(q_1)\,\tilde v^2(q_2)\,\mathcal{G}_0^2(p)\,\mathcal{G}_0(p+q_1)\,\mathcal{G}_0(p+q_2)$$

THEREFORE,

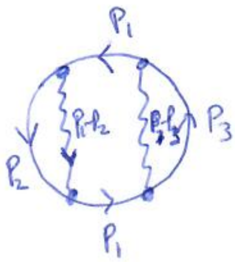
$$F_2 = -\frac{T^3}{V^2}\sum_{p_1 p_2 q} \tilde v^2(q)\,\mathcal{G}_0(p_1+q)\,\mathcal{G}_0(p_1)\,\mathcal{G}_0(p_2+q)\,\mathcal{G}_0(p_2) \to$$ 

$$+\frac{T^3}{2V^2}\sum_{q_1 q_2 p} \tilde v^2(q_1)\,\tilde v^2(q_2)\,\mathcal{G}_0(p)\,\mathcal{G}_0(p+q_1)\,\mathcal{G}_0(p-q_2)\,\mathcal{G}_0(p-q_1-q_2) \to$$ 

$$-\frac{T}{V^2}\sum_{q_1 q_2 p} \tilde v(q_1)\,\tilde v(q_2)\,\mathcal{G}_0^2(p)\,\mathcal{G}_0(p+q_1)\,\mathcal{G}_0(p+q_2) \to$$ 

$\to$ BECAUSE OF $\tilde v^2(q)$ OF THE 1ST TERM $\to F_2$ IS DOMINATED BY . IN THE OTHER TERMS $\tilde v(q_1)\tilde v(q_2)$ GIVES SMALLER CONTRIBUTION BECAUSE, WHEN $q_1 = q_2$ THE PHASE SPACE IS DIMINISHED BY THE GREEN'S FUNCTIONS $\mathcal{G}(p-q) \sim \Theta(k_F - |\vec p - \vec q|)$

LET US NOW COMPUTE THE VALUE OF F_2



$$\rightarrow \; -\frac{T}{V^2} \sum_{P_1 P_2 P_3} \tilde{v}(\vec{P}_1-\vec{P}_2)\,\tilde{v}(\vec{P}_1-\vec{P}_3)\,\mathcal{G}_0^2(P_1)\,\mathcal{G}_0(P_2)\,\mathcal{G}_0(P_3)$$

LET US PERFORM THE MATSUBARA FREQUENCY SUMMATION OVER THE INDEX n_1

- IT ONLY SHOWS UP IN $\mathcal{G}_0^2(P_1) = \mathcal{G}_0(\vec{P}_1, n_1)$

$$\Rightarrow \sum_{n_1} \mathcal{G}_0^2(\vec{P}_1, n_1) = \sum_n \frac{1}{\left(\frac{P_1^2}{2m} - \mu - i\omega_n\right)^2} = \sum_n h(\omega_n)$$

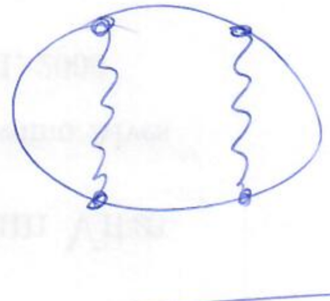
$$= -\sum_{\substack{\text{RESIDUES} \\ \text{OF} \\ h}} \text{RESIDUES}\left(\frac{\zeta\beta}{e^{\beta z}-\zeta} * \frac{1}{\left(\frac{P_1^2}{2m}-\mu-z\right)^2}\right)$$

BUT THE RESIDUE OF $\frac{1}{\left(\frac{P_1^2}{2m}-\mu-z\right)^2}$ AT THE ONLY POLE $z = \frac{P_1^2}{2m} - \mu$

IS ZERO,

$$\text{RES}\left(\frac{1}{(z-a)^2}\right)\Bigg|_{z=a} = 0$$

$\Rightarrow$



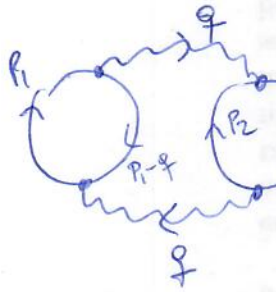
$= 0$

NO CONTRIBUTION FROM THIS DIAGRAM

HOW ABOUT THE OTHER 2 DIAGRAMS?

$$= \frac{T^3 V (4\pi)^2 e^4}{2(2\pi)^9} \int \frac{d\vec{q}_1}{q_1^2} \int \frac{d\vec{q}_2}{q_2^2} * \underbrace{\int d\vec{p} \sum_{\substack{n_1 n_2 \\ n}} \mathcal{G}_0(\vec{p}, n)\, \mathcal{G}_0(p-q_1)\, \mathcal{G}_0(p-q_2)\, \mathcal{G}_0(p+q_1+q_2)}_{\text{FUNCTION OF } \vec{q}_1 \text{ AND } \vec{q}_2}$$

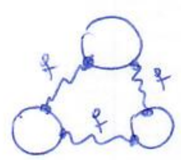
$$\sim \left(\int \frac{d\vec{q}}{q^2} \right)^2 \sim \left(\int_0^{k_{MAX}} q^2 \frac{dq}{q^2} \right)^2 \sim \text{FINITE CONTRIBUTION FOR } q \to 0$$



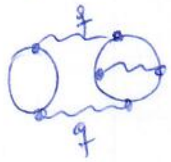
$$= \frac{TV}{2(2\pi)^9} (4\pi e^2)^2 \int \frac{d\vec{q}}{q^4} * \int d\vec{p}_1 d\vec{p}_2 \sum_{\substack{n_1 n_2 \\ n}} \mathcal{G}_0(p_1)\, \mathcal{G}_0(p_2)\, \mathcal{G}_0(p_1-q)\, \mathcal{G}_0(p_2+q)$$

$$\sim \int \frac{d\vec{q}}{q^4} \sim \int_0^{k_{MAX}} \frac{dq}{q^2} \to \infty \quad \text{CONTRIBUTION FOR } q \to 0$$

IN 3RD ORDER:



$$\sim \int \frac{d\vec{q}}{q^6} \sim \int \frac{dq}{q^4} \to \infty \quad (\text{MORE DIVERGENT})$$

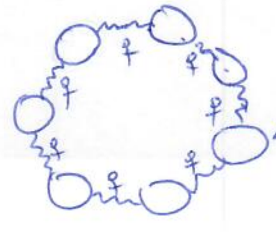


$$\sim \int \frac{d\vec{q}}{q^4} \sim \int \frac{dq}{q^2} \to \infty$$

$$\sim \int \frac{d\vec{q}}{q^2} \sim \int dq \to \text{FINITE}$$

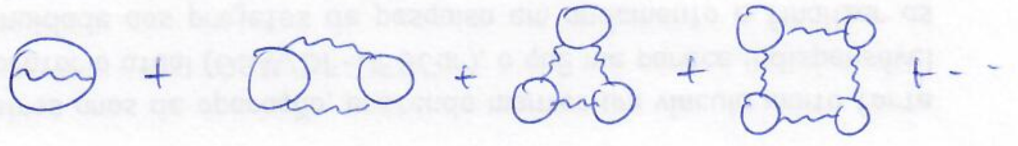
* IN GENERAL, THE MORE DIVERGENT DIAGRAMS ARE THOSE IN WHICH THERE ARE n "BUBBLES" OF PARTICLE-HOLE PROPAGATORS → ALL INTERACTION LINES CARRY THE SAME MOMENTUM.

THESE ARE THE RING DIAGRAMS →



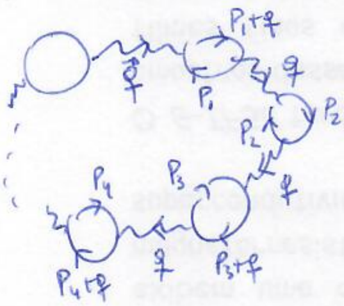
RANDOM PHASE APPROXIMATION (RPA)

WE HAVE IDENTIFIED THE LEADING DIAGRAMS (RING DIAGRAMS) IN EACH ORDER OF PERTURBATION THEORY. THE RPA CONSISTS IN SUMMING ALL OF THEM AND NEGLECTING THE OTHER ONES

$F \approx F_0 +$  $+ \dots$

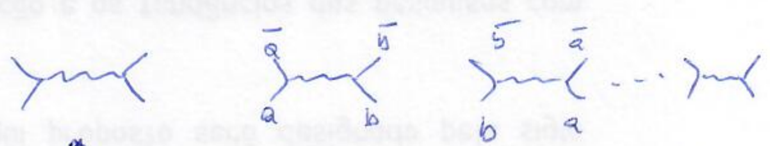
* WHERE WE ARE NOT PAYING ATTENTION TO THE COMBINATORIAL PREFACTORS THE PURPOSE IS ONLY TO IDENTIFY THE RING DIAGRAMS

n-TH ORDER RING DIAGRAM



- ALL INTERACTION LINES CARRY THE SAME MOMENTUM q
- THE COMBINATORIAL PREFACTOR IS SIMPLY $2^{n-1}(n-1)!$. THIS IS DUE TO THE CONTRACTIONS OF THE FOLLOWING KIND

THEREFORE, IN EACH ORDER OF P.T. WE HAVE

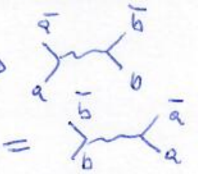


$-\frac{T(-1)^n}{n!}\langle S_{INT}^n\rangle_c$

FIX ONE S_{INT}

NOW, WE ASK WHAT ARE ALL POSSIBLE ARRANGEMENTS WITH THE REMAINING S_{INT}'s $= (n-1)!$

$\underset{\uparrow \text{ RPA}}{\cong} -\frac{T}{n}(-1)^n 2^{n-1}$ 

$2^{n-1} \rightarrow$ 

$$(-1)^n \sum_q \left[\frac{\beta T}{2V} \oint \sum_{\sigma,p} \tilde{v}(q)\, G_0(P)\, G_0(P+q)\right]^n$$

* NOTICE THAT IT IS NOT $n!$ BECAUSE OF THE "PERIODIC" BOUNDARY CONDITION OF THE RING DIAGRAM

COMES FROM PERMUTATION $\langle\bar{\psi}\psi\rangle_0 \rightarrow \langle\psi\bar{\psi}\rangle_0 = G_0$

$$\Rightarrow F_m^{(RPA)} = \frac{-2^m \, T^{m+1}}{2m \, V^m} \sum_{q} \left(\tilde{v}(\vec{q}) \sum_{p} \mathcal{G}_0(p) \mathcal{G}_0(p+q) \right)^m$$ 36

$$= \frac{-T}{2m} \sum_{q} \left(\frac{2T}{V} \tilde{v}(\vec{q}) \sum_{p} \mathcal{G}_0(p) \mathcal{G}_0(p+q) \right)^m$$

RECALLING THAT $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots = \sum_{m=1}^{\infty} -\frac{x^m}{m}$

$$\Rightarrow F^{(RPA)} = F_0 + \sum_{q} \frac{T}{2} \ln\left(1 - \tilde{v}(\vec{q}) \, \Pi^{(RPA)}(q)\right)$$

WHERE $\Pi^{(RPA)}(q) = \frac{2T}{V} \sum_{p} \mathcal{G}_0(p) \mathcal{G}_0(p+q)$

IS THE PROPER POLARIZATION OPERATOR (OR PROPER POLARIZATION INSERTION) IN THE RPA.

BEFORE COMPUTING Π^{RPA} (WHICH IS NOT SO SIMPLE) LET US ANALYZE ITS MEANING

- CONSIDER THE MEAN PARTICLE NUMBER $N = -\frac{\partial}{\partial \mu} F$ IN 1^{st} ORDER (SEE PAGE 25)

$$N^{(1)} = -\frac{\partial}{\partial \mu} F_1 = -\frac{\partial}{\partial \mu} \left(-\frac{T^2}{2V} \sum_{p,q,\sigma} \tilde{v}(\vec{q}) \mathcal{G}_0(p) \mathcal{G}_0(p+q) \right)$$

SINCE $\frac{\partial}{\partial \mu} \mathcal{G}_0(p) = \frac{\partial}{\partial \mu} \left(\frac{1}{\epsilon_p - \mu - i\omega_n} \right) = -\mathcal{G}_0^2(p)$

$$\Rightarrow \boxed{N^{(1)} = -\frac{2T^2}{V} \sum_{p,q} \tilde{v}(\vec{q}) \, \mathcal{G}_0^2(p) \mathcal{G}_0(p+q)}$$

HOW DO WE COMPARE $N^{(1)}$ IN THE RPA?

$$N^{(RPA)} = -\frac{\partial}{\partial\mu}\left(\frac{T}{2}\sum_{q}\ln\left(1-\tilde{v}(\vec{q})\,\Pi^{(RPA)}(q)\right)\right)$$

$$= -\frac{T}{2}\sum_{q}\frac{-\tilde{v}(\vec{q})}{1-\tilde{v}(\vec{q})\Pi}\underbrace{\left(\frac{\partial}{\partial\mu}\Pi^{(RPA)}(q)\right)}_{-\frac{2T}{V}\times 2\sum_{p}\mathcal{G}_0^2(p)\,\mathcal{G}_0(p+q)}$$

$$\Rightarrow \boxed{N^{(RPA)} = -\frac{2T^2}{V}\sum_{p,q}\tilde{v}_{eff}(\vec{q},\omega_n)\,\mathcal{G}_0^2(p)\,\mathcal{G}_0(p+q)}$$

THEREFORE, IN RPA, THE COULOMB INTERACTION IS REPLACED BY AN EFFECTIVE INTERACTION

$$\tilde{v}_{eff}(q) \equiv \frac{\tilde{v}(\vec{q})}{1-\tilde{v}(\vec{q})\Pi^{(RPA)}(q)} = \frac{1}{\tilde{v}^{-1}(\vec{q}) - \Pi^{(RPA)}(q)} \equiv \frac{\tilde{v}(\vec{q})}{\epsilon^{(RPA)}(q)}$$

WHERE WE HAVE INTRODUCED THE DIELECTRIC FUNCTION

$$\boxed{\epsilon(q) = 1-\tilde{v}(\vec{q})\,\Pi(q)}$$

* RECALL THAT $\vec{D}(\vec{q},\omega) = \epsilon(\vec{q},\omega)\,\vec{E}(\vec{q},\omega)$

$\vec{D}$: VACUUM FIELD

$\vec{E}$: ELECTRIC FIELD IN A MEDIUM

AS WE HAVE SAID ON PAGE 29, HIGHER ORDER TERMS IN PERTURBATION THEORY CONTAINS THE PHYSICS OF SCREENING, i.e., THE POLARIZATION OF THE ELECTRON GAS. THE FACT THAT $\epsilon^{(RPA)}(q) \neq 1$ IS THE PROOF OF THIS STATEMENT.

