SFI5711 - Solid State Physics (2020)
List of exercises #4

1. Plasmons

In the long-wavelength and high-frequency regime, the dielectric constant of the interacting electron gas is given by

$$\epsilon(\omega) = \epsilon_0 \left(1 - \left(\frac{\omega_p}{\omega}\right)^2\right),$$

with $\omega_p = \sqrt{ne^2/(\epsilon_0 m)}$ being the plasma frequency.

(a) Using Maxwell equations in neutral matter, show that a monochromatic incident electric field $\mathbf{E}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}) e^{-i\omega t}$ satisfies the wave-like equation

$$\nabla^2 \mathbf{E} = \frac{\omega_p^2 - \omega^2}{c^2} \mathbf{E}.$$

For which frequencies is this metal tranparent?

(b) Sodium forms a body-centered cubic crystalline structure. Each side of the cube is 4.23 Å. Given that each sodium atom contributes with one electron to the conduction band, calculate the wave-length below which sodium becomes transparent to light. In which region of the electromagnetic spectrum is this wave-length?

2. Screening in 2D

Consider a two-dimensional electron gas (2DEG) located in the $z = 0$ plane embedded in a material of permittivity ϵ_0. An external potential ϕ_{ext} (due to an external charge) is applied to the system.

(a) Show that the relationship between the induced potential $\phi_{\text{ind}}(x, y, 0)$ in the 2DEG and the induced surface charge density $\sigma_{\text{ind}}(x, y)$ is

$$\tilde{\phi}_{\text{ind}}(\mathbf{q}) = \frac{1}{2\epsilon_0 q} \tilde{\sigma}_{\text{ind}}(\mathbf{q}),$$

where $\mathbf{q} = (q_x, q_y)$ and $\tilde{\phi}_{\text{ind}}(\mathbf{q})$ is the Fourier transform in $\mathbf{q}$ of $\phi_{\text{ind}}(\mathbf{r}, z = 0)$, where $\mathbf{r} = (x, y)$. [*Hint*: Find the relation between $\tilde{\phi}_{\text{ind}}(\mathbf{q}, q_z)$ and $\tilde{\rho}_{\text{ind}}(\mathbf{q}, q_z)$. Then recall the volumetric charge density of the 2DEG is $\rho(x, y, z) = \sigma(\mathbf{r}) \delta(z)$.]

(b) Employ the Thomas-Fermi approach to calculate the dielectric constant of the 2DEG, defined as $\epsilon(\mathbf{q})/\epsilon_0 = \tilde{\phi}_{\text{ext}}/\tilde{\phi}$ where ϕ is the total potential. Show that it has the form

$$\epsilon(\mathbf{q}) = \epsilon_0 \left(1 + \frac{k_{TF}}{q}\right),$$

and compute the Thomas-Fermi wavevector k_{TF}.

(c) The previous result implies that the external potential due to a point charge Q is "screened", giving rise to the effective potential

$$\tilde{\phi}(\mathbf{q}) = \frac{Q}{2\epsilon_0 (q + k_{TF})}.$$

Show that this is the case and obtain the real-space expression for $\phi(\mathbf{r})$ in the limit $k_{TF} r \gg 1$. How strongly screened is the external charge?

3. Friedel oscillations for noninteracting systems

Friedel oscillations appear not only near external charges, but also near steps or barriers of the crystal. Consider electrons living in two dimensions and subject to a one- dimensional confining potential $V(x, y)$ that is zero for $0 \leq x \leq L$, and infinity otherwise.

(a) Solve the Schrödinger equation for this potential and calculate the charge density $\rho_{k_0}(\mathbf{r})$ for the electronic states with fixed momentum k_0

$$\rho_{k_0}(\mathbf{r}) = 2e \sum_{\mathbf{k}} |\psi(\mathbf{r})|^2 \delta(k - k_0)$$

where ψ is the wave-function. Assume periodic boundary conditions along the y direction with period L. Plot this charge density as a function of x, showing the existence of oscillations. (*Hint*: You can replace the sum over the discrete values of k_x and k_y by integrals as long as $x \ll L$, which is the range of interest).

(b) Integrate the charge density over all wave-vectors $k_0 < k_F$, $\rho = \int_0^{k_F} \rho_k \mathrm{d}k$, where k_F is some arbitrary "Fermi vector". Plot the integrated charge density ρ, comparing the amplitude of its oscillations with those of the previous item. (*Hint*: All the integrals are analytic, just use the properties of the Bessel functions).

4. Fermi liquid magnetic susceptibility

Consider an external magnetic field h applied along the z-direction of an isotropic Fermi liquid. It corresponds to a perturbation of the form

$$\delta E^{(0)}_{\mathbf{k}\sigma} = -\mu_B \sigma h,$$

where $\sigma = \pm 1$ and μ_B is the Bohr magneton.

(a) Show that the change in the energy dispersion is given by

$$\delta E_{\mathbf{k}\sigma} = -\frac{\mu_B \sigma h}{1 + F_0^a},$$

where F_0^a is the anti-symmetric part of Landau's f-function.

(b) The magnetization of the system is given by

$$M = \mu_B \sum_{\mathbf{k}} \left(n_{\mathbf{k}\uparrow} - n_{\mathbf{k}\downarrow}\right) = \mu_B \sum_{\mathbf{k}\sigma} \sigma \left(\delta n_{\mathbf{k}\sigma}\right).$$

Calculate the magnetic susceptibility $\chi = M/h$ and show that it is given by

$$\chi = \left(\frac{m^*}{m}\right) \frac{\chi^{(0)}}{1 + F_0^a},$$

where $\chi_0 = \mu_B^2 \rho_F^{(0)}$ is the Pauli susceptibility of the non-interacting electron gas.