Aulas passadas

Eq. de onda na corda: $$\frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = 0 \qquad c = \sqrt{\frac{T_0}{\mu}} \ \text{(velocidade da onda)}$$

Soluções (ondas propagantes): $$y(x,t) = f(x - ct) + g(x + ct)$$

Condições iniciais:

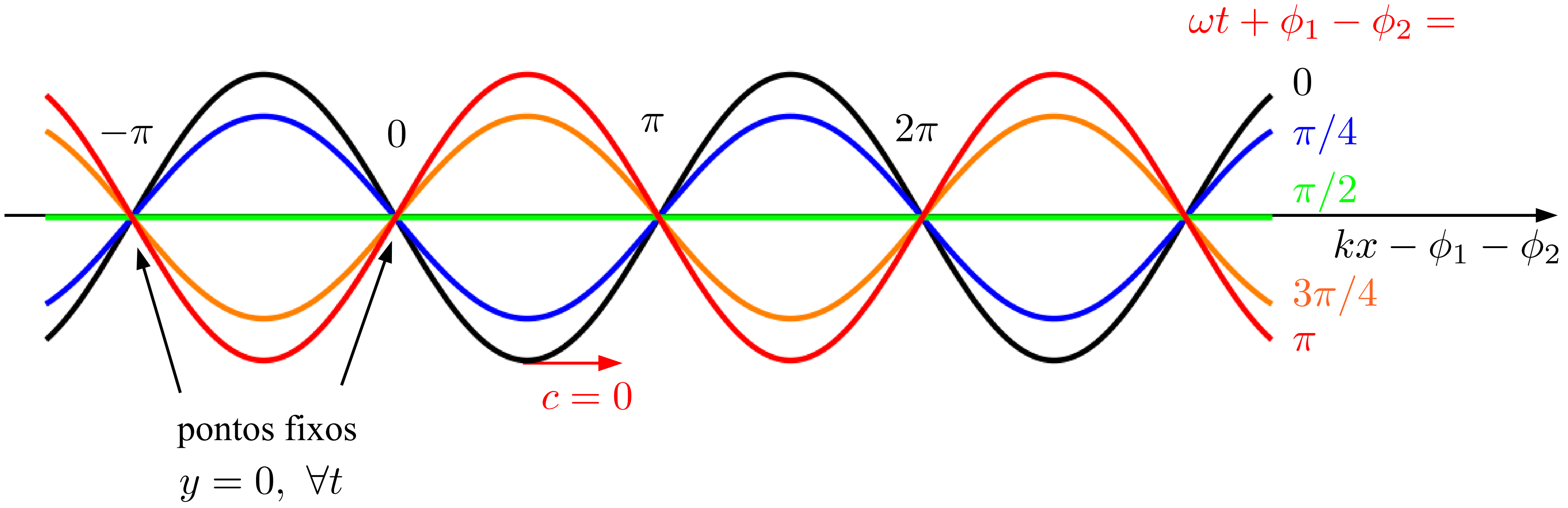
$$\begin{aligned} f(x) &= \frac{1}{2}y_0(x) - \frac{1}{2c}\int_{-\infty}^{x} v_0(z)\mathrm{d}z \\ g(x) &= \frac{1}{2}y_0(x) + \frac{1}{2c}\int_{-\infty}^{x} v_0(z)\mathrm{d}z \end{aligned}$$

Ondas harmônicas: $$y(x,t) = A\cos(kx - \omega t - \phi),\ \omega = ck$$

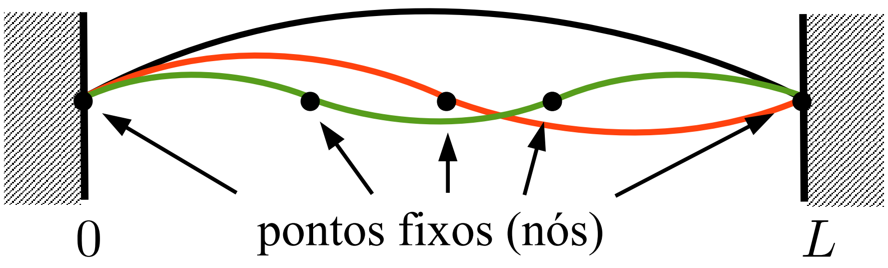
Aulas passadas

Ondas estacionárias:

$$\begin{aligned} y(x,t) &= A\sin(kx-\omega t-2\phi_1)+A\sin(kx+\omega t-2\phi_2) \\ &= 2A\sin(kx-\phi_1-\phi_2)\cos(\omega t+\phi_1-\phi_2) \end{aligned}$$

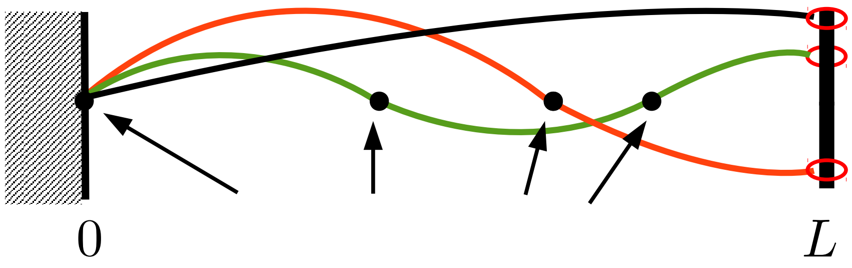


Aulas passadas



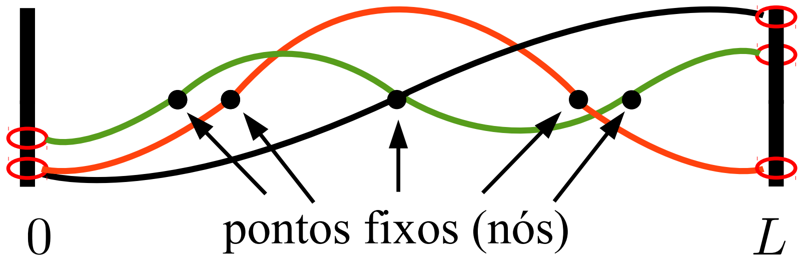
$$y(x,t) = \sum_{n=1}^{\infty} C_n \sin(k_n x)\cos(\omega_n t - \varphi_n)$$

$$k_n = \frac{\pi}{L} n, \quad n \in \mathbb{N}^*$$



$$y(x,t) = \sum_{n=1}^{\infty} C_n \sin(k_n x)\cos(\omega_n t - \varphi_n)$$

$$k_n = \frac{(2n-1)\pi}{2L}, \quad n \in \mathbb{N}^*$$



$$y(x,t) = \sum_{n=1}^{\infty} C_n \cos(k_n x)\cos(\omega_n t - \varphi_n)$$

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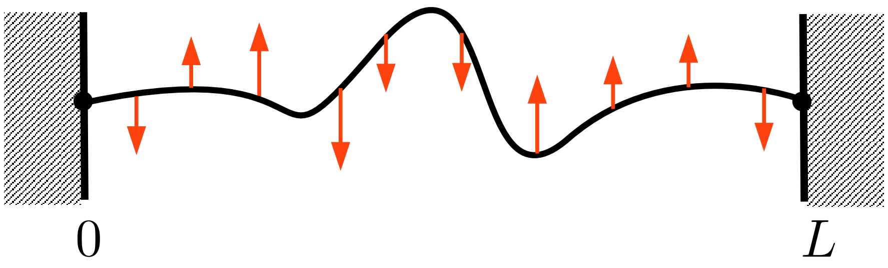
Aulas passadas



$$y(x,t) = \sum_{n=1}^{\infty} C_n \sin(k_n x)\cos(\omega_n t - \varphi_n)$$

$$= \sum_{n=1}^{\infty} \sin(k_n x)\left(A_n \cos(\omega_n t) + B_n \sin(\omega_n t)\right)$$

$$k_n = \frac{\pi}{L} n, \quad \omega_n = c k_n, \quad n \in \mathbb{N}^*$$

$\{A_n, B_n\}$ determinados pelas condições iniciais:

$$y_0(x) = \sum_{n=1}^{\infty} A_n \sin(k_n x) \longrightarrow A_n = \frac{2}{L}\int_0^L y_0(x)\sin(k_n x)\mathrm{d}x$$

$$v_0(x) = \sum_{n=1}^{\infty} B_n \omega_n \sin(k_n x) \longrightarrow B_n = \frac{2}{\omega_n L}\int_0^L v_0(x)\sin(k_n x)\mathrm{d}x$$

$$C_n^2 = A_n^2 + B_n^2, \quad \tan\varphi_n = B_n/A_n$$

1 – Série de Fourier: definição

Representação de uma função integrável de período L em termos de funções harmônicas:

$$f(x+L) = f(x)$$

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} C_n \cos(k_n x - \phi_n) = \sum_{n=0}^{\infty} A_n \cos(k_n x) + B_n \sin(k_n x) \quad \Rightarrow \\ &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(k_n x) + \sum_{n=1}^{\infty} b_n \sin(k_n x) \\ &= \frac{1}{2}c_0 + \sum_{n=1}^{\infty} c_n \cos(k_n x - \phi_n) \end{aligned}$$

$$\boxed{k_n = \tfrac{2\pi}{L} n, \quad n \in \mathbb{N}}$$

Conjunto discreto de funções harmônicas.

