SFI5711 - Solid State Physics (2020)
List of exercises #6

1. **Size of a Cooper pair**
   The typical energy spread of two electrons near the Fermi level forming a Cooper pair is given by the gap Δ.

   (a) *(3 points)* Use Heisenberg's uncertainty principle, $\delta p\, \delta x \approx \hbar$, to estimate the size of a Cooper pair as function of T_c and of the electronic density n.

   (b) *(2 points)* Find the values of T_c and n for Al (consult Ashcroft & Mermin) and use the expression you found in the previous item to estimate the Cooper pair size.

2. **Superconducting gap near T_c**
   (10 points) The gap equation at an arbitrary temperature is given by

$$1 = V_0 \rho_F \int_0^{\hbar\omega_D} \frac{\mathrm{d}\varepsilon}{\sqrt{\varepsilon^2 + \Delta^2}} \tanh\left(\frac{\sqrt{\varepsilon^2 + \Delta^2}}{2k_B T}\right)$$

   Consider a temperature below but very close to T_c, i.e. $T = T_c(1-t)$, with $t \ll 1$. For consistency, the gap Δ also has to be small, i.e. $\Delta \ll k_B T_c$. Expand the gap equation for both t and Δ and show that the gap function vanishes as T approaches T_c according to the power-law

$$\frac{\Delta(T)}{\Delta(T=0)} \approx 1.74\sqrt{1 - \frac{T}{T_c}}.$$

3. **BCS ground state energy**
   From the BCS solution, we found that the ground state energy at zero temperature (per electron) is given by

$$E_0 = \frac{1}{N}\sum_{\mathbf{k}} \left(\xi_{\mathbf{k}} - E_{\mathbf{k}} + \Delta_{\mathbf{k}} \left\langle c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow}\right\rangle\right).$$

   (a) *(3 points)* Determine the difference between the superconducting and normal state energies $\delta E_0 \equiv E_0 - E_0(\Delta = 0)$ as function of Δ. (*Hint*: the integration is restricted to $|\xi| < \hbar\omega_D$. Recall that $\hbar\omega_D \gg \Delta$).

   (b) *(3 points)* By minimizing the energy found in the previous item, show that one recovers the formula for Δ at zero temperature found in class. (*Hint*: use the weak-coupling approximation $\rho_F V_0 \ll 1$).

   (c) *(4 points)* Show that the thermodynamic critical field H_c necessary to kill superconductivity is given by

$$\frac{H_c}{\Delta(T=0)} = 2\sqrt{\pi\rho_F},$$

   where ρ_F is the density of states at the Fermi level. (*Hint:* the magnetic energy density is given by $H_c^2/(8\pi)$).

4. **d-wave superconductivity in the cuprates**
   In the family of high-temperature superconductors known as the cuprates (such as La_2CuO_4 and $YBa_2Cu_3O_7$), the superconducting gap $\Delta_{\mathbf{k}}$ changes as function of the momentum $\mathbf{k}$. In a very simplified model, we can consider a *two-dimensional* parabolic band $\xi_{\mathbf{k}} = \frac{\hbar^2 k^2}{2m} - \mu$ and a gap function of the form

$$\Delta_{\mathbf{k}} = \Delta_0 \cos 2\varphi,$$

   where φ is the azimuthal angle. This gap vanishes and changes sign in the so-called nodal points, located at $\varphi = \pi/4$ and its multiples.

   (a) *(3 points)* What is the orbital angular momentum of the d-wave Cooper pair? What is its spin state, singlet or triplet? (*Hint*: compare the gap function with spherical harmonics).

(b) *(5 points)* The density of states for the d-wave superconductor is defined as

$$\rho(E) = \int \frac{d^2k}{(2\pi)^2}\,\delta\left(E - \sqrt{\xi_{\mathbf{k}}^2 + \Delta_{\mathbf{k}}^2}\right).$$

Obtain the low-energy ($E \ll \Delta$) expression for the density of states, and show that it vanishes linearly with E. Why is the spectrum not gapped, like in the s-wave case $\Delta_{\mathbf{k}} = \Delta_0$?