THE QUESTIONS NOW ARE: IS THIS NEW EFFECTIVE INTERACTION PHYSICAL? DOES IT REMOVE THE VAN HOVE SINGULARITY AT K_F?

WHAT NEW PHYSICS IS BROUGHT BY $\tilde{v}_{eff}$?

EVIDENTLY, IN ORDER TO ~~SOLVE~~ ANSWER THEM WE WILL HAVE TO COMPUTE $\tilde{v}_{eff}$ WHICH IS NOT AN EASY TASK

COMPUTING THE PROPER POLARIZATION INSERTION $\Pi^{(RPA)}(\vec{q},\omega_n)$

$$\Pi^{(RPA)} = \frac{2T}{V}\sum_{\vec{k}}\sum_{m} \mathcal{G}_0(\vec{k},\omega_m)\,\mathcal{G}_0(\vec{k}+\vec{q},\omega_m+\omega_n)$$

$$= \frac{2T}{V}\sum_{\vec{k}}\sum_{m} \frac{1}{\xi^{(0)}_{\vec{k}}-i\omega_m} \cdot \frac{1}{\xi^{(0)}_{\vec{k}+\vec{q}}-i\omega_{m+n}}$$

WHERE $\xi^{(0)}_{\vec{k}} = \frac{k^2}{2m} - \mu$

$$= \frac{2T}{V}\sum_{\vec{k}} \frac{1}{\xi^{(0)}_{\vec{k}+\vec{q}}-\xi^{(0)}_{\vec{k}}-i\omega_n} \sum_{m=-\infty}^{\infty}\left(\frac{1}{\xi^{(0)}_{k}-i\omega_m} - \underbrace{\frac{1}{\xi^{(0)}_{\vec{k}+\vec{q}}-i\omega_{m+n}}}_{\text{SHIFT}}\right)$$

SEE PAGE 25

$$= \frac{2T}{V}\sum_{\vec{k}} \frac{1}{\xi^{(0)}_{\vec{k}+\vec{q}}-\xi^{(0)}_{k}-i\omega_n}\,\beta\left(\langle n_{\vec{k}}\rangle - \langle n_{\vec{k}+\vec{q}}\rangle\right)$$

THE INTEGRAL OVER $\vec{k}$ IS HARD. AT $T\to 0$, IT SIMPLIFIES TO

$$\Pi^{RPA} \to \frac{2}{V}\sum_{\vec{k}} \frac{\theta(-|\vec{k}+\vec{q}|+k_F) - \theta(k_F-k)}{\xi^{0}_{\vec{k}+\vec{q}}-\xi^{(0)}_{k}-i\omega_n}$$

IN THE CONTINUUM LIMIT

$$\Pi^{(RPA)} \to \frac{2}{(2\pi)^3} \int d\vec{k} \; \frac{\Theta(k_F - |\vec{k}+\vec{q}|) - \Theta(k_F - k)}{\xi^{(0)}_{\vec{k}+\vec{q}} - \xi^{(0)}_{\vec{k}} - i\omega_m}$$

$$= \frac{2}{(2\pi)^3} \int d\vec{k} \; \frac{\Theta(|\vec{k}+\vec{q}| - k_F)\Theta(k_F - k) - \Theta(k_F - |\vec{k}+\vec{q}|)\Theta(k - k_F)}{\omega + \xi^{(0)}_{\vec{k}} - \xi^{(0)}_{\vec{k}+\vec{q}}}$$

WHERE $\omega = i\omega_m$ (A WICK ROTATION IN THE MATSUBARA FREQUENCY)

ACTUALLY, THERE ARE SOME SUBTLETIES IN DOING THE WICK ROTATION WHICH WE WILL NOT DISCUSS HERE.

IT TURNS OUT THAT THE $\vec{k}$-INTEGRAL CAN BE PERFORMED (SEE FETTER & WALECKA, SECT. 12)

THE RESULT IS THE FOLLOWING

$$\mathrm{Re}\left(\Pi^{(RPA)}\right) = \frac{2 m k_F}{(2\pi)^2} \left\{ -1 + \frac{1}{2\bar{q}} \left[1 - \left(\frac{\bar{\omega}}{\bar{q}} - \frac{\bar{q}}{2} \right)^2 \right] \ln \left| \frac{1 + \frac{\bar{\omega}}{\bar{q}} - \frac{\bar{q}}{2}}{1 - \frac{\bar{\omega}}{\bar{q}} + \frac{\bar{q}}{2}} \right| - \frac{1}{2\bar{q}} \left[1 - \left(\frac{\bar{\omega}}{\bar{q}} + \frac{\bar{q}}{2} \right)^2 \right] \ln \left| \frac{1 + \frac{\bar{\omega}}{\bar{q}} + \frac{\bar{q}}{2}}{1 - \frac{\bar{\omega}}{\bar{q}} - \frac{\bar{q}}{2}} \right| \right\}$$

WHERE
$$\begin{cases} \bar{q} = \dfrac{q}{k_F} \\ \bar{\omega} = \dfrac{\omega}{k_F^2/m} \end{cases}$$

THE IMAGINARY PART IS

$$\mathrm{Im}\left(\Pi^{(RPA)}\right) = -\frac{mk_F}{4\pi\bar{q}} * \begin{cases} 1-\left(\frac{\bar{\omega}}{\bar{q}}-\frac{\bar{q}}{2}\right)^2, & \text{IF } q>2k_F \text{ AND } \bar{q}>\left|\bar{\omega}-\frac{\bar{q}^2}{2}\right| \\ 2\bar{\omega}, & \text{IF } q<2k_F \text{ AND } 0\leq\bar{\omega}\leq\frac{1}{2}\bar{q}^2-\bar{q} \\ 0, & \text{OTHERWISE} \end{cases}$$

* NOTICE $\Pi^{RPA}(\vec{q},\omega) = \Pi^{RPA}(\vec{q},-\omega)$. HERE, WE ARE CONSIDERING ONLY THE CASE $\omega \geq 0$.

OUR TASK NOW IS MAKE SENSE OR THESE RESULTS

- THE STATIC LIMIT $\omega \to 0$

$$\mathrm{Re}\left(\Pi^{RPA}\right) = \frac{2mk_F}{(2\pi)^2}\left\{-1+\frac{1}{2\bar{q}}\left(1-\frac{\bar{q}^2}{4}\right)\ln\left|\frac{1-\bar{q}/2}{1+\bar{q}/2} * \frac{1-\bar{q}/2}{1+\bar{q}/2}\right|\right\}$$

$$= -\frac{mk_F}{\pi^2}\left\{\frac{1}{2}-\frac{1}{4}\left(\frac{\bar{q}}{2}-\frac{2}{\bar{q}}\right)\ln\left|\frac{1+\bar{q}/2}{1-\bar{q}/2}\right|\right\}$$

$$= -\frac{mk_F}{\pi^2}F\left(\frac{q}{2k_F}\right), \quad F(x) = \text{LINDHARD FUNCTION}$$

(SEE PAGE 26)

BACK INTO THE EFFECTIVE INTERACTION

$$\tilde{v}_{eff}(\vec{q},0) = \frac{1}{\frac{q^2}{4\pi e^2}+\frac{mk_F}{\pi^2}F\left(\frac{q}{2k_F}\right)} = \frac{\pi^2/mk_F}{\left(q/q_{TF}\right)^2+F\left(q/2k_F\right)}$$

WHERE $q_{TF}^2 = \frac{4me^2k_F}{\pi}$ → THOMAS-FERMI WAVEVECTOR

$$= \frac{4}{\pi}\left(\frac{4}{9\pi}\right)^{\frac{1}{3}} r_s k_F^2 \approx 0.66\, r_s k_F^2$$

$$\Rightarrow \boxed{q_{TF} \approx 0.81\sqrt{r_s}\, k_F}$$

* NOTICE THE RESULT $\sim \sqrt{r_s}$ $\rightarrow$ THIS CANNOT BE OBTAINED BY SIMPLE PERTURBATION THEORY. (WE HAVE SUMMED AN INFINITY OF DIAGRAMS.)

HOW THE COULOMB'S LAW IS MODIFIED?

$$q_{TF}^2 \frac{\tilde{v}_{eff}(\vec{q},0)}{4\pi e^2} = \frac{1}{(q/q_{TF})^2 + F\left(\frac{q}{2k_F}\right)}$$

$$\Rightarrow \frac{q_{TF}^2}{4\pi e^2} v_{eff}(\vec{r},0) = \int \frac{d\vec{q}}{(2\pi)^3} \frac{e^{-i\vec{q}\cdot\vec{r}}}{(q/q_{TF})^2 + F(q/2k_F)} = \frac{2\pi}{(2\pi)^3} \int \frac{q^2 \sin\theta \, e^{-iqr\cos\theta}}{(q/q_{TF})^2 + F\left(\frac{q}{2k_F}\right)} d\theta dq$$

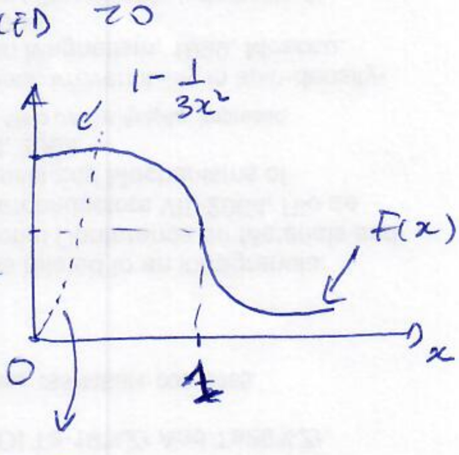
$$= \frac{2}{(2\pi)^2} \times \frac{1}{r} \int_0^\infty \frac{q \sin qr}{(q/q_{TF})^2 + F\left(\frac{q}{2k_F}\right)} dq$$

$$\Rightarrow \boxed{v_{eff}(\vec{r},0) = \frac{e^2}{r} \times \frac{2}{\pi} \left(\frac{2k_F}{q_{TF}}\right)^2 \int_0^\infty \frac{x \sin(2k_F r x)}{\left(\frac{2k_F}{q_{TF}}\right)^2 x^2 + F(x)} dx}$$

IN THE PERTURBATIVE REGIME $r_s \ll 1 \Rightarrow k_F \gg q_{TF}$

THE INTEGRAL CAN BE WELL APPROXIMATED TO

$$\int_0^\infty \frac{x \sin(2k_F r x)}{\left(\frac{2k_F}{q_{TF}}\right)^2 x^2 + 1} dx = \frac{\pi}{2} \left(\frac{q_{TF}}{2k_F}\right)^2 e^{-r q_{TF}}$$



$\left(\frac{2k_F}{q_{TF}}\right)^2 x^2$ FOR $q_{TF} \ll k_F$

FOR $q_{TF} \ll k_F$, $F(x)$ IS ONLY IMPORTANT FOR $x \lesssim \frac{q_{TF}}{2k_F} \ll 1$

IN THIS REGION, $F(x)$ CAN BE APPROXIMATED TO

$F(x) \simeq 1$

(42)

THEREFORE, $\boxed{v_{eff}(\vec{r}, 0) \simeq \frac{e^2}{r} e^{-r/\lambda_{TF}}}$

WHERE $\lambda_{TF} = 1/q_{TF}$ ≡ THOMAS-FERMI ~~WAVELENGTH~~ (OR DEBYE) SCREENING LENGTH.

THE EFFECTIVE INTERACTION IS THEN OF YUKAWA TYPE.