Coeficientes de "Fourier":

$$a_n = \frac{2}{L}\int_0^L f(x)\cos(k_n x)\mathrm{d}x$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin(k_n x)\mathrm{d}x$$

$$c_n^2 = a_n^2 + b_n^2, \quad \tan\phi_n = b_n/a_n$$

Note que $b_0 = 0$

1 – Série de Fourier: detalhes matemáticos

Determinação do coeficiente a_0

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(k_n x) + \sum_{n=1}^{\infty} b_n \sin(k_n t)$$

$$\int_0^L \mathrm{d}x f(x) = \frac{L}{2}a_0 + \sum_{n=1}^{\infty} a_n \int_0^L \mathrm{d}x \cos(k_n x) + \sum_{n=1}^{\infty} b_n \int_0^L \mathrm{d}x \sin(k_n x)$$

$$\Rightarrow a_0 = \frac{2}{L}\int_0^L f(x)\mathrm{d}x = \frac{2}{L}\int_0^L f(x)\cos(k_0 x)\mathrm{d}x$$

1 – Série de Fourier: detalhes matemáticos

Determinação do coeficiente $a_{n\neq 0}$

Usando as identidades

$$\int_0^L \cos(k_n x)\cos(k_m x)\mathrm{d}x = \frac{L}{2}\delta_{n,m} \qquad m, n \neq 0$$

$$\int_0^L \cos(k_n x)\sin(k_m x)\mathrm{d}x = 0$$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(k_n x) + \sum_{n=1}^{\infty} b_n \sin(k_n x)$$

$$\int_0^L \mathrm{d}x \cos(k_m x) f(x) = \frac{1}{2}\int_0^L \mathrm{d}x \cos(k_m x) a_0 + \sum_{n=1}^{\infty} a_n \int_0^L \mathrm{d}x \cos(k_m x)\cos(k_n x) + \sum_{n=1}^{\infty} b_n \int_0^L \mathrm{d}x \cos(k_m x)\sin(k_n t)$$

$$\Rightarrow a_m = \frac{2}{L}\int_0^L f(x)\cos(k_m x)\mathrm{d}x$$

1 – Série de Fourier: detalhes matemáticos

Determinação do coeficiente b_n

Usando as identidades

$$\int_0^L \sin(k_n x)\sin(k_m x)\mathrm{d}x = \frac{L}{2}\delta_{n,m} \qquad m, n \neq 0$$

$$\int_0^L \cos(k_n x)\sin(k_m x)\mathrm{d}x = 0$$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(k_n x) + \sum_{n=1}^{\infty} b_n \sin(k_n x)$$

$$\int_0^L \mathrm{d}x \sin(k_m x) f(x) = \frac{1}{2}\int_0^L \mathrm{d}x \sin(k_m x) a_0 + \sum_{n=1}^{\infty} a_n \int_0^L \mathrm{d}x \sin(k_m x)\cos(k_n x) + \sum_{n=1}^{\infty} b_n \int_0^L \mathrm{d}x \sin(k_m x)\sin(k_n t)$$

$$\Rightarrow b_m = \frac{2}{L}\int_0^L f(x)\sin(k_m x)\mathrm{d}x$$

1 – Série de Fourier: detalhes matemáticos

Prova das identidades

$$\int_0^L \cos(k_n x)\cos(k_m x)\mathrm{d}x = \frac{L}{2}\delta_{n,m}$$

$$\int_0^L \sin(k_n x)\sin(k_m x)\mathrm{d}x = \frac{L}{2}\delta_{n,m} \qquad m, n \neq 0$$

$$\int_0^L \cos(k_n x)\sin(k_m x)\mathrm{d}x = 0$$

Usando que $\cos(A-B) + \cos(A+B) = 2\cos(A)\cos(B)$

$$\int_0^L \cos(k_n x)\cos(k_m x)\mathrm{d}x = \int_0^L \cos(2\pi n x/L)\cos(2\pi m x/L)\mathrm{d}x =$$

$$= \int_0^L \frac{\cos\left((n-m)\frac{2\pi x}{L}\right) + \cos\left((n+m)\frac{2\pi x}{L}\right)}{2}\mathrm{d}x \underset{\substack{\uparrow \\ n,m\in\mathbb{N}^*}}{=} \int_0^L \frac{\cos\left((n-m)\frac{2\pi x}{L}\right)}{2}\mathrm{d}x = \frac{L}{2}\delta_{n,m}$$

1 – Série de Fourier: detalhes matemáticos

Prova das identidades

$$\int_0^L \cos(k_n x)\cos(k_m x)\mathrm{d}x = \frac{L}{2}\delta_{n,m}$$

$$\int_0^L \sin(k_n x)\sin(k_m x)\mathrm{d}x = \frac{L}{2}\delta_{n,m} \qquad m,n \neq 0$$

$$\int_0^L \cos(k_n x)\sin(k_m x)\mathrm{d}x = 0$$

Usando que $\cos(A-B) - \cos(A+B) = 2\sin(A)\sin(B)$

$$\int_0^L \sin(k_n x)\sin(k_m x)\mathrm{d}x = \int_0^L \sin(2\pi n x/L)\sin(2\pi m x/L)\mathrm{d}x =$$

$$= \int_0^L \frac{\cos\left((n-m)\frac{2\pi x}{L}\right) - \cos\left((n+m)\frac{2\pi x}{L}\right)}{2}\mathrm{d}x \underset{n,m\in\mathbb{N}^*}{=} \int_0^L \frac{\cos\left((n-m)\frac{2\pi x}{L}\right)}{2}\mathrm{d}x = \frac{L}{2}\delta_{n,m}$$

1 – Série de Fourier: detalhes matemáticos

Prova das identidades

$$\int_0^L \cos(k_n x)\cos(k_m x)\mathrm{d}x = \frac{L}{2}\delta_{n,m}$$

$$\int_0^L \sin(k_n x)\sin(k_m x)\mathrm{d}x = \frac{L}{2}\delta_{n,m} \qquad m, n \neq 0$$

$$\int_0^L \cos(k_n x)\sin(k_m x)\mathrm{d}x = 0$$

Usando que $\sin(A+B) - \sin(A-B) = 2\cos(A)\sin(B)$

$$\int_0^L \cos(k_n x)\sin(k_m x)\mathrm{d}x = \int_0^L \cos(2\pi n x/L)\sin(2\pi m x/L)\mathrm{d}x =$$

$$= \int_0^L \frac{\sin\left((n-m)\frac{2\pi x}{L}\right) - \sin\left((n+m)\frac{2\pi x}{L}\right)}{2}\mathrm{d}x = 0$$

1 – Série de Fourier: detalhes matemáticos

Em suma,

$$\int_0^L \cos(k_n x)\cos(k_m x)\mathrm{d}x = \frac{L}{2}\delta_{n,m}$$

$$\int_0^L \sin(k_n x)\sin(k_m x)\mathrm{d}x = \frac{L}{2}\delta_{n,m} \qquad m,n \neq 0$$

$$\int_0^L \cos(k_n x)\sin(k_m x)\mathrm{d}x = 0$$

significa que

$\boxed{\{\cos(k_n x), \sin(k_n x)\} \text{ são funções ortogonais no intervalo } 0 < x < L}$

