Problem set 1 - Many-body theory SFI7534

1. **Harmonic oscillator.** For the simple Harmonic oscillator $H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$,

   (a) show that the correlation function
   $$\frac{1}{2}\left\langle\{x(t), x(0)\}\right\rangle = \frac{\hbar\cos\omega t}{2m\omega}\left(1 + 2\left\langle n\right\rangle\right),$$
   where $\langle\cdots\rangle$ is the canonical thermal average.

   (b) What are the physical meaning of these 2 term on the right hand side?

   (c) In the high-temperature limit, does your result agree with the equipartition theorem? What is the high-temperature limit?
   (*Hint*: Recall that $e^X Y e^{-X} = Y + [X,Y] + \frac{1}{2!}[X,[X,Y]] + \ldots$, and that $\left\langle a^\dagger a\right\rangle = 1/(e^{\beta\hbar\omega} - 1)$.)

2. **Field operator.** The mth Eigenstate of a free particle in a 1D box of size L is $\langle x|m\rangle = \sqrt{\frac{2}{L}}\sin\left(m\pi x/L\right)$. Let $c_m^\dagger$ and c_m be the corresponding creation and annihilation operators of spinless fermions. Adopting the states in the Fock space as $|n_1, n_2, \ldots\rangle$, with n_m denoting the occupation in the mth Eigenstate, answer the following:

   (a) Compute $c_6 c_4^\dagger c_6^\dagger c_4 c_2 \left|1,1,1,1,1,0,0,\ldots\right\rangle$.

   (b) What operator must be applied to $\left|1,1,1,1,1,0,0,\ldots\right\rangle$ in order to obtain $\left|1,1,0,1,0,0,1,0,0,\ldots\right\rangle$?

   (c) Compute $\left\{\psi(x), c_m^\dagger\right\}$, where $\psi(x)$ is a fermionic field operator.

3. **Coherent states.** Consider the bosonic state in the Fock space
   $$|\phi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle,$$
   where $|n\rangle = \frac{\left(b^\dagger\right)^n}{\sqrt{n!}}|0\rangle$, with $b^\dagger$ being the creation operator related to the single-boson state $|\psi_b\rangle$ in the Hilbert space.

   (a) What are the coefficients c_n such that $|\phi\rangle = e^{Eb^\dagger}|0\rangle$ with $E \in \mathbb{C}$?

   (b) Show that $|\phi\rangle$ is an Eigenstate of the annihilation operator b with Eigenvalue E, $b|\phi\rangle = E|\phi\rangle$.

   (c) Show that this basis is not orthonormal $\langle\phi|\varphi\rangle = e^{E^*F}$, where $b|\varphi\rangle = F|\varphi\rangle$, and that it is overcomplete
   $$\mathbb{I} = \frac{1}{\pi}\int \mathrm{d}\left(\Re E\right)\mathrm{d}\left(\Im E\right) e^{-E^*E}|\phi\rangle\langle\phi|.$$

   (d) Compute the average number $\langle n\rangle = \left\langle\phi\left|b^\dagger b\right|\phi\right\rangle / \langle\phi|\phi\rangle$.

4. Particle density operator. Let $\tilde{\rho}(\mathbf{q}) = \int e^{i\mathbf{q}\cdot\mathbf{r}}\rho(\mathbf{r})\mathrm{d}\mathbf{r}$ be the Fourier transform of the particle density operator $\rho(\mathbf{r})$. For a system of N particles, answer the following:

   (a) Write $\tilde{\rho}(\mathbf{q})$ in first quantization.

   (b) Write $\tilde{\rho}(\mathbf{q})$ in second quantization for bosonic and fermionic operators in momentum Eigenstates.

   (c) Let $A = \int \mathrm{d}\mathbf{q}\tilde{\rho}(-\mathbf{q})\tilde{\rho}(\mathbf{q})$. Show that, in second quantization, A is a constant when acted in the one-particle Hilbert space.

5. **Simple interacting system.** Let
   $$H_0 = \sum_{i=1}^{N}\frac{1}{2m}p^2 + U(\mathbf{r}_i)$$
   be the Hamiltonian of a N free (spinless bosonic or fermionic) particles.

(a) Write H_0 is second quantization in term of the field operators $\psi(\mathbf{r})$ and $\psi^\dagger(\mathbf{r})$.

(b) In the Heisenberg representation, find the movement equation for those fields evolving according to H_0. (Does it resembles the Schrödinger's equation?)

(c) Consider now the effects of interactions, i.e., $H = H_0 + H_{\text{int}}$ where

$$H_{\text{int}} = \frac{1}{2}\sum_{i\neq j} V(|\mathbf{r}_i - \mathbf{r}_j|).$$

Write H_{int} in second quantization and find the new movement equation for the field operators for the cases of bosonic and fermionic particles.

(d) Discuss the case of punctual contact interactions for fermionic particles, i.e., $V(|\mathbf{r}_i - \mathbf{r}_j|) = g\delta(\mathbf{r}_i - \mathbf{r}_j)$. Does your answer change when considering fermions with spins? Explain.

6. **Current density.** The current density operator $\mathbf{J}(\mathbf{r})$ is defined from the continuity equation $\partial_t\rho(\mathbf{r}) = -\nabla\cdot\mathbf{J}(\mathbf{r})$, where $\rho(\mathbf{r}) = \psi^\dagger(\mathbf{r})\psi(\mathbf{r})$ is the corresponding density operator and $\partial_t\rho(\mathbf{r}) = i\hbar^{-1}[H, \rho(\mathbf{r})]$. Show that, for the system Hamiltonian in problem 5, the current operator is

$$\mathbf{J}(\mathbf{r}) = -\frac{i\hbar}{2m}\left[\psi^\dagger(\mathbf{r})\left(\nabla\psi(\mathbf{r})\right) - \left(\nabla\psi^\dagger(\mathbf{r})\right)\psi(\mathbf{r})\right].$$

7. **1D bosonic system.** Consider the Hamiltonian

$$H = \sum_{i=1}^{N} J_1\left(b_i^\dagger b_{i+1} + \text{h.c.}\right) + J_2\left(b_i^\dagger b_{i+1}^\dagger + \text{h.c.}\right),$$

where $b_i^\dagger$ creates a boson on site i of a ring of N sites and lattice constant a.

(a) Give physical meaning for all the terms in H.

(b) Diagonalize H.
(*Hint*: Defining $b_j = N^{-1/2}\sum_q e^{iqaj}b_q$, write H in the momentum space and perform a transformation $b_q = u_q c_q + v_q c_{-q}^\dagger$ choosing the coefficients u_q and v_q such that c_q is also a bosonic operator.)

(c) What does happen when $J_1 = J_2$?