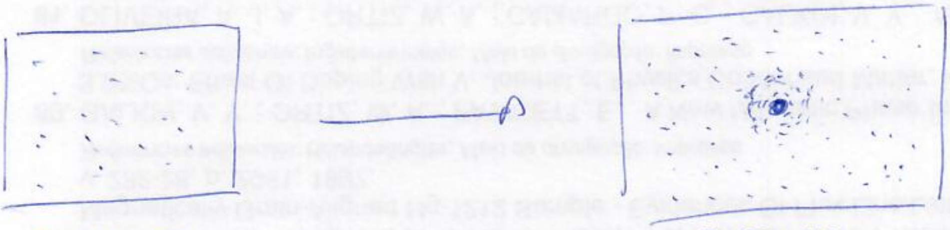
IN OTHER WORDS, THE COULOMB POTENTIAL IS SCREENED (OR SHIELDED) DUE TO THE MEDIUM POLARIZATION AS MENTIONED EARLIER

IN THE MOMENTUM SPACE, $\tilde{v}_{eff} \simeq \frac{4\pi e^2}{q^2 + q_{TF}^2}$

NOTICE THAT THE $\frac{1}{q^2}$ DIVERGENCY IS REMOVED BY THE INTERACTIONS. AS A CONSEQUENCE, THE VAN HOVE SINGULARITY AT k_F DOES NOT EXIST. (AS DISCUSSED AROUND PAGE 28/29).

HEURISTIC INTERPRETATION OF THOMAS-FERMI SCREENING

ADD A TEST CHARGE ON THE JELLIUM MODEL AT THE ORIGIN



ENERGIES

$\epsilon_{\vec p}$

ENERGIES

ELECTRIC POTENTIAL DUE TO THE TEST CHARGE

$$\epsilon_{\vec p} + \delta\epsilon \simeq \epsilon_{\vec p} + eV(\vec r)$$

DIELECTRIC FUNCTION

$$\epsilon_{\vec p} + v_{eff}(\vec r)$$

$$\Rightarrow \delta\rho = \rho_{IND}(\vec r) = -e\int \frac{d\vec p}{(2\pi)^3}\langle n(\epsilon_p+\delta\epsilon)\rangle - \langle n(\epsilon_p)\rangle$$

$$= -e\int \frac{d\vec p}{(2\pi)^3}\langle n(\epsilon_{\vec p} + v_{eff}(\vec r))\rangle - \langle n(\epsilon_p)\rangle$$

$$\approx \langle n(\epsilon_p)\rangle + v_{eff}(\vec r)\frac{\partial}{\partial\epsilon_p}\langle n(\epsilon_p)\rangle$$

$$= e\, v_{eff}(\vec r)\int \frac{d\vec p}{(2\pi)^3}\frac{\partial}{\partial \epsilon_p}\langle n(\epsilon_p)\rangle$$

$T\to 0$: $\approx \theta(\mu-\epsilon_p)$

$$\int d\epsilon_p\, \rho(\epsilon_p) \qquad \underbrace{\qquad\qquad}_{-\delta(\epsilon_p-\mu)}$$

DENSITY OF STATES

$$\Rightarrow \boxed{\rho_{IND} = -e\,\rho(E_F)\, v_{eff}(\vec r)}$$

USING POISSON'S EQUATION

$$\nabla^2 v_{eff} = -4\pi e\left(e\delta(\vec r) + \rho_{IND}(\vec r)\right)$$

$$\Rightarrow \boxed{\tilde v_{eff} = \frac{4\pi e^2}{q^2+4\pi e^2\rho(E_F)}}$$

FRIEDEL OSCILLATIONS

LET US NOW RETURN TO V_{eff} ON PAGE 41.

IN THE $2k_F\lambda \gg 1$ LIMIT, THE LOG-DIVERGENT DERIVATIVE OF $F(x)$ WILL PLAY AN IMPORTANT WHOLE.

OUR 1st TASK IS TO HIGHLIGHT THE IMPORTANCE OF THIS NON-ANALICITY.

PERHAPS, THE EASIEST WAY IS BY STUDING THE SINGULARITY STRUCTURE OF THE INTEGRAND

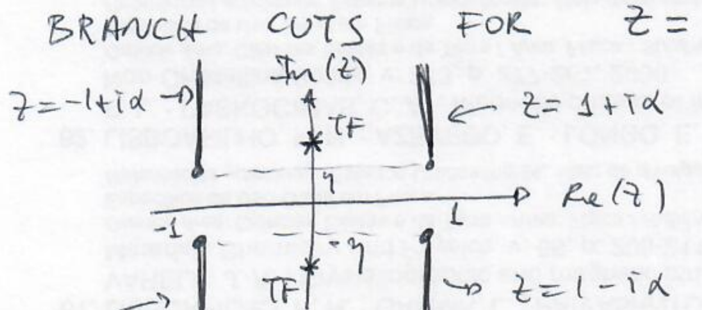
$$I = \int_0^{\infty} \frac{\sin(Ax)\, x\, dx}{(Bx)^2 + F(x)}, \qquad F(x) = \frac{1}{2} + \frac{1}{8}\left(\frac{1}{x} - x\right) \ln\left(\left(\frac{1+x}{1-x}\right)^2\right)$$

$$A = 2k_F\lambda$$

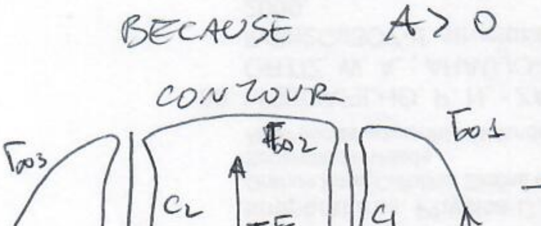
$$= -\frac{i}{2}\int_{-\infty}^{\infty} \frac{e^{iAx}\, x\, dx}{(Bx)^2 + F(x)} \qquad B = 2k_F/q_{TF}$$

$$I = \lim_{\eta \to 0^+} -\frac{i}{2}\int_{-\infty}^{\infty} \frac{e^{iAz}\, z\, dz}{(Bz)^2 + F_\eta(z)}, \qquad F_\eta(z) = \frac{1}{2} + \frac{1}{8}\left(\frac{1}{z} - z\right) \ln\left(\frac{(1+z)^2 + \eta^2}{(1-z)^2 + \eta^2}\right)$$

→ BRANCH CUTS FOR $z = \pm 1 \pm i\alpha$, $\forall \alpha > \eta$



BECAUSE $A > 0$, WE COULD USE THE FOLLOWING CONTOUR



$$\to \int_{-\infty}^{\infty} + \int_{\Gamma_{\infty 1}} + \int_{C_1} + \int_{\Gamma_{\infty 2}} + \int_{C_2} + \int_{\Gamma_{\infty 3}} = \oint$$

$$= 2\pi i\, \mathrm{Res}(TF)$$

WHERE THERE IS ONLY ONE ~~REED~~ POLE INSIDE IT. THE ONE CORRESPONDING TO THE THOMAS-FERMI APPROXIMATION, WHICH IS SOLUTION OF

THE EQUATION $(B i\alpha)^2 + F(i\alpha) = 0$ 45

$$\Rightarrow -B^2\alpha^2 + \frac{1}{2} + \frac{1}{8}\left(\frac{-i}{\alpha} - i\alpha\right) \ln\left(\frac{(1+i\alpha)^2}{(1-i\alpha)^2}\right) = 0$$

$$-B^2\alpha^2 + \frac{1}{2} + \frac{1}{8}\left(\frac{1}{\alpha}+\alpha\right)(-i)\ln\left(\frac{(Re^{i\theta})^2}{(Re^{-i\theta})^2}\right)$$

$$+4i\theta, \quad \theta = \text{ARCTAN}(\alpha)$$

$$\Rightarrow -B^2\alpha^2 + \frac{1}{2} + \frac{1}{2}\left(\frac{1}{\alpha}+\alpha\right)\text{ARCTAN}(\alpha) = 0$$

FOR $B \gg 1$, THE SOLUTION IS FOR $\alpha \ll 1$

$$\Rightarrow -(B\alpha)^2 + \frac{1}{2} + \frac{1}{2}\left(\frac{1}{\alpha}+\alpha\right)\left(\alpha + O(\alpha^3)\right) \simeq 0$$

$$\frac{1}{2} + O(\alpha^2)$$

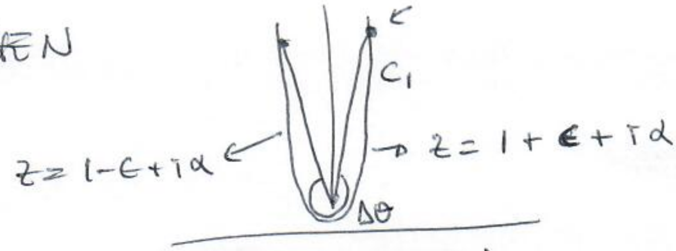
$$\Rightarrow \left(B^2 + O(1)\right)\alpha^2 \simeq 1 \Rightarrow \alpha \approx \frac{1}{B}$$

THE CORRESPONDING RESIDUE GIVES THE THOMAS-FERMI CONTRIBUTION $\sim e^{2Az} = e^{-2A\left(\frac{1}{B}\right)} = e^{-\frac{A}{B}} = e^{-9\pi r}$

HERE, WE ARE AFTER THE INTEGRALS OVER THE PATHS C_1 AND C_2 [SINCE THE ONES OVER $\Gamma_{\alpha_{1,2,3}}$ VANISH (BECAUSE e^{2Az}, $\mathrm{Im}(z) \to \infty$)].

BECAUSE OF THE BRANCH CUTS, C_1 AND C_2 HAVE TO BE COMPUTED CAREFULLY. WE NEED TO REALIZE THAT ONLY THE LOGS MATTER FOR THE BRANCH CUTS SINCE THE OTHER FUNCTIONS IN THE INTEGRAND ARE ANALYTIC ALONG THE BRANCH CUTS

THEN



$$\int_{C_1} dz = \int_{\infty}^{\eta} \frac{e^{iA(1+i\alpha)}\,(1+i\alpha)\, i\,d\alpha}{(B(1+i\alpha))^2 + F(1+\epsilon+i\alpha)} + \int_{\eta}^{\infty} \frac{e^{iA(1+i\alpha)}\,(1+i\alpha)\, i\,d\alpha}{(B(1+i\alpha))^2 + F(1-\epsilon+i\alpha)}$$

$$= -\left[\int_{\eta}^{\infty} e^{iA(1+i\alpha)}\,(1+i\alpha)\, i\,d\alpha \;\frac{F(1-\epsilon+i\alpha) - F(1+\epsilon+i\alpha)}{\left((Bz)^2 + F(+)\right)\left((Bz)^2 + F(-)\right)}\right]$$

WHERE $\quad F(-) + F(+) = \frac{1}{8}\left(\frac{1}{z} - z\right) \ln\left(\frac{\epsilon+i\alpha}{-\epsilon+i\alpha}\right)^2 , \quad \epsilon \to 0$

$$= \frac{1}{8}\left(\frac{1}{z} - z\right)\left(-4\pi i\right)$$

$\epsilon + i\alpha \to \alpha e^{i\pi/2}$

$-\epsilon + i\alpha \to \alpha e^{i\left(\frac{\pi}{2} + 2\pi\right)}$

THE RESULTING INTEGRAL IS CUMBERSOME AND IS SHORTLY SHOWN IN FETTER AND WALECKA SECT. 14.

THEY RESULT IN AN ADDITIONAL CONTRIBUTION TO THE THOMAS-FERMI ONE WHICH DOMINATES FOR LARGE $2k_F r$ SINCE IT DECAYS ALGEBRAICALLY WITH r.

$$V_{eff}(\vec{r}\to\infty) = \frac{e^2}{r}\,\frac{2}{\pi}\left(\frac{2k_F}{q_{TF}}\right)^2 I$$

RECALL $\quad a_0 = \frac{4k_F}{\pi q_{TF}^2}$

$$\simeq V_{eff}^{(THOMAS\text{-}FERMI)} + V_{eff}^{(FRIEDEL)}$$

WHERE

$$\boxed{V_{eff}^{(FRIEDEL)} \simeq \frac{e^2}{a_0}\,\frac{2}{\pi}\,\frac{1}{\left(\frac{1}{2}\left(\frac{q_{TF}}{2k_F}\right)^2 + 1\right)^2}\,\frac{\cos 2k_F r}{(2k_F r)^3}}$$

LARGE $2k_F r$

THIS FRIEDEL CONTRIBUTION DOMINATES FOR LARGE DISTANCES AS CAN BE EASILY CONFIRMED VIA NUMERICAL INTEGRATION

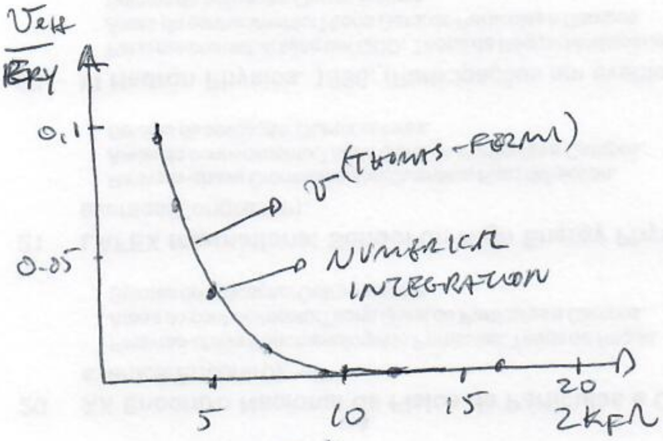
$$V_{eff} = \frac{e^2}{a_0}\left(\frac{4}{\pi^2}\left(\frac{2k_F}{q_{TF}}\right)^2 \frac{1}{2k_F r}\int_0^\infty \frac{x \sin((2k_F r)x)\,dx}{x^2 + \left(\frac{q_{TF}}{2k_F}\right)^2 F(x)}\right)$$

$\frac{e^2}{a_0} \rightarrow E_{Ry} = 13.6\text{ eV}$

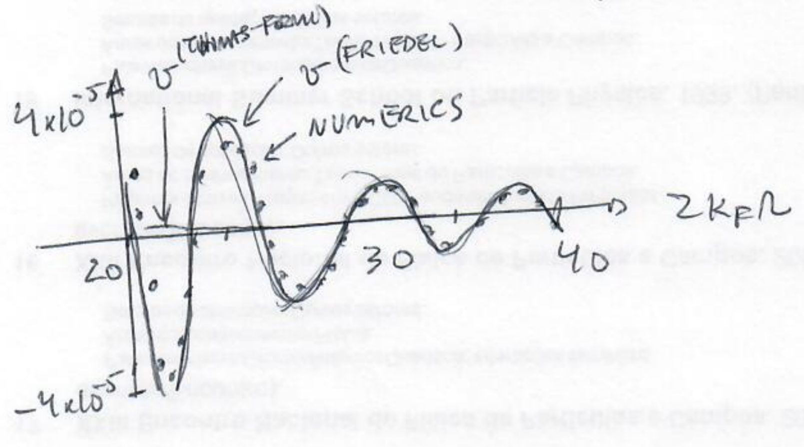
(GABOVICH ET AL., SOV. PHYS. JETP 48, 124 (1978))

WHERE,

$$F(x) = \frac{1}{2} + \frac{1}{4}\left(\frac{1}{x} - x\right)\ln\left|\frac{1+x}{1-x}\right|$$



HERE, $q_{TF} = k_F$



HERE, $q_{TF} = k_F$

- SMALLER q_{TF} MAKES THE CROSSOVER BETWEEN v^{T-F} AND $v^{FRIEDEL}$ HAPPENING AT LARGER $2k_F r$

- NOTICE HOW SMALL THE FRIEDEL OSCILLATIONS ARE. NONETHELESS, THEY ARE MEASURED IN EXPERIMENTS AS IN ~~THE~~ QUANTUM CORALS IN STM.