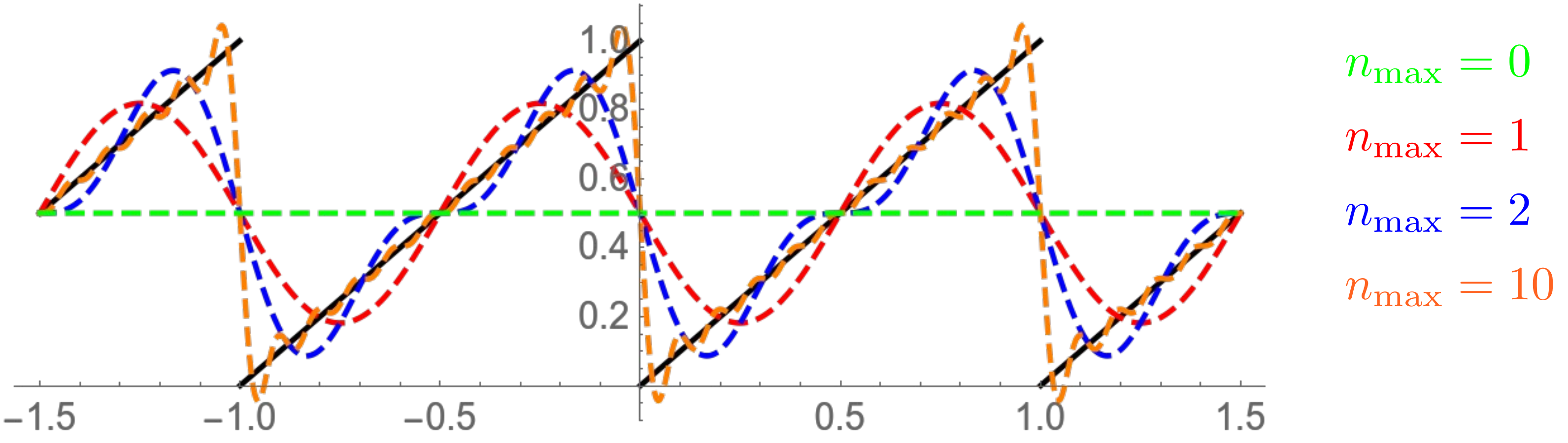
1 – Série de Fourier: 1º exemplo

$$f(x) = x, \quad L = 1$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos(k_n x)\mathrm{d}x = 2\int_0^1 x\cos(2\pi n x)\mathrm{d}x = \delta_{n,0}$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin(k_n x)\mathrm{d}x = 2\int_0^1 x\sin(2\pi n x)\mathrm{d}x = -\frac{1}{n\pi}$$

$$f(x) = \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{n_{\max}} \frac{\sin(2\pi n x)}{n}$$



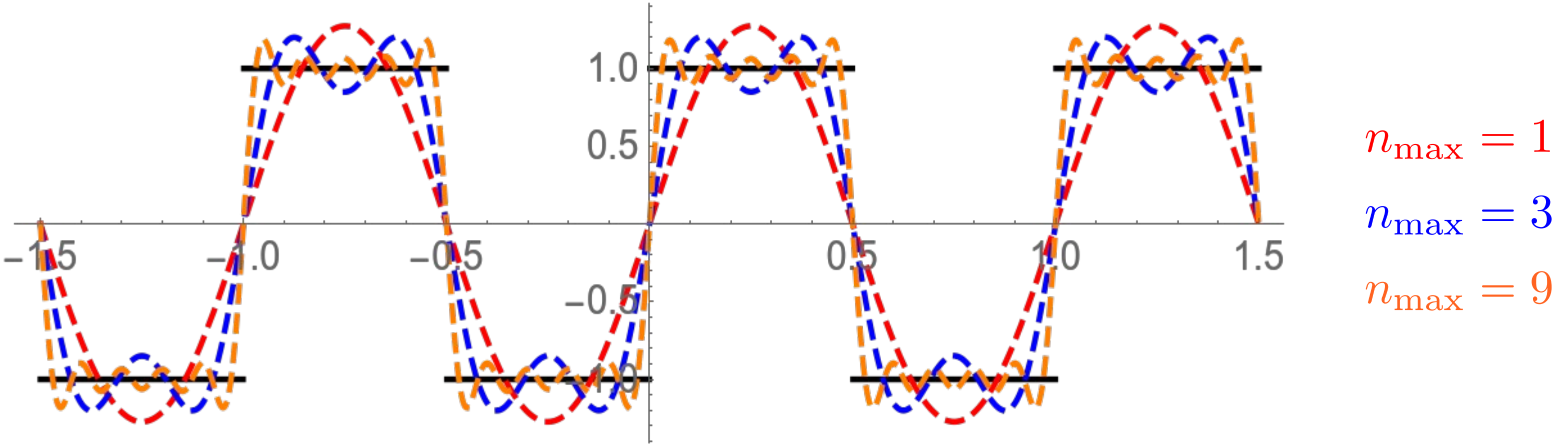
1 – Série de Fourier: 2º exemplo

$$f(x) = \mathrm{sign}(x), \quad L = 1, \quad -\frac{L}{2} < x < \frac{L}{2}$$

$$a_n = 2\int_{-1/2}^{1/2} \mathrm{sign}(x)\cos(2\pi n x)\mathrm{d}x = 0$$

$$b_n = 4\int_{0}^{1/2} \sin(2\pi n x)\mathrm{d}x = \frac{2(1-(-1)^n)}{n\pi}$$

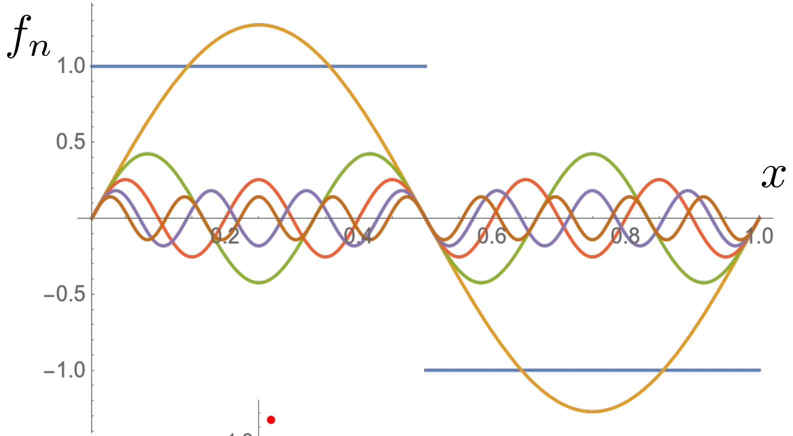
$$f(x) = \frac{2}{\pi}\sum_{n=1}^{n_{\max}} (1-(-1)^n)\frac{\sin(2\pi n x)}{n}$$

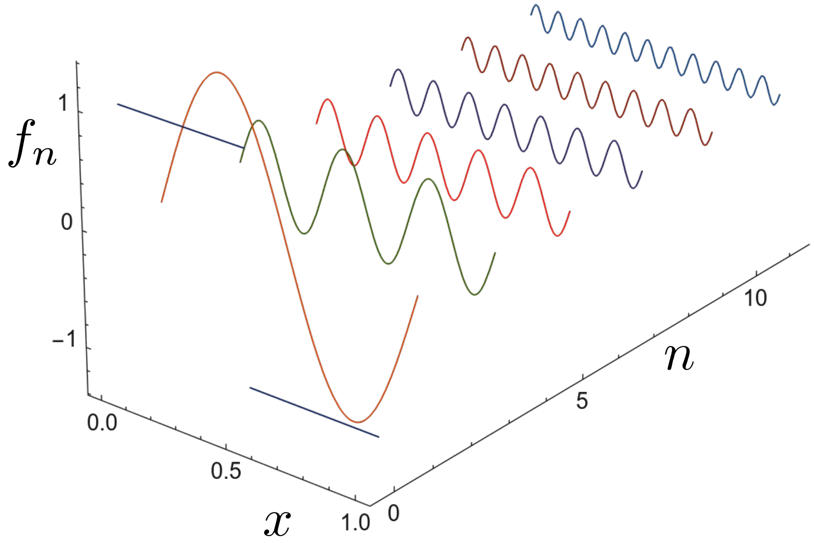


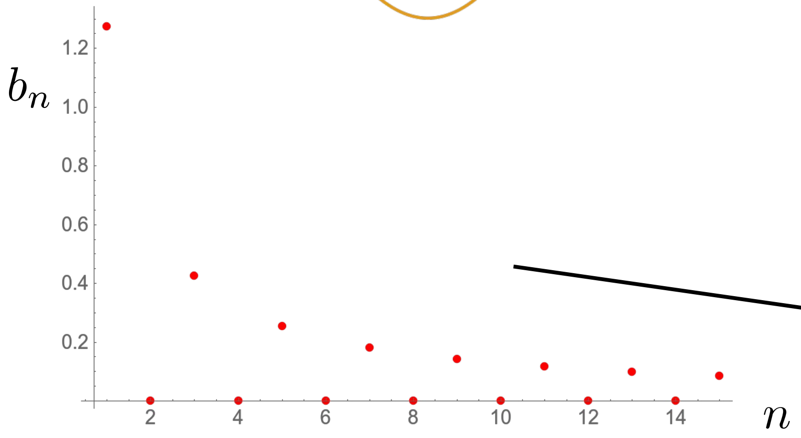
1 – Série de Fourier: 2º exemplo

$$f(x) = \operatorname{sign}(x), \quad L = 1, \quad -\frac{L}{2} < x < \frac{L}{2}$$

$$f(x) = \frac{2}{\pi} \sum_{n=1}^{n_{\max}} (1 - (-1)^n) \frac{\sin(2\pi n x)}{n}$$



