Problem set 4 - Many-body theory SFI7534

1. **Gaussian integral of complex variables.**

   (a) Let $\mathbb{H}$ be an Hermitean positive definite $n \times n$ matrix. Then show that

   $$\int \prod_{i=1}^{n} \frac{\mathrm{d}\left(\Re z_i\right)\mathrm{d}\left(\Im z_i\right)}{\pi} e^{-(\mathbf{z}^*)^T\cdot\mathbb{H}\cdot\mathbf{z}+(\mathbf{z}^*)^T\cdot\mathbf{J}+(\mathbf{J}^*)^T\cdot\mathbf{z}} = (\mathrm{Det}\,\mathbb{H})^{-1} e^{(\mathbf{J}^*)^T\cdot\mathbb{H}^{-1}\cdot\mathbf{J}},$$

   where $\mathbf{J} \in \mathbb{C}^n$ is a complex vector.

   (b) Use this result to show that

   $$\langle z_{i1}^* \dots z_{im}^* z_{j1} \dots z_{jm}\rangle = \sum_{\text{all pairings}} \mathbb{H}^{-1}_{j1,iP_1} \dots \mathbb{H}^{-1}_{jm,iP_m},$$

   where $\{P_1, \dots, P_m\}$ is a permutation of $\{1, \dots, m\}$, and the average is defined as

   $$\langle A\rangle = \frac{\int \prod_{i=1}^{n} \frac{\mathrm{d}(\Re z_i)\mathrm{d}(\Im z_i)}{\pi} e^{-(\mathbf{z}^*)^T\cdot\mathbb{H}\cdot\mathbf{z}} A}{\int \prod_{i=1}^{n} \frac{\mathrm{d}(\Re z_i)\mathrm{d}(\Im z_i)}{\pi} e^{-(\mathbf{z}^*)^T\cdot\mathbb{H}\cdot\mathbf{z}}}.$$

2. **Gaussian integral of Grassmann variables.**

   (a) Let $\mathbb{H}$ be an Hermitean positive definite $n \times n$ matrix (actually, this is not necessary). Then show that

   $$\int \prod_{i=1}^{n} \mathrm{d}\bar{\eta}_i \mathrm{d}\eta_i e^{-(\bar{\boldsymbol{\eta}})^T\cdot\mathbb{H}\cdot\boldsymbol{\eta}+(\bar{\boldsymbol{\eta}})^T\cdot\boldsymbol{\xi}+\left(\bar{\boldsymbol{\xi}}\right)^T\cdot\boldsymbol{\eta}} = (\mathrm{Det}\,\mathbb{H}) e^{\left(\bar{\boldsymbol{\xi}}\right)^T\cdot\mathbb{H}^{-1}\cdot\boldsymbol{\xi}},$$

   where $\boldsymbol{\eta}$, $\bar{\boldsymbol{\eta}}$, $\boldsymbol{\xi}$ and $\bar{\boldsymbol{\xi}}$ are independent Grassmann vectors.

   (b) Use this result to show that

   $$\langle \eta_{i1}\eta_{i2} \dots \eta_{im}\bar{\eta}_{jm} \dots \bar{\eta}_{j2}\bar{\eta}_{j1}\rangle = \sum_{\text{all pairings}} (-1)^P\, \mathbb{H}^{-1}_{i1,jP_1} \dots \mathbb{H}^{-1}_{im,jP_m},$$

   where $\{P_1, \dots, P_m\}$ is a permutation of $\{1, \dots, m\}$, P is the number of transpositions in this permutation, and the average is defined as in the previous problem.

3. **Coherent states.** Consider the coherent states $a_i\left|\boldsymbol{\psi}\right\rangle = \psi_i\left|\boldsymbol{\psi}\right\rangle$, $\left\langle\boldsymbol{\psi}\right| a_i^\dagger = \left\langle\boldsymbol{\psi}\right|\bar{\psi}_i$ with $\boldsymbol{\psi}$ and $\bar{\boldsymbol{\psi}}$ being independent (complex for bosons, and Grassmann for fermions) vectors. Show

   (a) the completeness relation

   $$\mathbb{I} = \int \mathrm{d}\left(\bar{\boldsymbol{\psi}}, \boldsymbol{\psi}\right) e^{-\bar{\boldsymbol{\psi}}\cdot\boldsymbol{\psi}} \left|\boldsymbol{\psi}\right\rangle\left\langle\boldsymbol{\psi}\right|,$$

   (b) the trace

   $$\mathrm{tr}\left\{A\right\} = \int \mathrm{d}\left(\bar{\boldsymbol{\psi}}, \boldsymbol{\psi}\right) e^{-\bar{\boldsymbol{\psi}}\cdot\boldsymbol{\psi}} \left\langle\zeta\boldsymbol{\psi}\right| A \left|\boldsymbol{\psi}\right\rangle,$$

   (c) and that, for a normal ordered operator $O \equiv O(a_1^\dagger, \dots, a_n^\dagger, a_1, \dots, a_n)$, the matrix element is

   $$\left\langle\boldsymbol{\psi}\right| O \left|\boldsymbol{\psi}'\right\rangle = O(\bar{\psi}_1, \dots, \bar{\psi}_n, \psi_1', \dots, \psi_n') e^{\bar{\boldsymbol{\psi}}\cdot\boldsymbol{\psi}'}.$$

   Here, $\zeta = 1$ for bosons and $\zeta = -1$ for fermions, and $\mathrm{d}\left(\bar{\boldsymbol{\psi}}, \boldsymbol{\psi}\right) = \prod_i \mathrm{d}\left(\bar{\psi}_i, \psi_i\right)$, where $\mathrm{d}\left(\bar{\psi}_i, \psi_i\right) = \pi^{-1}\mathrm{d}\left(\Re\psi_i\right)\mathrm{d}\left(\Im\psi_i\right)$ for bosons, and $\mathrm{d}\left(\bar{\psi}_i, \psi_i\right) = \mathrm{d}\bar{\psi}_i\mathrm{d}\psi_i$ for fermions.