SCANNING TUNNELING MICROSCOPY

TECHNICAL DETAILS

PERHAPS, THE EASIEST WAY OF COMPUTING THE FRIEDEL PART OF THE EFFECTIVE POTENTIAL IS BRINGING FORWARD THE SINGULARITY OF $F(x)$. THIS CAN BE DONE BY INTEGRATION BY PARTS

$$I = \int_0^\infty \frac{x \sin(Ax)\, dx}{(Bx)^2 + F(x)} = \cancel{-\frac{\cos Ax}{A} \frac{x}{(Bx)^2+F}\Big|_0^\infty}^{\,0} + \frac{1}{A}\int \cos Ax \left(\frac{x}{(Bx)^2+F}\right)' dx$$

$$= \cancel{\frac{\sin Ax}{A^2}\left(\frac{x}{(Bx)^2+F}\right)'\Big|_0^\infty}^{\,0} - \frac{1}{A^2}\int_0^\infty \sin(Ax)\left(\frac{x}{(Bx)^2+F}\right)'' dx$$

AND

$$\left(\frac{x}{(Bx)^2+F}\right)'' = \frac{2B^4x^3 + 2(3B^2+F')(-F+xF')}{((Bx)^2+F)^3} - \frac{x F''}{((Bx)^2+F)^2}$$

THE 2ND TERM IS THE MOST IMPORTANT BECAUSE

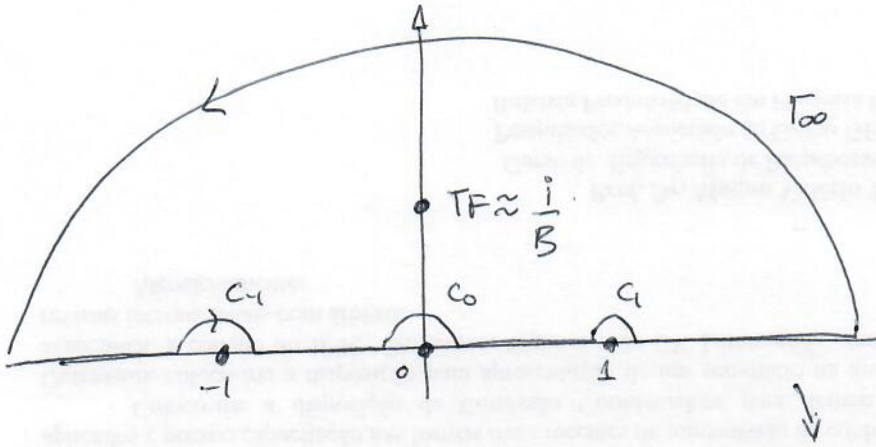
$$F'' = \frac{1}{x^2(x^2-1)} + \frac{\ln\left(\left(\frac{1+x}{1-x}\right)^2\right)}{4x^3}$$

F'' IS STRONGLY DIVERGENT AT THE SINGULARITY $x = \pm 1$
(THE $q = 2k_F$ SINGULARITY)

THEN, WE APPROXIMATE

$$I \simeq \frac{1}{A^2}\int_0^\infty \frac{\sin(Ax)\, x F''}{((Bx)^2+F)^2} dx = \frac{1}{A^2}\int_0^\infty \frac{\sin(Ax)\, dx}{x\left((Bx)^2+F(x)\right)^2 (x^2-1)}$$

THE INTEGRAL IS THEN DOMINATED BY THE POLES @ $x = \pm 1$



$$I \sim \frac{i}{2A^2} \int_{-\infty}^{\infty} \frac{e^{2Az}\, dz}{z\left((Bz)^2 + F\right)^2 (z^2 - 1)}$$

NEGLECT THE BRANCH CUTS → THE POLES $z = \pm 1$ SHOULD DOMINATE

$$\int_{-\infty}^{\infty} + \int_{C_{-1}} + \int_{C_0} + \int_{C_1} + \int_{T_\infty} = 2\pi i \left(\text{RESIDUE AT } TF \approx \frac{i}{B} \right)$$

$\int_{T_\infty} \to 0$

GIVES THE THOMAS-FERMI CONTRIBUTION

$$\Rightarrow \int_{-\infty}^{\infty} = -\int_{C_{-1}} - \int_{C_1} + \left(\text{THOMAS-FERMI} - \int_{C_0} \right)$$

NOT FRIEDEL

$$= \frac{2\pi i}{2} \left(\text{RESIDUES AT } x = \pm 1 \right)$$

$$= \pi i \left[\frac{e^{-iA}}{2(B^2 + F(1))^2} + \frac{e^{iA}}{2(B^2 + F(1))^2} \right] = \pi i \frac{\cos A}{\left(B^2 + \frac{1}{2}\right)^2}$$

$$\Rightarrow I^{(\text{FRIEDEL})} = \frac{\pi}{2A^2} \frac{\cos A}{\left(B^2 + \frac{1}{2}\right)^2}$$

THUS, $$v_{eff}^{FRIEDEL} \simeq \frac{e^2}{a_0} \frac{4}{\pi^2} \left(\cancel{\frac{2k_F}{q_{TF}}}\right)^2 \frac{1}{2k_F r} I^{FRIEDEL} \left(\frac{2k_F}{q_{TF}}\right)^4$$ (50)

$$= \frac{e^2}{a_0} \frac{4}{\pi^2} \frac{1}{2k_F r} \frac{\pi}{2} \frac{\left(\frac{2k_F}{q_{TF}}\right)^4}{\left(\left(\frac{2k_F}{q_{TF}}\right)^2 + \frac{1}{2}\right)^2} \frac{\cos 2k_F r}{(2k_F r)^2}$$

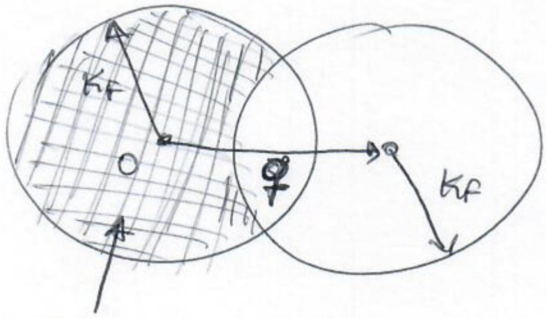
$$\boxed{v_{eff}^{FRIEDEL} \simeq \frac{e^2}{a_0} \frac{2}{\pi\left(\left(\frac{q_{TF}}{2k_F}\right)^2 + \frac{1}{2}\right)^2} \frac{\cos(2k_F r)}{(2k_F r)^3}}$$

PHYSICAL ORIGIN OF THE FRIEDEL OSCILLATIONS

→ MATHEMATICALLY → $q = 2k_F$ NON-ANALICITY

• $2k_F$ SINGULARITY COMES FROM PAGE 39.

$$\int d\vec{k}\ \Theta\left(|\vec{k}+\vec{q}| - k_F\right)\Theta(k_F - k) \cdots$$



REGION WHERE THE INTEGRAND IS NON-ZERO

• THUS, THE NON-ANALICITY COMES FROM A <u>WELL-DEFINED FERMI SURFACE</u>.

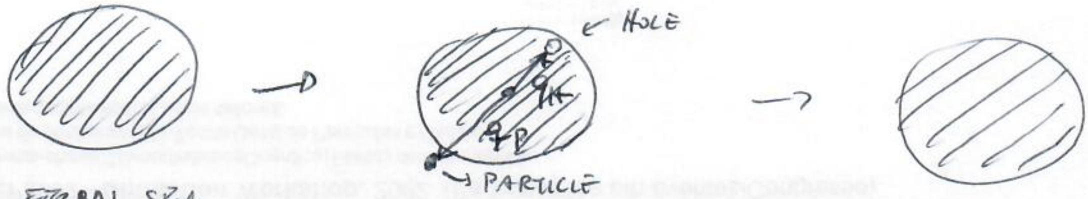
• AT FINITE TEMPERATURES → $\int d\vec{k}\ \Theta\Theta \to \int dk \langle n_{\vec{k}} \rangle \langle 1 - n_{\vec{k}+\vec{q}} \rangle$

↓

SMOOTH FUNCTION

⇒ FRIEDEL OSCILLATIONS WILL BE SMEARED OUT.

THE WELL-DEFINED FERMI SURFACE ALLOWS FOR LOW-ENERGY PROCESSES OF TYPE



FERMI SEA
GROUND STATE

$|\vec{q}_P - \vec{q}_H| \approx 2k_F$

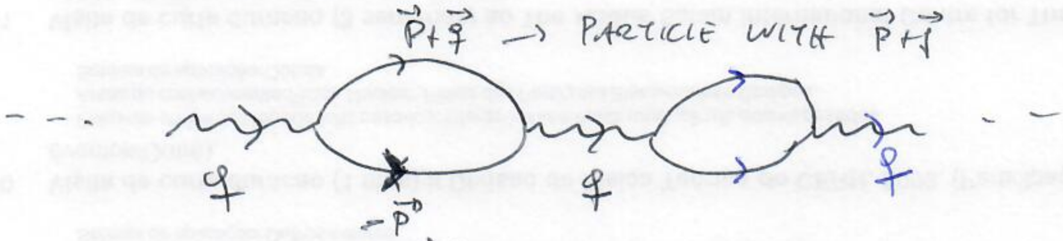
ΔE IS SMALL

VIRTUAL POLARIZATION OF THE GROUND STATE

RETURN TO THE GS

(IN THE RING DIAGRAM)

$\vec{p}+\vec{q}$ → PARTICLE WITH $\vec{p}+\vec{q}$



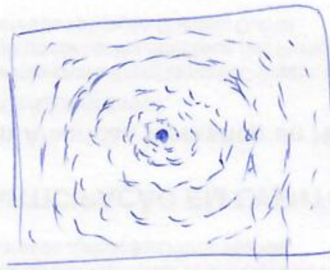
HOLE WITH $-\vec{p}$

IN $d=1$, BECAUSE THE PHASE SPACE FOR $q = 2k_F$ IS MUCH LARGER COMPARATIVELY, THE $q = 2k_F$ SINGULARITY IN THE LINDHARD FUNCTION SHOULD BE BIGGER

DENSITY RESPONSE OF THE ELECTRON GAS

ON PAGE 47, WE MENTIONED THAT FRIEDEL OSCILLATIONS ARE VERIFIED EXPERIMENTALLY. HOW IS THAT POSSIBLE? CERTAINLY IT IS HARD TO PROBE THE EFFECTIVE INTERACTIONS BETWEEN ELECTRONS DIRECTLY.

WHAT IS DONE IS ADDING AN IMPURITY OF CHARGE Q AT THE CENTER OF THE ELECTRON GAS AND MEASURE THE DISTRIBUTION OF INDUCED CHARGE.



$\rightarrow| |\leftarrow$

$\frac{1}{2k_F}$

ELECTRONIC DENSITY CHANGE

$$\underbrace{(-e)\,\rho_{IND}(\vec{r},t)}_{\text{CHARGE RESPONSE DUE TO THE PRESENCE OF THE IMPURITY CHARGE Q.}} = \langle \delta n(\vec{r},t)\rangle \underbrace{(-e)}_{\text{ELECTRON CHARGE}}$$

$\rightarrow$ EXPERIMENTALLY MEASUREABLE QUANTITY

$\Rightarrow \rho$ IS THE DENSITY RESPONSE (AFTER A QUICK TRANSIENT, IT DOES NOT DEPEND ON TIME.)

(IF THE IMPURITY CHARGE IS NOT AT REST, ρ_{IND} MAY DEPEND ON TIME.)