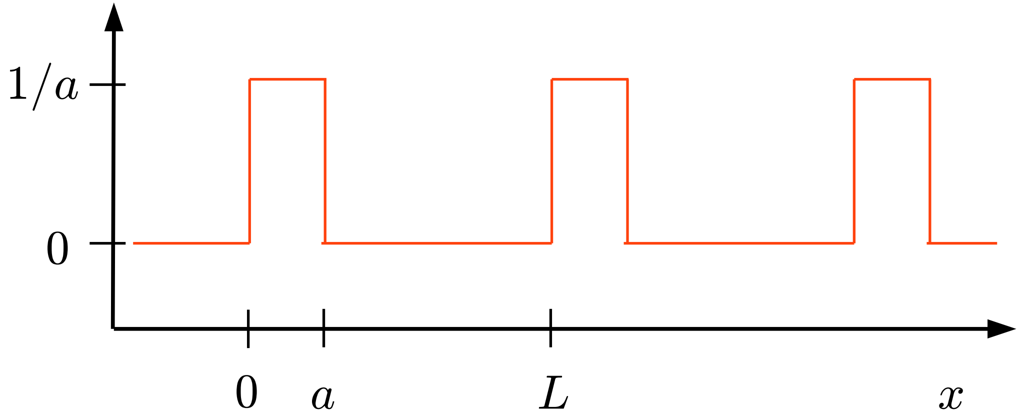




Espectro de “frequências”

1 – Série de Fourier: 3º exemplo

$$f(x) = \frac{1}{a}\Theta(x)\Theta(a-x), \qquad 0 < x < L$$



$$a_n = \frac{2}{L}\int_0^a \frac{1}{a}\cos(2\pi n x)\mathrm{d}x = \frac{\sin(2an\pi/L)}{an\pi}$$

$$= \left(\frac{2}{L}\right)\frac{\sin(k_n a)}{k_n a}$$

$$b_n = \frac{2}{L}\int_0^a \frac{1}{a}\sin(2\pi n x)\mathrm{d}x = \frac{2\sin^2(an\pi/L)}{an\pi}$$

$$= \left(\frac{4}{L}\right)\frac{\sin^2(k_n a/2)}{k_n a}$$

$$\langle x^m \rangle = \frac{\int_0^L x^m f(x)\,\mathrm{d}x}{\int_0^L f(x)\,\mathrm{d}x}$$

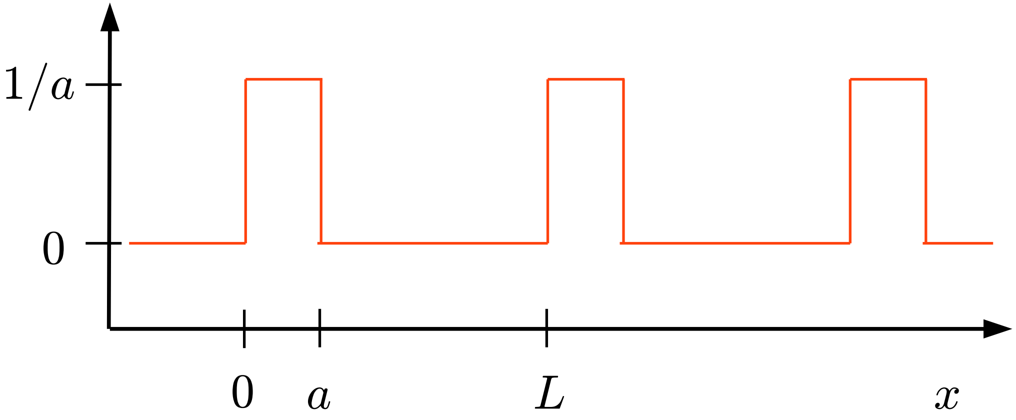
$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \frac{a}{\sqrt{12}}$$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos(k_n x) + \sum_{n=1}^{\infty} b_n \sin(k_n t)$$

$$= \frac{1}{2}c_0 + \sum_{n=1}^{\infty} c_n \cos(k_n x - \phi_n)$$

1 – Série de Fourier: 3º exemplo

$$f(x) = \frac{1}{a}\Theta(x)\Theta(a-x), \qquad 0 < x < L$$



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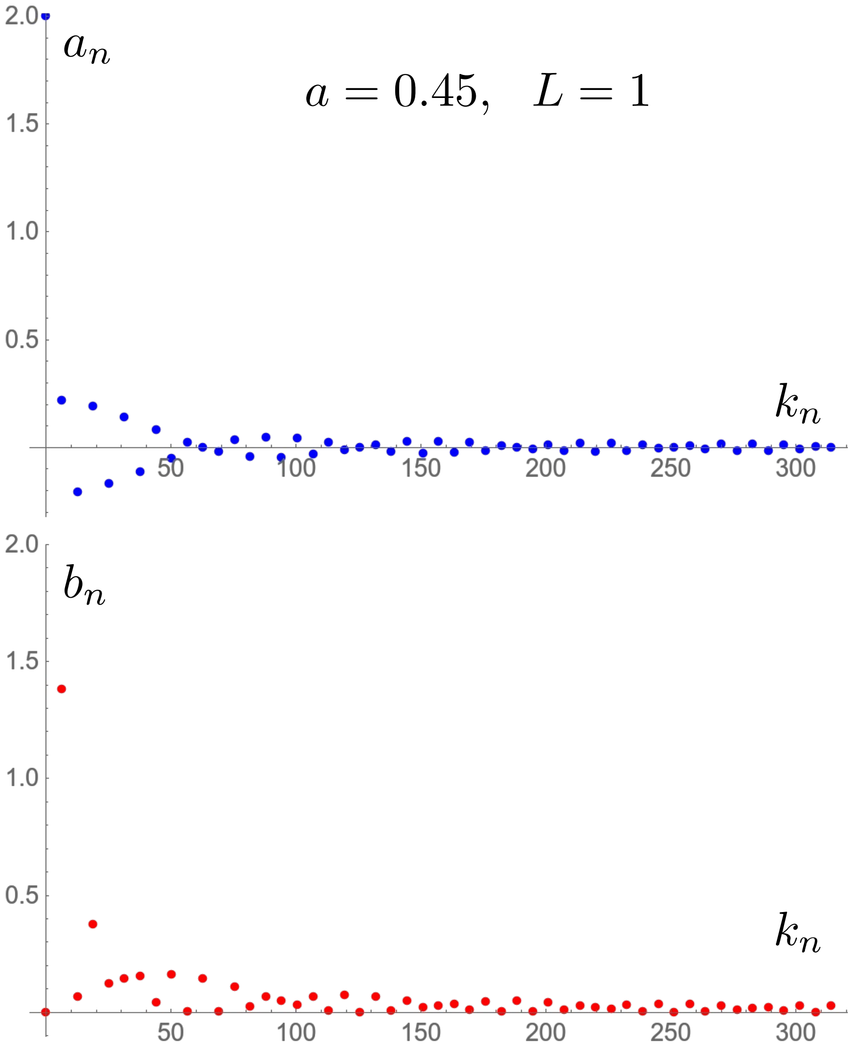