LET US CONSIDER THE SIMPLEST CASE $\rightarrow$ $\rho_{IND}(\vec{r},t) = \rho_{IND}(\vec{r})$,

BUT LET US KEEP A MORE GENERAL FORMALISM

$$\Rightarrow \tilde{\rho}_{IND}(\vec{q},\omega) = \langle \delta\tilde{n}(\vec{q},\omega)\rangle = \tilde{\chi}(\vec{q},\omega)\,\tilde{\phi}_{EXT}(\vec{q},\omega)$$

$\downarrow$

LINEAR RESPONSE THEORY

RESPONSE = SUSCEPTIBILITY * PERTURBATION

THE INDUCED CHARGE CREATES AN INDUCED ~~B~~ELECTRIC POTENTIAL: ~~[illegible]~~

$$\phi_{IND}(\vec{r},t) = \int d\vec{r}' \, \frac{e^2 \rho(\vec{r}',t)}{|\vec{r}-\vec{r}'|} \qquad \text{(COULOMB'S LAW)}$$

$$\Rightarrow \tilde{\phi}_{IND}(\vec{q},\omega) = \tilde{v}(\vec{q}) \, \tilde{\rho}_{IND}(\vec{q},\omega) = \tilde{V}(\vec{q}) \, \tilde{\chi}(\vec{q},\omega) \, \tilde{\phi}_{EXT}(\vec{q},\omega)$$

$$\text{WHERE } \tilde{v}(\vec{q}) = \frac{4\pi e^2}{q^2}$$

RECALL THAT $\phi_{EXT}(\vec{r},t) = \frac{eQ}{r} \rightarrow \tilde{\phi}_{EXT} = -\frac{4\pi Q e \, \delta(\omega)}{q^2}$

IN ADDITION

$$\phi_{TOT} = \phi_{IND} + \phi_{EXT} \rightarrow \tilde{\phi}_{TOT} = \left(1 + \tilde{v}(\vec{q}) \, \tilde{\chi}(\vec{q},\omega)\right) \tilde{\phi}_{EXT}(\vec{q},\omega)$$

$$\equiv \frac{\tilde{\phi}_{EXT}(\vec{q},\omega)}{\tilde{\varepsilon}(\vec{q},\omega)}$$

WITH THE DIELECTRIC FUNCTION $\tilde{\varepsilon}(\vec{q},\omega) = \frac{1}{1+\tilde{v}(\vec{q})\,\tilde{\chi}(\vec{q},\omega)}$

COMPARING WITH $\tilde{\varepsilon} = 1 - \tilde{v}\,\Pi$ (SEE PAGE 37)

~~[illegible]~~ WE HAVE THAT

$$\boxed{\tilde{\chi}(\vec{q},\omega) = \frac{\Pi(\vec{q},\omega)}{1-\tilde{v}(\vec{q})\,\Pi(\vec{q},\omega)}}$$

$$\Rightarrow \boxed{\rho_{IND}(\vec{r}) = \int \frac{d\vec{q}}{(2\pi)^3} \, e^{-i\vec{q}\cdot\vec{r}} \, \tilde{\chi}(\vec{q},0) \, \phi_{EXT}(\vec{q})}$$

IN THE RPA,

$$\rho_{IND}^{(RPA)}(\vec{r}) = \int \frac{d\vec{q}}{(2\pi)^3} \frac{\Pi^{RPA}(\vec{q},0)}{1 - \frac{4\pi e^2}{q^2}\Pi^{RPA}(\vec{q},0)} \left(\frac{-4\pi Q e}{q^2}\right) e^{-i\vec{q}\cdot\vec{r}}$$

$$= \int \frac{d\vec{q}}{(2\pi)^3} \frac{\frac{m k_F}{\pi^2} F\left(\frac{q}{2k_F}\right)}{q^2 + \frac{4e^2 m k_F}{\pi} F\left(\frac{q}{2k_F}\right)} (4\pi Q e)\, e^{-i\vec{q}\cdot\vec{r}}$$

RECALL $\Pi^{RPA}_{(\vec{q},0)} = -\frac{m k_F}{\pi^2} F(\vec{q}/2k_F)$ (SEE PAGE 40)

* NOTICE THAT $\int -e\,\rho_{IND}(\vec{r})\,d\vec{r}$ = TOTAL INDUCED CHARGE

$$= (-e) \left.\frac{\frac{m k_F}{\pi^2} F(q/2k_F)}{q^2 + \frac{4e^2 m k_F}{\pi} F(q/2k_F)} (4\pi Q e)\right|_{q=0} = \frac{(-e)(-4\pi Q e)}{-4\pi e^2} \frac{\Pi(0,0)}{\Pi(0,0)}$$

$= -Q \rightarrow$ TOTAL SCREENING OF THE CHARGE IMPURITY

(GENERAL RESULT. DOES NOT DEPEND ON RPA.)

* $$\rho_{IND}^{(RPA)}(\vec{r}) = \int \frac{d\vec{q}}{(2\pi)^3} e^{-i\vec{q}\cdot\vec{r}} \frac{4\pi Q e\, F(q/2k_F)}{(q/q_{TF})^2 + F(q/2k_F)} \frac{m k_F}{\pi^2} \cdot \frac{\pi}{4e^2 m k_F}$$

IN THE THOMAS-FERMI APPROXIMATION

$\Rightarrow F(q/2k_F) \simeq F(0) = 1$

$\boxed{q_{TF} \ll k_F \text{ AND } 2k_F r \text{ NOT TOO LARGE}}$

$$\Rightarrow \rho_{IND}^{RPA}(\vec{r}) = \int \frac{d\vec{q}}{(2\pi)^3} e^{-i\vec{q}\cdot\vec{r}} \frac{4\pi Q (q_{TF})^2}{4e(q^2 + q_{TF}^2)} = \frac{Q}{e} \frac{q_{TF}^2}{4\pi r} e^{-r q_{TF}}$$

(COMPARE WITH V_{eff} IN PAGE 42)

- AS DISCUSSED ON PAGE 43, THIS RESULT CAN BE UNDERSTOOD CLASSICALY BY ASSUMING ELECTROSTATIC EQUILIBRIUM AND SOLVING THE POISSON'S EQUATION

- IN ADDITION, QUANTUM MECHANICAL ~~[illegible]~~ EFFECTS ~~SAY~~ SAY THAT THIS IS NOT THE WHOLE STORY FOR $2k_F r \gg 1$ THE SINGULARITY AT $q = 2k_F$ PLAYS ITS TRICK YIELDING FRIEDEL OSCILLATIONS IN THE ELECTRONIC DENSITY.

(SEE FETTER + WALECKA SEC 14, AGAIN)

$$\rho_{IND}^{(RPA)} = \frac{Q}{e}\int \frac{d\vec{q}}{(2\pi)^3} e^{-i\vec{q}\cdot\vec{r}} \frac{q_{TF}^2 F(q/2k_F)}{q^2 + q_{TF}^2 F(q/2k_F)}$$

$$= \frac{Q}{e}\int \frac{d\vec{q}}{(2\pi)^3} e^{-i\vec{q}\cdot\vec{r}} \left(1 - \frac{\left(\frac{q}{q_{TF}}\right)^2}{\left(\frac{q}{q_{TF}}\right)^2 + F(q/2k_F)}\right) = \frac{2Q}{(2\pi)^2 e r}\int_0^\infty dq\, q \overbrace{\sin(qr)}\left(1 - \frac{\left(q/q_{TF}\right)^2}{\left(\frac{q}{q_{TF}}\right)^2 + F\left(\frac{q}{2k_F}\right)}\right)$$

THE SINGULAR PART (FRIEDEL OSCILLATIONS) COME FROM THE 2ND TERM. REPEATING THE DERIVATION ON PAGES 48 AND 49,

$$\rho_{IND}^{RPA/FRIEDEL} \sim -\frac{2Q(2k_F)\frac{Q}{q_{TF}}}{(2\pi)^2 e r}\int_0^\infty dx \frac{\left(\frac{2k_F}{q_{TF}}\right)^3 x^3 \sin(2k_F r x)}{\left(\frac{2k_F}{q_{TF}}\right)^2 x^2 + F(x)} \sim \frac{-2Q(2k_F)^5}{(2\pi)^2 e (2k_F r) q_{TF}^2}\int_0^\infty dx \frac{x^3 \sin(Ax) F''}{A^2((Bx)^2 + F)^2}$$

$$\sim \frac{-2Q(2k_F)^5}{(2\pi)^2 e (2k_F r)^3 q_{TF}^2}\int_0^\infty dx \frac{x \sin(Ax)}{(x^2-1)((Bx)^2+F)^2} = \frac{-2Q}{(2\pi)^2 e}\left(\frac{2k_F}{q_{TF}}\right)^2 \frac{(2k_F)^3}{(2k_F r)^3} \times \frac{-i}{2}\int_{-\infty}^\infty \frac{x e^{iAx}\, dx}{(x^2-1)((Bx)^2+F)^2}$$

$$\sim \frac{-2Q}{(2\pi)^2 e}\left(\frac{2k_F}{q_{TF}}\right)^2 \frac{(2k_F)^3}{(2k_F r)^3} \cdot \frac{-i}{2}\, \frac{2\pi i}{2}\left[\frac{e^{iA}}{2(B^2+\frac{1}{2})^2} + \frac{e^{-iA}}{2(B^2+\frac{1}{2})^2}\right]$$

$$= \frac{-Q}{4\pi e}\left(\frac{2k_F}{q_{TF}}\right)^2 \frac{(2k_F)^3}{(2k_F r)^3} \frac{\cos(2k_F r)}{\left(\left(\frac{2k_F}{q_{TF}}\right)^2 + \frac{1}{2}\right)^2} = \boxed{-\frac{4}{\pi}\frac{Q}{e}\frac{(q_{TF}/k_F)^2}{((q_{TF}/k_F)^2 + 8)^2}\frac{\cos(2k_F r)}{r^3}}$$

CONSIDER THE GREEN'S FUNCTION $\mathcal{G}_{\alpha\beta} \equiv \langle \psi_\alpha \bar{\psi}_\beta \rangle \equiv$ 

- IN POSITION REPRESENTATION, $\mathcal{G} = \langle \psi(\vec{x}, \omega_m)\, \bar{\psi}(\vec{x}', \omega_m)\rangle$
$= \langle \psi(x)\, \bar{\psi}(x')\rangle$

FOR TRANSLATIONAL INVARIANT SYSTEMS $\rightarrow \mathcal{G}(x,x') = \mathcal{G}(x-x')$

WHICH, IN THE MOMENTUM REPRESENTATION BECOMES

$$\mathcal{G} = \langle \psi_k \bar{\psi}_q \rangle = \delta_{k,q}\, \mathcal{G}_{\vec{k},\omega_m}$$

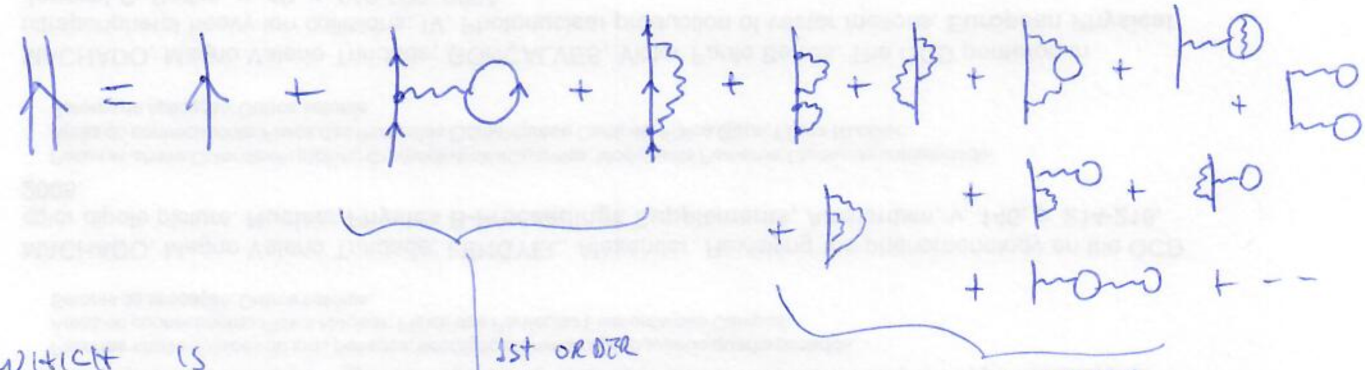
PERTURBATIVE SERIES: (SEE PAGE 12)

$$\langle O \rangle = \frac{\langle O\rangle_0 + \langle O(-S_{int})\rangle_0 + \frac{1}{2!}\langle O(-S_{int})^2\rangle_0 + \cdots}{1 + \langle -S_{int}\rangle_0 + \frac{1}{2!}\langle (-S_{int})^2\rangle_0 + \cdots}$$

$$= \langle O\rangle_0 + \langle O(-S_{int})\rangle_0^c + \frac{1}{2!}\langle O(-S_{int})^2\rangle_0^c + \frac{1}{3!}\langle O(-S_{int})^3\rangle_0^c + \cdots$$

LINKED CLUSTER THEOREM (ONLY THE CONNECTED DIAGRAMS)