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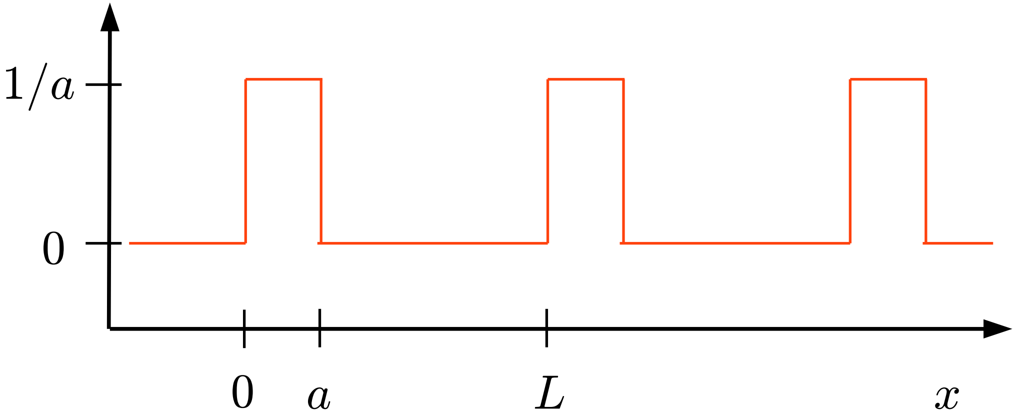

$$\langle x^m \rangle = \frac{\int_0^L x^m f(x)\,\mathrm{d}x}{\int_0^L f(x)\,\mathrm{d}x}$$

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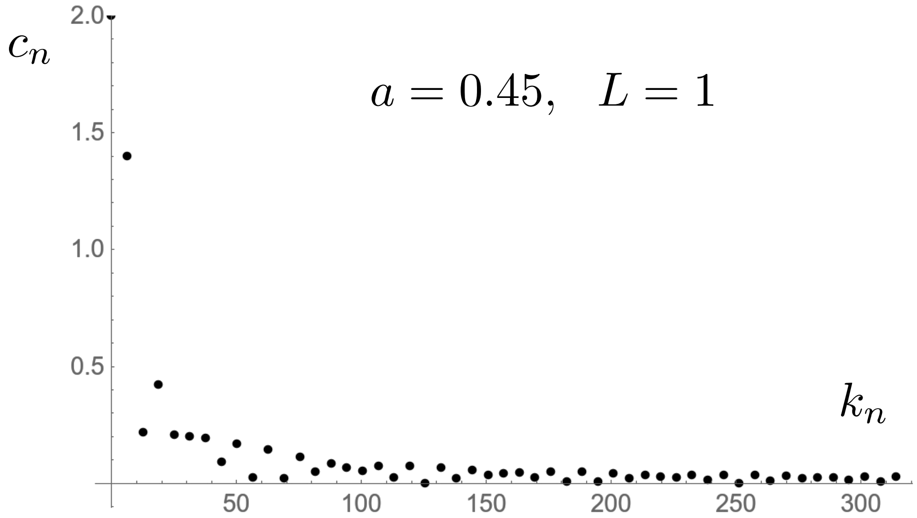

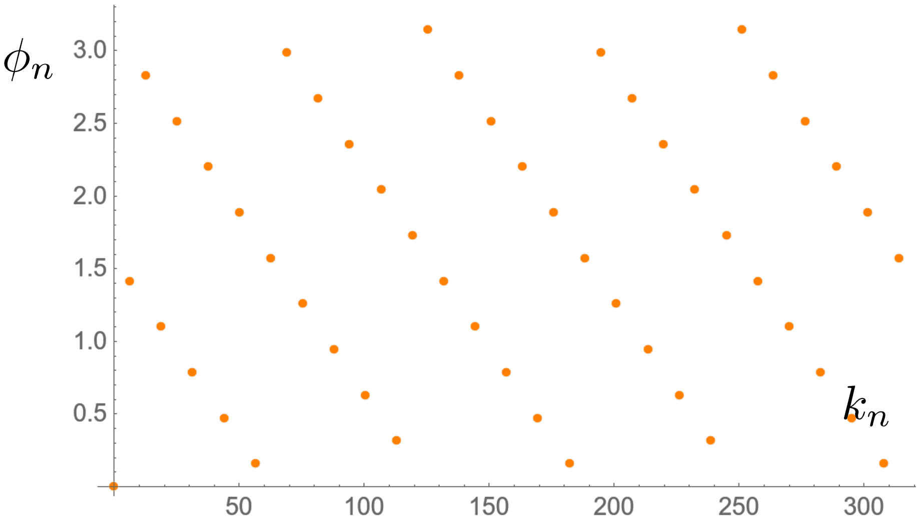

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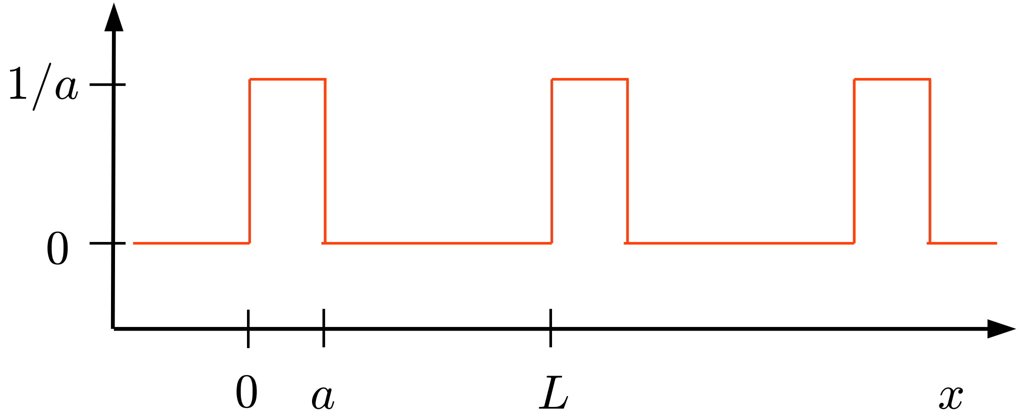

$$\langle x^m \rangle = \frac{\int_0^L x^m f(x)\,\mathrm{d}x}{\int_0^L f(x)\,\mathrm{d}x}$$

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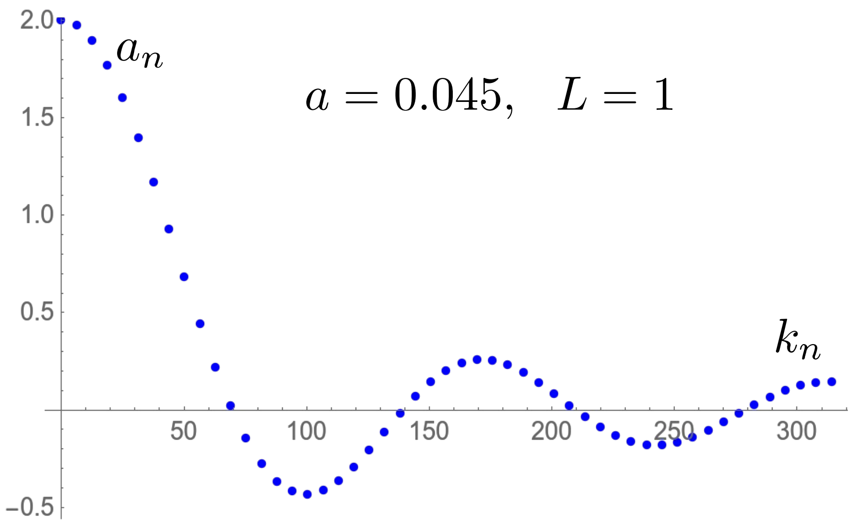

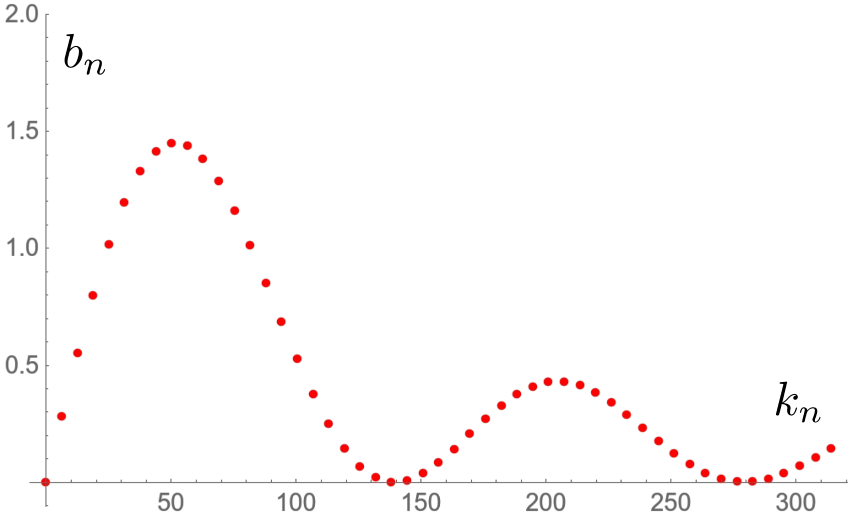

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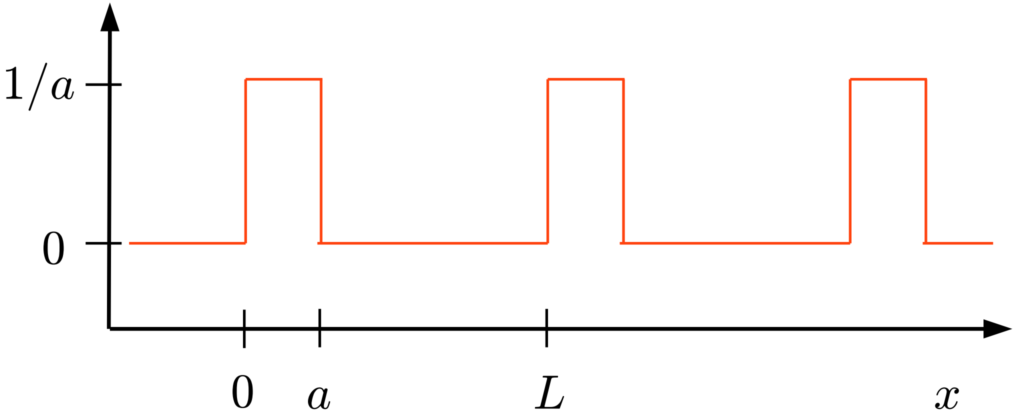

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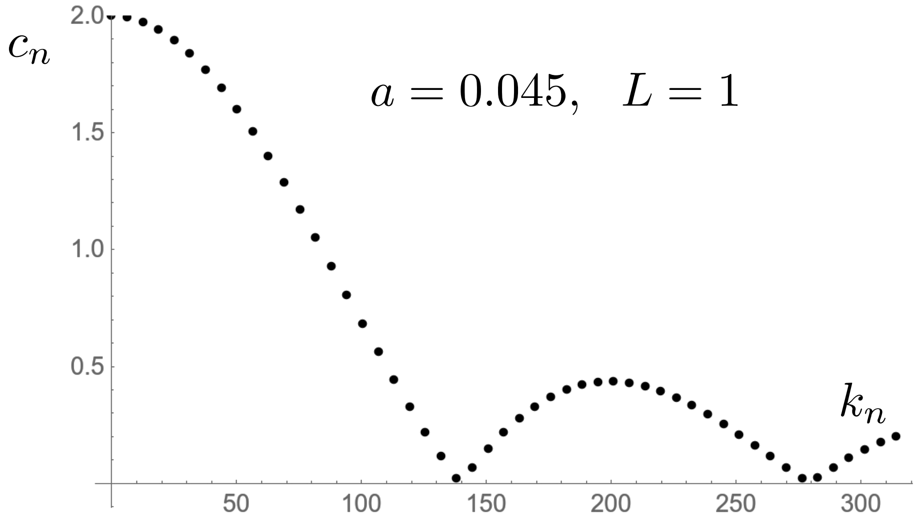

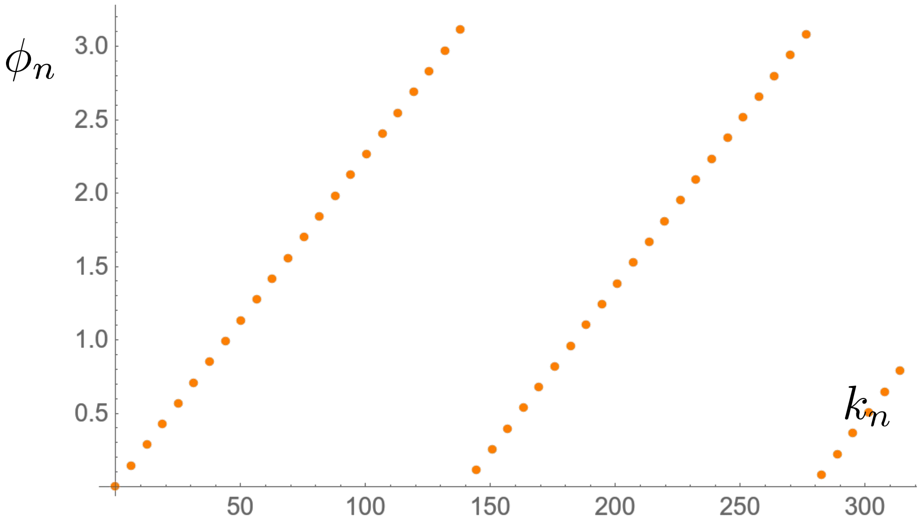

1 – Série de Fourier: 3º exemplo

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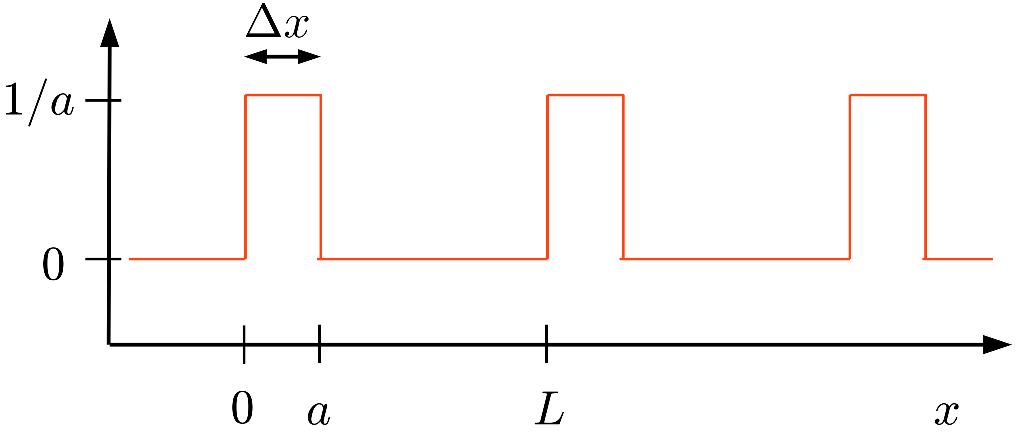

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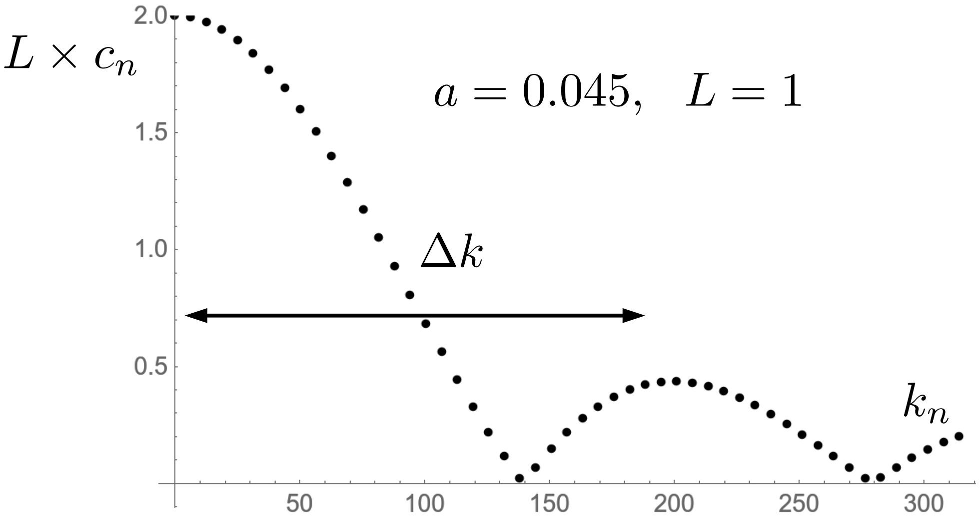