THUS, FOR THE GREEN'S FUNCTION, IT BECOMES



WHICH IS 1ST ORDER

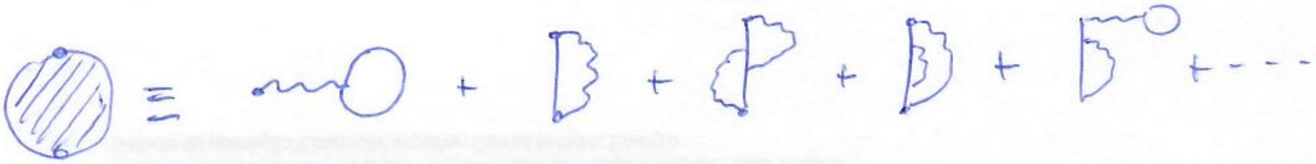
2ND ORDER

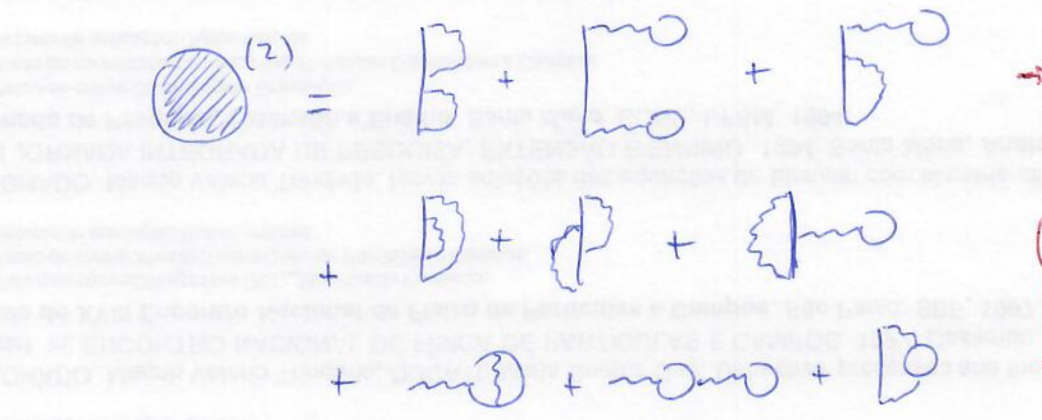
$$\mathcal{G} = \mathcal{G}_0 - \langle \psi\bar{\psi} V \bar{\psi}\bar{\psi}\psi\psi\rangle_0^c + \frac{1}{2!}\langle \psi\bar{\psi} V \bar{\psi}\bar{\psi}\psi\psi V \bar{\psi}\bar{\psi}\psi\psi\rangle_0^c + \cdots$$

THIS DEFINES THE SELF-ENERGY

WHERE  $= \tilde{\Sigma} =$ SELF-ENERGY

THUS, $\boxed{G_{\alpha\beta} = G_{0\alpha\beta} + \int d\gamma\, d\delta\, G_{0\alpha\gamma}\, \Sigma_{\gamma\delta}\, G_{0\delta\beta}}$





IMPROPER DIAGRAMS

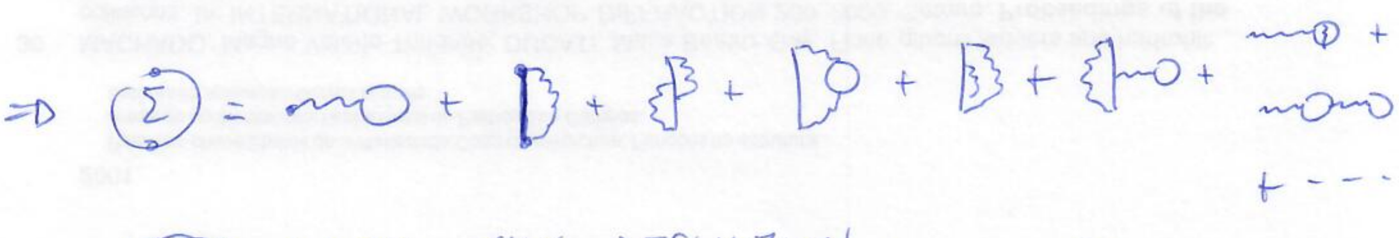
PROPER DIAGRAMS

* IMPROPER DIAGRAMS ≡ DIAGRAMS THAT CAN BE SEPARATED INTO 2 DIAGRAMS BY CUTTING A FERMION LINE.

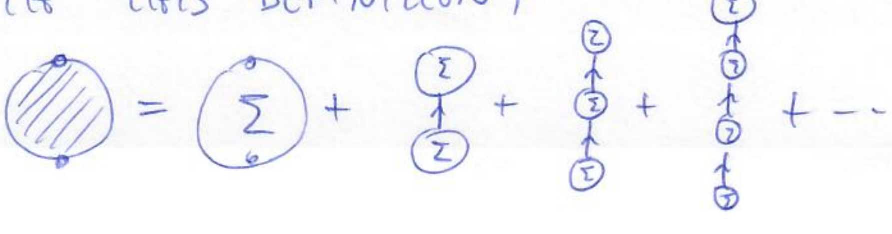
- THE PART OF THE DIAGRAM THAT CANNOT BE REDUCED INTO SMALLER DIAGRAMS BY CUTTING A FERMION LINE IS CALLED PROPER SELF-ENERGY PART

PROPER SELF-ENERGY: $\Sigma \equiv$ 

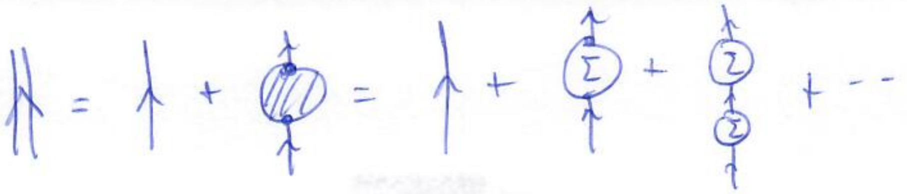
≡ SUM OF ALL SELF-ENERGY PROPER DIAGRAMS

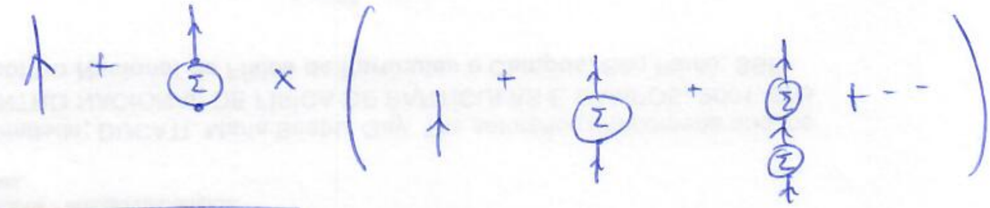


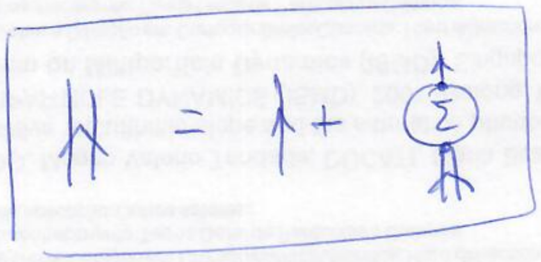
WITH THIS DEFINITION,

THEN 

THEREFORE,







DYSON'S EQUATION

$$\Rightarrow \mathcal{G}(x-x') = \mathcal{G}_0(x-x') + \int dx_1\, dx_2\, \mathcal{G}_0(x-x_1) \sum(x_1-x_2)\, \mathcal{G}_0(x_2-x')$$

FOURIER TRANSFORMING : $\sum(k) = \sum(\vec{k}, \omega_m) = \int dk\, e^{ikx} \sum(x)$

$$\Rightarrow \mathcal{G}(k) = \mathcal{G}_0(k) + \mathcal{G}_0(k) \sum(k)\, \mathcal{G}(k)$$

$$\Rightarrow \boxed{\mathcal{G}_\sigma(\vec{k}, \omega_m) = \frac{1}{\mathcal{G}_0^{-1}(\vec{k}, \omega_m) - \sum_\sigma(\vec{k}, \omega_m)} = \frac{1}{\epsilon^{(0)}_{\vec{k}} - \mu - i\omega_m - \sum_\sigma(\vec{k}, \omega_m)}}$$

$\Rightarrow \epsilon_{\vec{k}} = \epsilon^{(0)}_k - \sum(\vec{k}, \omega_m)$ → THIS EXPLAINS THE NAME "PROPER SELF ENERGY"

MINUS SIGN IS JUST A CONVENTION, IF THE READER DESIRES AN EQUATION OF TYPE $\epsilon_k = \epsilon_k^{(0)} + \sum$, THEN MAKE $\sum \to -\sum$

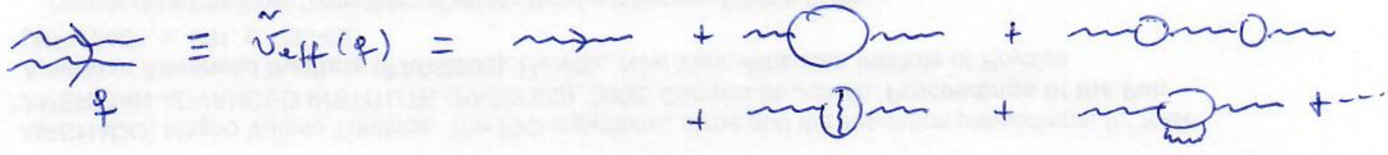
ALTERNATIVELY, YOU CAN CHANGE THE DEFINITION OF $\mathcal{G} \to -\mathcal{G}$, i.e., $\mathcal{G} = -\langle \psi \bar{\psi} \rangle$

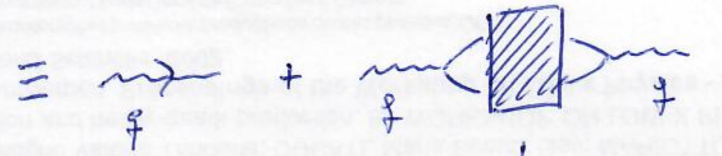
IMPORTANT POINTS

- FULL INFORMATION ABOUT THE GREEN'S FUNCTION IS ENCODED IN $\sum$
- COMPUTING $\sum$ IN SOME ORDER OF APPROXIMATION DOES NOT MEAN THAT $\mathcal{G}$ IS COMPUTED IN THE SAME ORDER & ACTUALLY AN INFINITE NUMBER OF DIAGRAMS IS ENCODED.

DYSON'S EQUATION IS SIMPLY A CLEVER WAY OF USING THE STRUCTURE OF THE DIAGRAMS IN ORDER TO SUM AN INFINITE CLASS OF PERTURBATION TERMS (FEYNMAN DIAGRAMS) IN A COMPACT FORM.

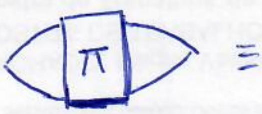
WE CAN ALSO APPLY IT TO THE INTERACTION LINE



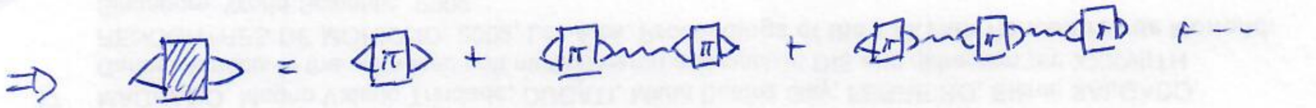


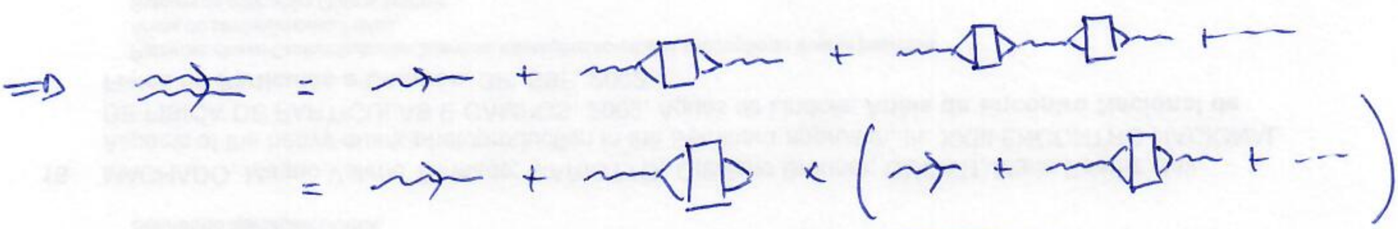
POLARIZATION INSERTION $\equiv \tilde{\Pi}(q)$

$\tilde{\Pi}$ CAN BE OBTAINED VIA JOINING IRREDUCIBLES (PROPER) POLARIZATION INSERTIONS



= PROPER POLARIZATION INSERTION





ANOTHER DYSON'S EQUATION

IN OTHER WORDS, $\tilde{v}_{eff}(q) = \tilde{v}(q) + \tilde{v}(q)\,\Pi(q)\,\tilde{v}_{eff}(q)$

$$\Rightarrow \boxed{\tilde{v}_{eff}(q) = \frac{\tilde{v}(q)}{1-\tilde{v}(q)\Pi(q)}}$$

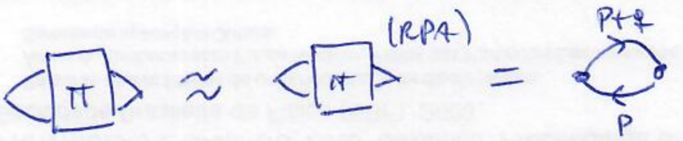
COMPARE WITH $\tilde{v}_{eff}$ ON PAGE 37.