Definindo

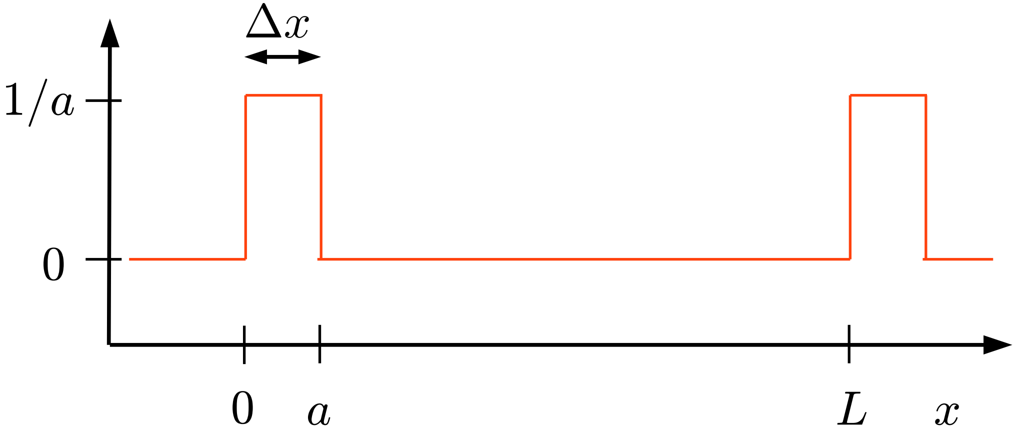
$$\langle k^m \rangle = \frac{\sum_{n=0}^{\infty} k^m c_n}{\sum_{n=0}^{\infty} c_n}$$

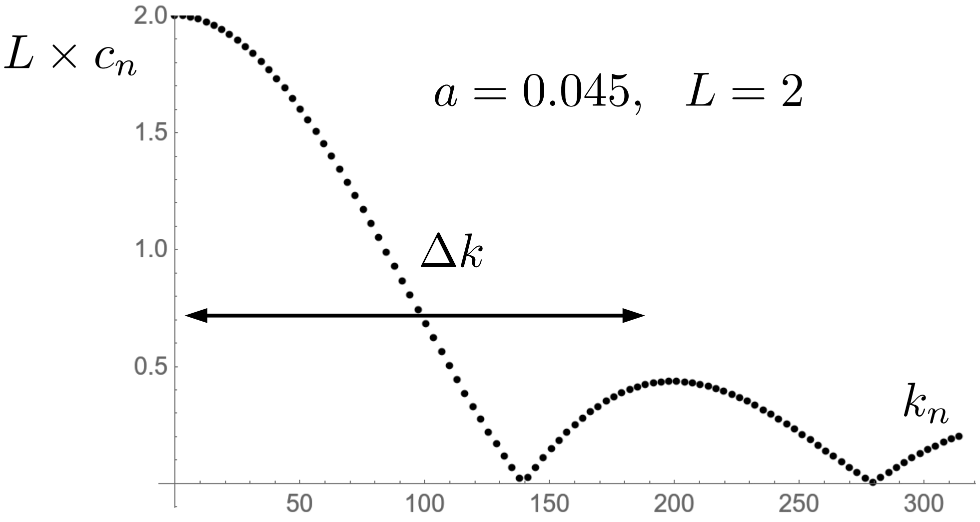
$$\Delta k \sim 1/a$$

Relação de incerteza: $\boxed{\Delta x \Delta k \sim 1}$

1 – Série de Fourier: 3º exemplo

$$f(x) = \frac{1}{a}\Theta(x)\Theta(a-x), \qquad 0 < x < L$$





$$\langle x^m \rangle = \frac{\int_0^L x^m f(x)\,\mathrm{d}x}{\int_0^L f(x)\,\mathrm{d}x}$$

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \frac{a}{\sqrt{12}}$$

Definindo

$$\langle k^m \rangle = \frac{\sum_{n=0}^{\infty} k_n^m c_n}{\sum_{n=0}^{\infty} c_n}$$

$$\Delta k \sim 1/a$$

Relação de incerteza: $\Delta x \Delta k \sim 1$

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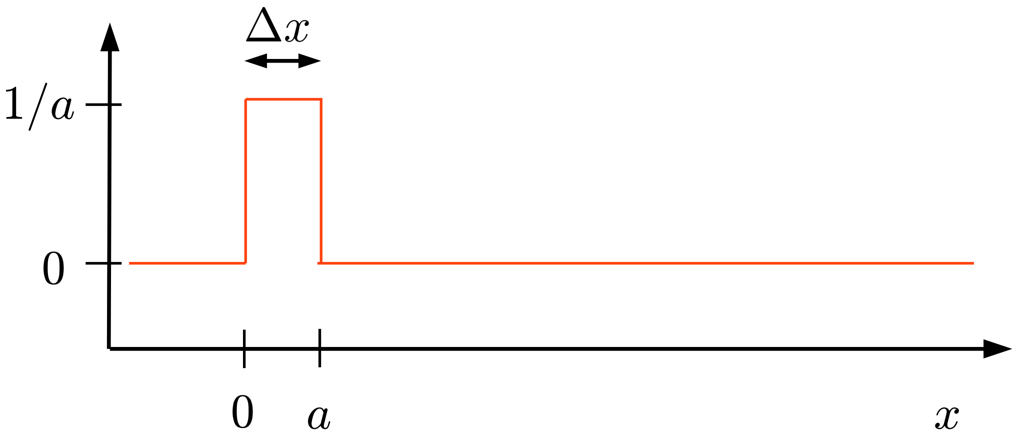
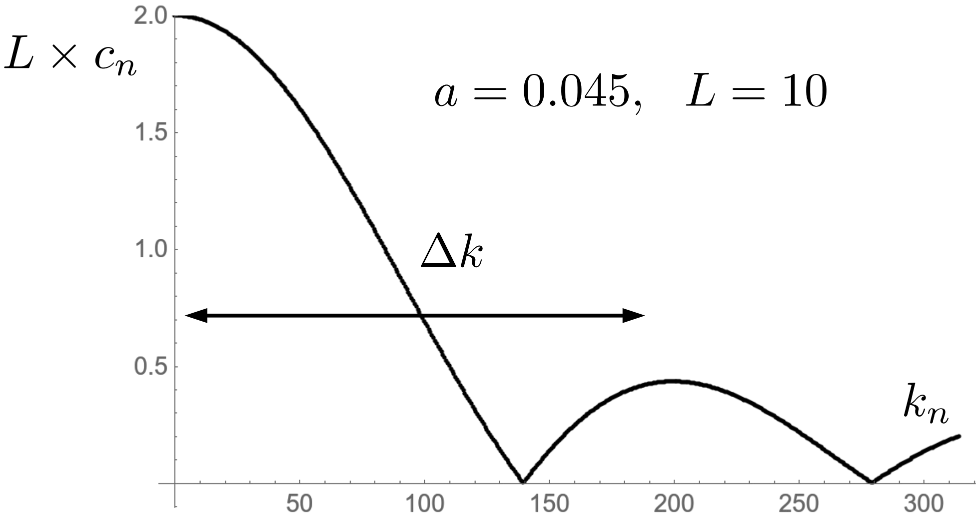
$$f(x) = \frac{1}{a}\Theta(x)\Theta(a-x), \qquad 0 < x < L$$





$$\langle x^m \rangle = \frac{\int_0^L x^m f(x)\,\mathrm{d}x}{\int_0^L f(x)\,\mathrm{d}x}$$

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \frac{a}{\sqrt{12}}$$

Definindo

$$\langle k^m \rangle = \frac{\sum_{n=0}^{\infty} k_n^m c_n}{\sum_{n=0}^{\infty} c_n}$$

$$\Delta k \sim 1/a$$

Relação de incerteza: $\Delta x \Delta k \sim 1$

2 – O limite contínuo: Transformada de Fourier

Representação de uma função integrável (não necessariamente periódica) em termos de funções harmônicas:

$$
\begin{aligned}
f(x) =& \frac{1}{2}c_0 + \sum_{n=1}^{\infty} c_n \cos\left(k_n x - \phi_n\right) \\
=& \frac{1}{2}c_0 + \frac{1}{2}\sum_{n=1}^{\infty} c_n \left(e^{i(k_n x - \phi_n)} + e^{-i(k_n x - \phi_n)}\right) \\
=& \sum_{n=-\infty}^{\infty} \tilde{c}_n e^{ik_n x} = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \Delta k \left(\tilde{c}_n L\right) e^{ik_n x} \\
\rightarrow& \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k \tilde{f}\left(k\right) e^{ikx}
\end{aligned}
$$

$$k_n = \tfrac{2\pi}{L} n, \quad n \in \mathbb{N}$$

Conjunto discreto de k’s

$$\Delta k = k_{n+1} - k_n = \tfrac{2\pi}{L} \rightarrow 0$$

$k \rightarrow$ variável contínua

$\boxed{\text{Superposição de ondas planas com amplitudes } \tilde{f}(k)}$

2 – O limite contínuo: Transformada de Fourier

Relação entre $f(x)$ e $\tilde{f}(k)$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k \tilde{f}\,(k)\, e^{ikx} \;\Rightarrow\; \int_{-\infty}^{\infty} \mathrm{d}x f(x) e^{-iqx} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k \tilde{f}\,(k) \overbrace{\int_{-\infty}^{\infty} \mathrm{d}x e^{i(k-q)x}}^{2\pi\delta(k-q)}$$

$$\Rightarrow\; \tilde{f}\,(q) = \int_{-\infty}^{\infty} \mathrm{d}x f(x) e^{-iqx}$$