THIS IS WHY $\Pi(q)$ IS CALLED PROPER POLARIZATION.
THE EFFECTIVE INTERACTION BETWEEN PARTICLES IS "DRESSED" BY MANY VIRTUAL POLARIZATIONS OF THE VACUUM (EXCITATIONS OF THE GROUND STATE)

CREATION / ANNIHILATION OF PARTICLE / ANTI PARTICLES (OR PARTICLE / HOLE) PAIRS

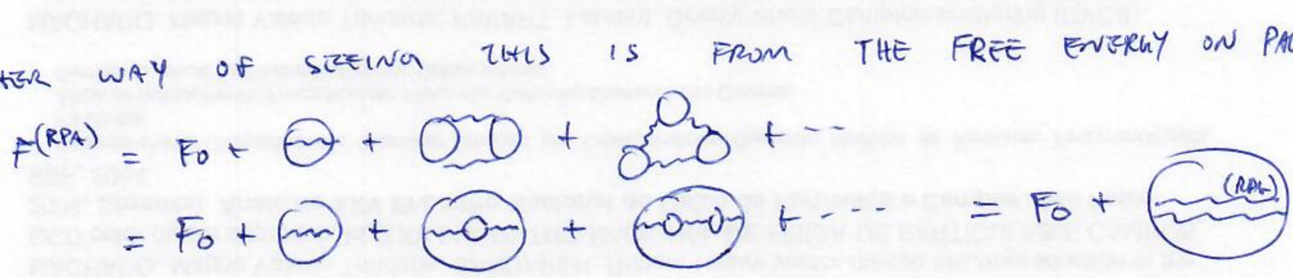
FINALLY, NOTICE THAT

THE RPA CORRESPONDS TO

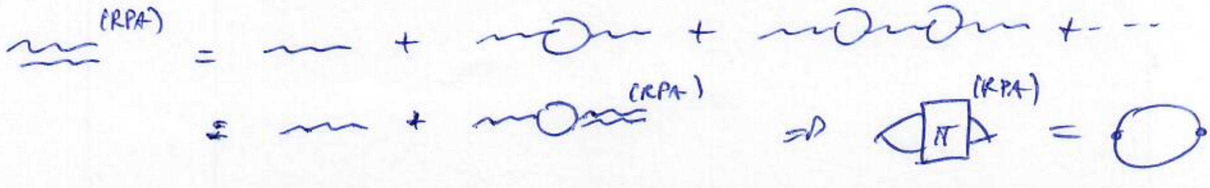


= SINGLE PARTICLE-HOLE BUBBLE (SEE PAGE 36)

ANOTHER WAY OF SEEING THIS IS FROM THE FREE ENERGY ON PAGE 35



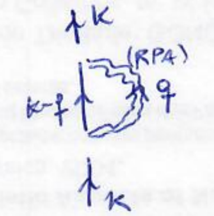
WHERE



HOW ABOUT THE PROPER SELF ENERGY IN RPA?
(ONLY RING DIAGRAMS)

$\Rightarrow \quad \Sigma^{(RPA)} =$  $+ \cdots$

$$\boxed{\Sigma^{(RPA)} = \text{} }$$ 

 $= \sum_q \mathcal{G}_0(k-q)\, \tilde{v}_{eff}^{(RPA)}(q)$

SINGULARITY AT $2k_F$
IN THE STATIC LIMIT

- IT IS A GOOD EXERCISE TO COMPUTE Σ IN THE HARTREE - FOCK APPROXIMATION

$\Sigma^{(HF)} = \Sigma^{(HF)} =$  $= \cancel{\tilde{v}(q=0) \sum_q \mathcal{G}_0(q)} + \sum_q \mathcal{G}_0(k-q)\,\tilde{v}(q)$

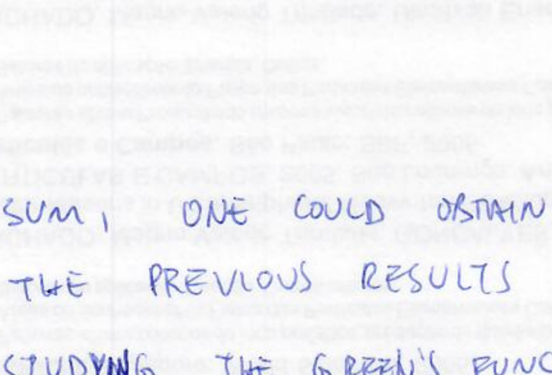
$= \sum_{\vec{q}, i\omega_n} \mathcal{G}_0(q)\, \tilde{v}(\vec{k}-\vec{q})$

$\overset{T=0}{=} \ \cdots \sum_{\vec{q}} \tilde{v}(\vec{k}-\vec{q})\, \theta(k_F - q)$

COMPARE WITH PAGE 28
(SINGULARITY AT K_F)

IN SUM, ONE COULD OBTAIN ALL THE PREVIOUS RESULTS BY STUDYING THE GREEN'S FUNCTION AND THE SELF-ENERGY.

NOTICE THE SAME FEYNMAN DIAGRAMS FROM THE FREE ENERGY AND GREEN'S FUNCTION (JUST OPEN A FERMION LINE) IN ALL CASES



PLASMON MODES

CLASSICAL OVERVIEW:

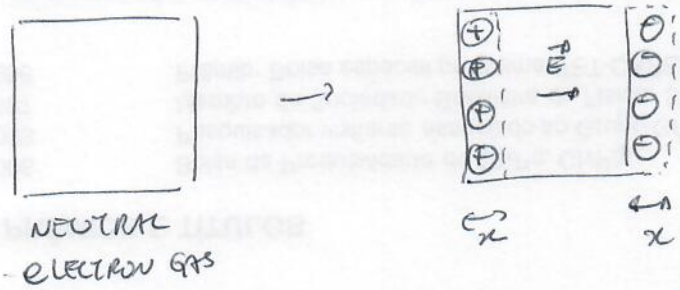
$$\begin{cases} \nabla \cdot \vec{E} = 4\pi\rho \\ m\ddot{\vec{r}} = -e\vec{E} \end{cases}$$

NET CHARGE DENSITY

SINCE $\vec{J} = -e\, n\, \dot{\vec{r}}$ PARTICLE MEAN DENSITY

$$\Rightarrow \dot{\vec{J}} = \frac{ne^2}{m}\vec{E}$$

CONTINUITY EQUATION: $\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{J} = 0 \Rightarrow \frac{\partial^2}{\partial t^2}\rho = -\nabla\cdot\dot{\vec{J}} = -\frac{ne^2}{m}\nabla\cdot\vec{E}$



$x \rightarrow$ DISPLACE THE e^- BY AN AMOUNT x WITH RESPECT TO THE POSITIVE BACKGROUND

THUS, $\frac{\partial^2}{\partial t^2}\rho = -\frac{ne^2}{m}4\pi\rho \Rightarrow \boxed{\omega_p = \sqrt{\frac{4\pi n e^2}{m}} \rightarrow \text{PLASMON FREQUENCY}}$

ALTERNATIVELY: DISPLACEMENT BY $x \rightarrow$ GENERATE SURFACE DENSITY $\sigma = exn$

$\sigma \rightarrow$ GENERATES $\vec{E} \rightarrow E = 4\pi e n x$

$E \rightarrow$ GENERATES A RESTORING FORCE ON THE CHARGES

$$m\ddot{x} = -eE = -4\pi e^2 n x \rightarrow \boxed{\omega_p^2 = \frac{4\pi n e^2}{m}}$$

PLASMON FREQUENCY $\Rightarrow$ CLASSICAL OSCILLATIONS OF ELECTRONIC DENSITY

$\downarrow$

QUANTIZATION OF THESE $\underbrace{\text{DENSITY WAVES}}_{\text{COLLECTIVE MODES}} \rightarrow$ PLASMONS

$\Rightarrow$ THERE WILL BE RESONANCES CORRESPONDING TO THESE COLLECTIVE MODES. THIS IS DIFFERENT FROM THE "SIMPLER" PARTICLE-HOLE EXCITATIONS.

RESONANCES SHOULD APPEAR IN THE RESPONSE FUNCTIONS (i.e. SUSCEPTIBILITY). ACCORDING TO PAGE 53 $\rightarrow$

$$\tilde{\chi}(\vec{q},\omega) = \frac{\Pi(\vec{q},\omega)}{1-\tilde{v}(\vec{q})\Pi(\vec{q},\omega)} = \frac{\Pi(\vec{q},\omega)}{\varepsilon(\vec{q},\omega)}$$

$\Rightarrow$ RESONANCES APPEAR WHEN THE DIELECTRIC FUNCTION $\varepsilon(\vec{q},\omega) = 0$

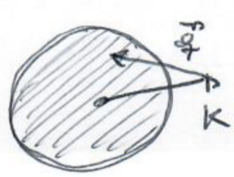
- NOTICE THIS IS REAL FREQUENCY, THIS IS BECAUSE RESONANCES APPEAR IN REAL TIME t, NOT IMAGINARY TIME τ. SINCE THE ANALYTICALLY CONTINUATION IS $\boxed{t = -i\tau}$, WE THEN CHANGE THE MATSUBARA FREQUENCY TO REAL FREQUENCY VIA $\boxed{i\omega_n = \omega}$

- WE NOW SEARCH FOR SOLUTIONS OF THE EQUATION $1 = \tilde{v}(\vec{q})\,\Pi(\vec{q},\omega)$ IN RPA.

FROM PAGE 38,

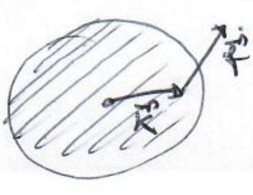
$$\Pi^{(RPA)}(\vec{q},\omega) = \frac{2}{V}\sum_{\vec{k}} \frac{\Theta(k_F - |\vec{k}+\vec{q}|) - \Theta(k_F - k)}{\xi^{(0)}_{\vec{k}+\vec{q}} - \xi^{(0)}_{\vec{k}} - \omega}$$

FROM NUMERATOR: CONTRIBUTIONS COME ONLY WHEN



$k > k_F$ AND $|\vec{k}+\vec{q}| < k_F$

OR



$k < k_F$ AND $|\vec{k}+\vec{q}| > k_F$

IN THE 1st CASE, $\xi^{(0)}_{k+q} = \frac{|\vec{k}+\vec{q}|^2}{2} < \xi^{(0)}_k$

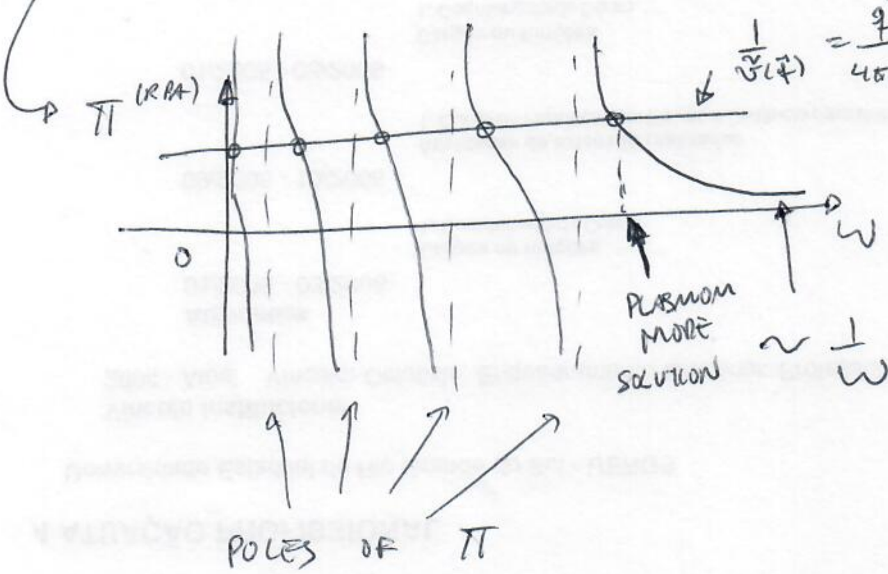
$$\Rightarrow \Pi^{(RPA)} \propto \sum_k \frac{1}{-|\xi^{(0)}_k - \xi^{(0)}_{k+q}| - \omega}$$ → POLES AT NEGATIVE FREQUENCIES

IN THE 2ND CASE,

$$\Pi^{(RPA)} \propto \sum_k \frac{-1}{|\xi^{(0)}_{k+q} - \xi^{(0)}_k| - \omega}$$ → POLES AT POSITIVE FREQUENCIES

↓

THESE ARE THE ONES WE ARE LOOKING FOR



$\sim \frac{1}{\omega}$ → THIS IS WHEN ω IS VERY LARGE BIGGER THAN ANY RESONANCE OF THE NON-INTERACTING SYSTEM (FOR FIXED q)

$\omega = (\xi^{(0)}_{k+q} - \xi^{(0)}_k)$ → FOR A FINITE SYSTEM, THE NO. OF POLES IS FINITE (DISCRETE k's)

THE SYSTEM'S RESONANCES HAPPEN WHEN $\Pi^{(RPA)} = \frac{1}{\tilde{v}(q)} = \frac{q^2}{4\pi e^2}$

MANY SOLUTIONS → ALMOST ALL OF THEM CORRESPOND TO PARTICLE-HOLE EXCITATIONS

THEY FORM A CONTINUUM IN THE THERMODYNAMIC LIMIT.

POLES OF $\pi^{(RPA)}(\vec{q},\omega)$:

$$\omega = \xi^{(0)}_{\vec{k}+\vec{q}} - \xi^{(0)}_{k} = \frac{|\vec{k}+\vec{q}|^2 - k^2}{2m} = \frac{q^2 + 2\vec{k}\cdot\vec{q}}{2m}$$

CONSTRAINT: $\omega \geq 0$, $k < k_F$, $|\vec{k}+\vec{q}| > k_F$

FOR A GIVEN q, WHAT IS THE MAXIMUM VALUE FOR ω?

$\rightarrow$ MAXIMIZE $\omega \Rightarrow \vec{k}\cdot\vec{q} = kq \rightarrow \vec{k} \parallel \vec{q}$

$\Rightarrow \omega = \frac{q^2 + 2kq}{2m} \rightarrow$ MAXIMIZE EVEN MORE $\Rightarrow k = k_F$

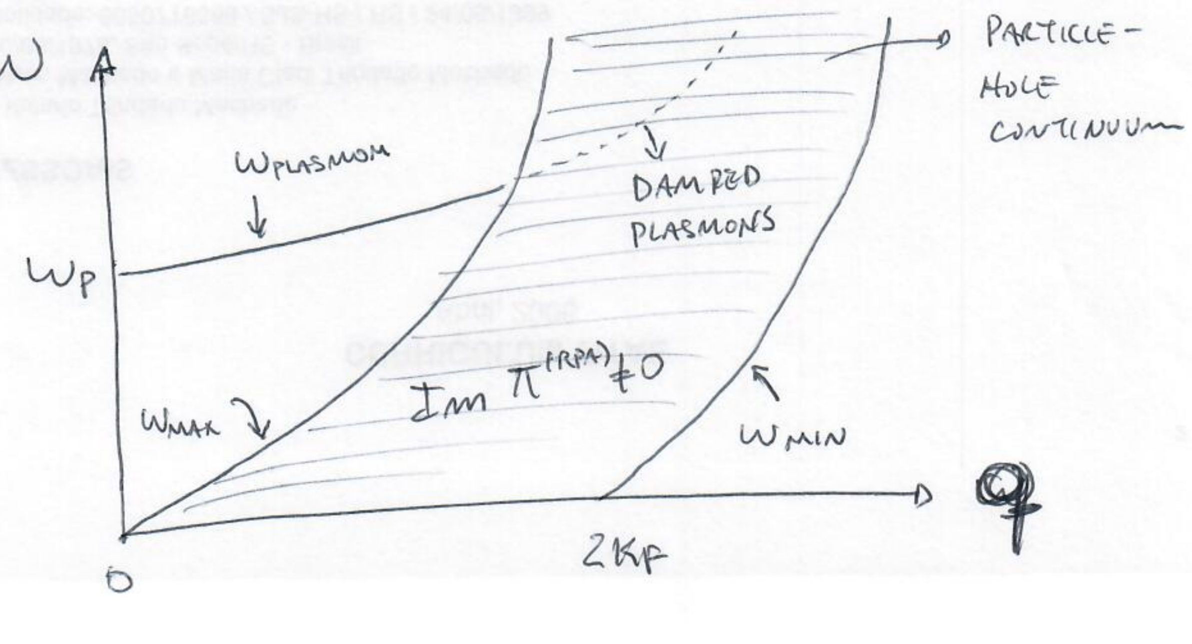
$$\Rightarrow \boxed{\omega_{MAX}(q) = \frac{q^2 + 2k_F q}{2m}}$$

HOW ABOUT THE MINIMUM VALUE? $\Rightarrow \vec{k}\cdot\vec{q} = -kq$

$$\omega_{min} = \frac{q^2 - 2kq}{2m} = \frac{q(q - 2k)}{2m}$$

IF $q < 2k_F \Rightarrow$ THERE IS ALWAYS ~~A~~ POSSIBLE k FOR MAKING $\omega_{min} = 0$

IF $q > 2k_F \Rightarrow \omega_{MIN}(q) = \frac{q^2 - 2k_F q}{2m}$



LET US FOCUS ON THE PLASMON MODE SOLUTION (PAGE 64) WHERE ω IS BEYOND THE PARTICLE-HOLE CONTINUUM (HIGHEST ω SOLUTION).

FROM PAGE 63, WE HAVE THAT

$$\Pi^{(RPA)} = \frac{2}{V}\oint \sum_{\vec{k}} \left(\frac{\theta(k_F - k)}{\xi^{(0)}_{\vec{k}} - \xi^{(0)}_{\vec{k}-\vec{q}} - \omega} - \frac{\theta(k_F - k)}{\xi^{(0)}_{k+q} - \xi^{(0)}_{\vec{k}} - \omega} \right)$$

$$= \frac{2}{V} \sum_{k<k_F} \frac{\xi^{(0)}_{\vec{k}+\vec{q}} + \xi^{(0)}_{\vec{k}-\vec{q}} - 2\xi^{(0)}_{k}}{\omega^2 - \left(\xi^{(0)}_{k+q} - \xi^{(0)}_{k-q}\right)\omega + \left(\xi^{(0)}_{k+q} - \xi^{(0)}_{\vec{k}}\right)\left(\xi^{(0)}_{k} - \xi^{(0)}_{k-q}\right)}$$

$$= \frac{2}{V}\frac{q^2}{m} \sum_{k<k_F} \frac{1}{\omega^2 - \frac{2\vec{k}\cdot\vec{q}}{m}\omega + \frac{q^4 - 4(\vec{k}\cdot\vec{q})^2}{(2m)^2}}$$

IN THE SMALL q LIMIT

$$\Pi^{(RPA)} \approx \frac{2q^2}{Vm} \sum_{k<k_F} \frac{1}{\omega^2}\left(1 + 2\frac{\vec{k}\cdot\vec{q}}{m\omega} + 3\left(\frac{\vec{k}\cdot\vec{q}}{m\omega}\right)^2 + \cdots\right)$$

$$\rightarrow \frac{2q^2}{m\omega^2}\frac{1}{(2\pi)^2}\int_0^\pi \sin\theta\, d\theta \int_0^{k_F} k^2 dk \left(1 + \frac{2kq}{m\omega}\cos\theta + \frac{3k^2q^2}{m^2\omega^2}\cos^2\theta + \cdots\right)$$

$$= \frac{2q^2}{m\omega^2}\frac{1}{(2\pi)^2}\left(\frac{2}{3}k_F^3 + 0 + \frac{3q^2}{m^2\omega^2} \times \frac{2}{3} \times \frac{k_F^5}{5} + \cdots\right)$$

$$= \frac{q^2 k_F^3}{3\pi^2 m\omega^2}\left(1 + \frac{3}{5}\frac{q^2k_F^2}{m^2\omega^2} + \cdots\right)$$

RECALL THAT $2\int \frac{d\vec{k}}{(2\pi)^3} = \frac{N}{V} = n = \frac{k_F^3}{3\pi^2}$

$$\Rightarrow \quad \Pi^{(RPA)}(\vec{q},\omega) \approx \frac{n q^2}{m\omega^2}\left(1+\frac{3}{5}\left(\frac{k_F q}{m\omega}\right)^2\right)$$ (67)

FOR $k_F q \ll m\omega$

THE SYSTEM'S PLASMON RESONANCE IS THEN AT

$$1 = \tilde{v}(\vec{q})\,\Pi(\vec{q},\omega) = \frac{4\pi n e^2}{m\omega^2}\left(1+\frac{3}{5}\left(\frac{k_F q}{m\omega}\right)^2\right)$$

$$= \left(\frac{\omega_p}{\omega}\right)^2\left(1+\frac{3}{5}\left(\frac{k_F}{m\omega}q\right)^2\right)$$

FOR $k_F q \ll m\omega \Rightarrow 1 \approx \left(\frac{\omega_p}{\omega}\right)^2\left(1+\frac{3}{5}\left(\frac{k_F}{m\omega_p}q\right)^2\right)$

SINCE $\frac{k_F^2}{m^2\omega_p^2} = \frac{k_F^2\, m}{m^2\, 4\pi n e^2} = \frac{k_F^2}{m\, 4\pi \underbrace{k_F^3}_{3\pi^2} e^2} = \frac{3\pi}{4 e^2 m k_F} = \frac{3}{q_{TF}^2}$

$$\Rightarrow \boxed{\omega \approx \omega_p\left(1+\frac{9}{10}\left(q/q_{TF}\right)^2+\dots\right)}$$

PLASMON DISPERSION RELATION

COLLECTIVE MODE OF THE INTERACTING e^- GAS

ASSOCIATED OPERATOR : $c^+_{k+q}\, c_k \rightarrow$ BOSON

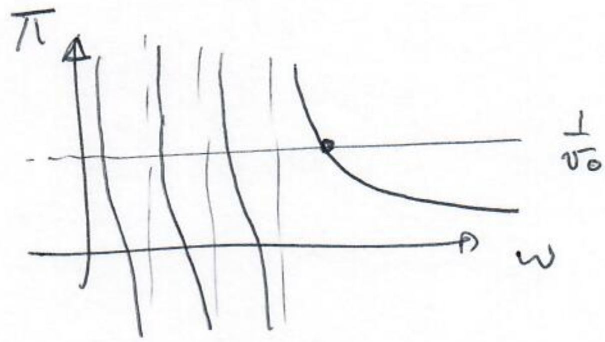
ZERO SOUND

IS THERE AN ANALOGOUS COLLECTIVE MODE FOR

NEUTRAL FERMI GAS (AS IN ^3He)?

- ASSUME $v(\vec{r}) = v_0\,\delta(\vec{r}) \Rightarrow \tilde{v}(q) = v_0$

~~NEUTRAL~~ RESONANCES:



- FOR $v_0 \gg 1$ (FOLLOWING THE PREVIOUS STEPS, PAGE 67)

$$1 \approx v_0 \frac{n\, q^2}{m\, \omega^2}\left(1 + \dots\right)$$

$$\Rightarrow \boxed{\omega = c\,q}\,, \quad \text{SOUND VELOCITY} \quad c = \sqrt{\frac{v_0 n}{m}}$$

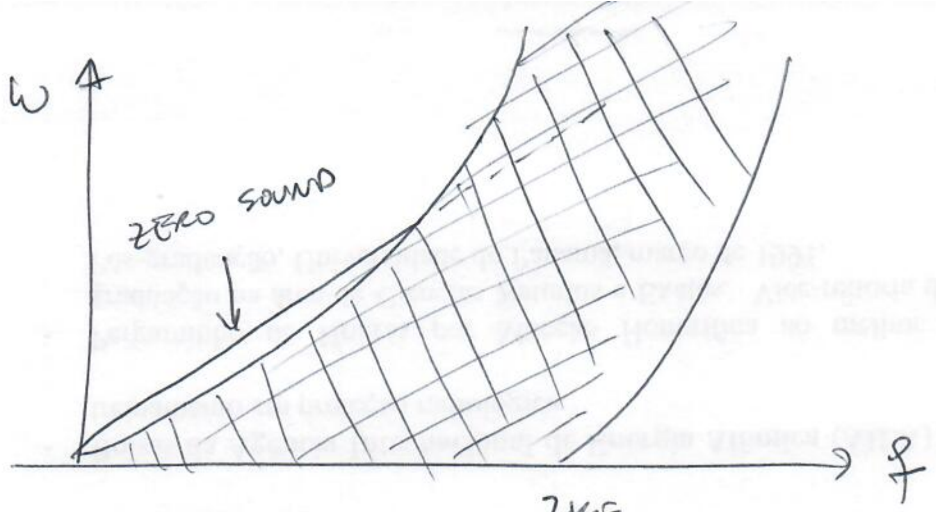
- FOR $v_0 \ll 1 \rightarrow$ INTERCEPT IS IN THE DIVERGENCY AND INDEPENDS ON v_0

$$\Rightarrow \omega = \xi^{(0)}_{k+q} - \xi^{(0)}_{k} = \frac{2\vec{k}\cdot\vec{q} + q^2}{2m} \approx \frac{2k_F}{2m}\, q = c\,q\,, \quad \boxed{c = \frac{k_F}{m} = v_F}$$

FOR A BETTER JOB: $\Pi \approx \frac{m k_F}{\pi^2}\left(\frac{1}{2}\ln\left|\frac{x+1}{x-1}\right| - 1\right)$, $x = \frac{\omega}{2k_F q/2m}$

$$\text{RESONANCE} = \frac{\pi^2}{m k_F \tilde{v}(q)} = \frac{1}{2}\ln\left|\frac{x+1}{x-1}\right| - 1 \xrightarrow{q \ll k_F} \boxed{\omega = v_F\left(1 + 2e^{-\frac{2\pi^2}{m k_F v_0} - 2}\right) q}$$

ZERO SOUND DISPERSION RELATION



ZERO SOUND IS VERY DIFFERENT FROM ORDINARY SOUND (1st SOUND)

ZERO SOUND	1st SOUND
∘ $\omega = c_0 q$	∘ $\omega = c_q q$ ↓ ALTHOUGH $c_q \approx c$ FOR $q \ll 1$
∘ GENERATED BY e-e INTERACTIONS (PHONON IN THE FERMI LIQUID)	∘ GENERATED BY COLLISIONS WITH A LATTICE
∘ LOW COLLISION RATE $\omega\tau \gg 1$	∘ HYDRODYNAMIC LIMIT $\omega\tau \ll 1$ (HIGH COLLISION RATE)
$\tau \sim \frac{1}{T^2}$ ≡ MEAN TIME BETWEEN COLLISIONS ↓ ONLY LIVES IN THE LOW-TEMPERATURE REGIME	

* IT HAS BEEN OBSERVED EXPERIMENTALLY IN ^3He