2 – Transformada de Fourier: definições

Transformada:

$$\tilde{f}(k) \equiv \mathcal{F}\left[f(x)\right] = \int_{-\infty}^{\infty} \mathrm{d}x f(x) e^{-ikx}$$

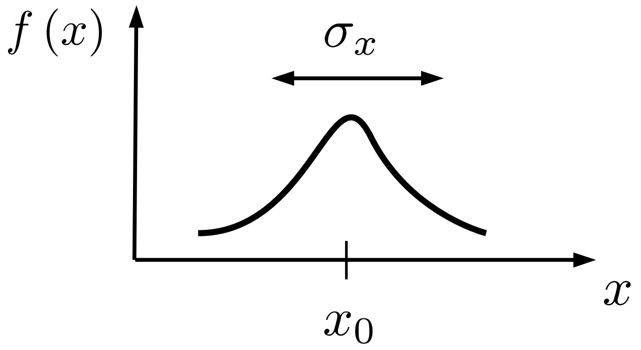
Transformada inversa:

$$f(x) = \mathcal{F}^{-1}\left[\tilde{f}(k)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} \mathrm{d}k$$

2 – Transformada de Fourier: 1º exemplo

Gaussiana:

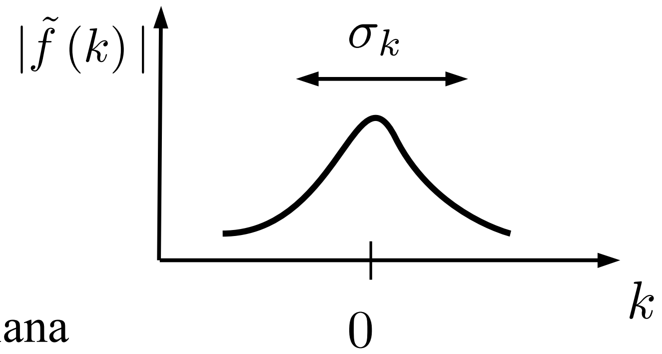
$$f\left(x\right) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{(x-x_0)^2}{2\sigma_x^2}}$$



Transformada:

$$\tilde{f}\left(k\right) = \frac{1}{\sqrt{2\pi}\sigma_x} \int_{-\infty}^{\infty} e^{-\frac{(x-x_0)^2}{2\sigma_x^2} - ikx} \mathrm{d}x = e^{-ikx_0} \times e^{-\frac{k^2}{2\sigma_k^2}}$$

fase   Gaussiana

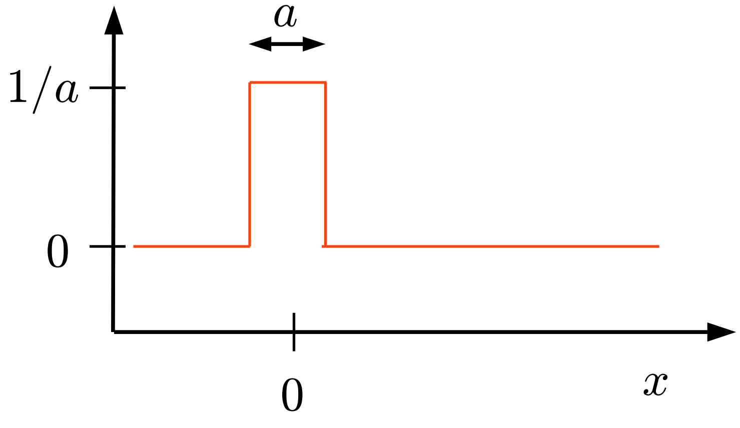


$$\sigma_k = 1/\sigma_x$$

$$\boxed{\text{Relação de incerteza:} \quad \Delta x \Delta k = \sigma_x \sigma_k = 1}$$

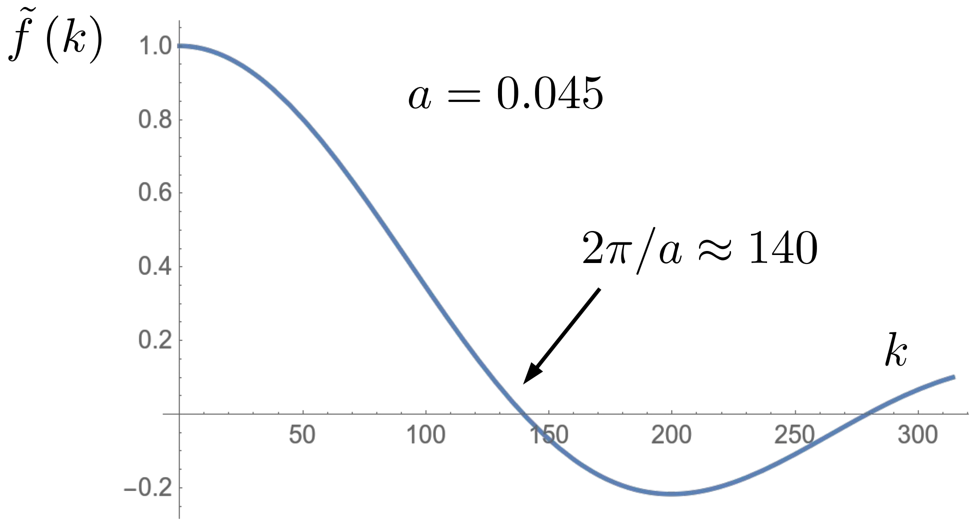
2 – Transformada de Fourier: 2º exemplo

Pulso quadrado: $f(x) = \frac{1}{a}\Theta(a/2 - |x|)$



Transformada:

$$\tilde{f}(k) = \frac{1}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} e^{-ikx}\mathrm{d}x = \frac{2\sin\left(\frac{1}{2}ka\right)}{ka}$$



3 – Transformada de Fourier e equações diferenciais

$$\sum_n A_n \frac{\mathrm{d}^n f}{\mathrm{d}x^n} = 0, \ \Rightarrow \sum_n A_n \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int \frac{\tilde{f}}{2\pi} e^{ikx} \mathrm{d}k = \int \left(\sum_n A_n \left(ik\right)^n \right) \frac{\tilde{f}}{2\pi} e^{ikx} \mathrm{d}k = 0$$

Logo, $\sum_n A_n \left(ik\right)^n = 0$

Exemplo: $m \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + kx = 0$ Usando $x\left(t\right) = \int \tilde{x}\left(\omega\right) e^{-i\omega t} \mathrm{d}\omega$

$$\Rightarrow \int \left(-m\omega^2 + k\right) \tilde{x}\left(\omega\right) e^{-i\omega t} \mathrm{d}\omega = 0, \ \forall t$$

$\Rightarrow \tilde{x}\left(\omega\right) = 0$ Exceto para $\omega = \pm\sqrt{\frac{k}{m}}$

Apenas 1 modo permitido

(2 frequências de mesmo módulo)

3 – Transformada de Fourier e a equação de onda

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = 0$$

Usando que $$y\left(x,t\right) = \int_{-\infty}^{\infty} \tilde{y}\left(k,t\right) e^{ikx} \mathrm{d}k = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{\tilde{y}}\left(k,\omega\right) e^{i(kx-\omega t)} \mathrm{d}k\mathrm{d}\omega$$

Então $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(-k^2+\omega^2/c^2\right) \tilde{\tilde{y}}\left(k,\omega\right) e^{i(kx-\omega t)} \mathrm{d}k\mathrm{d}\omega = 0, \ \forall x,t$$

Logo, qualquer onda plana é permitida desde que $\boxed{\omega = \pm ck}$

Relação de dispersão