(c) *(3 points)* The behavior of the density of states at low energies is manifested in the low-temperature behavior of the specific heat. In fact, the specific heat inside the superconducting state is given by

$$C = 2k_B \int_0^{\infty} dE\,\rho(E)\left(\frac{E}{2k_BT}\right)^2 \cosh^{-2}\left(\frac{E}{2k_BT}\right).$$

First, use the expression for $\rho(E)$ we found in class for the s-wave superconductor $\Delta_{\mathbf{k}} = \Delta_0$ to obtain the low-temperature ($k_BT \ll \Delta_0$) behavior of the specific heat. (*Hint*: you only need to evaluate the integral in the limit $k_BT \ll \Delta_0$).

(d) *(4 points)* Now, use the expression you found in item (b) for the density of states at low energies to determine the specific heat behavior at low temperatures ($k_BT \ll \Delta_0$) inside of a d-wave superconductor. How does it compare to your result for the s-wave superconductor?

5. **s^{+-} superconductivity in the iron pnictides**
The other known family of high-temperature superconductors, the iron pnictides (such as LaFeAsO and $BaFe_2As_2$), are multi-band compounds. In this case, the superconducting gap can have different signs on different bands. Let us consider a very simplified model, with two parabolic two-dimensional bands: a hole-like band $\xi_{1,\mathbf{k}}$, with density of states ρ_1, and an electron-like band $\xi_{2,\mathbf{k}}$, with density of states ρ_2. The gap equations form a 2×2 system of equations

$$\Delta_\lambda = -\frac{1}{N}\sum_{\lambda'=1}^{2} V_{\lambda\lambda'}\Delta_{\lambda'}\sum_{\mathbf{k}}\frac{1}{2E_{\lambda',\mathbf{k}}}\tanh\left(\frac{E_{\lambda',\mathbf{k}}}{2k_BT}\right),$$

where $\lambda, \lambda' = 1, 2$ are band indices. The momentum integration is limited to $|\xi_{\lambda',\mathbf{k}}| < W$, where W is some energy scale associated with the pairing mechanism, which replaces the Debye energy $\hbar\omega_D$. The interaction potential $V_{\lambda\lambda'}$ and the gap functions Δ_λ do not depend on the momentum, i.e. we are considering s-wave states.

(a) *(5 points)* Consider that the only non-vanishing interaction is the inter-band interaction V_{12}, i.e. $V_{11} = V_{22} = 0$. Show that T_c is given by

$$T_c = 1.13\,\frac{W}{k_B}\exp\left(-\frac{1}{|V_{12}|\sqrt{\rho_1\rho_2}}\right).$$

(b) *(5 points)* Show that, if V_{12} is an attractive interaction, the signs of the two gap functions Δ_1 and Δ_2 are the same (the so-called s^{++} state), whereas if V_{12} is a repulsive interaction, their signs are different (known as the s^{+-} state). Determine the absolute value of the ratio between the two gap functions Δ_1/Δ_2 immediately below T_c in both cases. Does this ratio depend on whether the superconducting state is s^{+-} or s^{++}?

6. **Magnetic instability of the one-dimensional electron gas**
(10 points) The criterion for a Stoner instability at the wave vector $\mathbf{q}$ is $\chi(\mathbf{q})U = 1$. Here, $\chi(\mathbf{q})$ is the Lindhard function:

$$\chi(\mathbf{q}) = \frac{1}{v}\sum_{\mathbf{k}}\frac{f(\varepsilon_{\mathbf{k}+\mathbf{q}}) - f(\varepsilon_{\mathbf{k}})}{\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k}+\mathbf{q}}}$$

v is the volume and $f(x)$ is the Fermi-Dirac distribution function. For the one-dimensional electron gas, where $\varepsilon_{\mathbf{k}} = \hbar^2k^2/(2m)$, show that

$$\chi(\mathbf{q}) = \frac{m}{\pi\hbar^2 q}\ln\left(\frac{2k_F + q}{2k_F - q}\right).$$

Plot this function and verify that it diverges at $q = \pm 2k_F$. Therefore, the 1D electron gas in unstable towards a magnetically ordered state whose magnetization has a $2k_F$ modulation.

7. **Stoner continuum**
The metallic (itinerant) ferromagnet has two types of excitations: spin waves and the particle-hole spin-flip pair. In the latter, an electron at the Fermi level ε_F is excited from the spin-up band $\varepsilon_{\mathbf{k}\uparrow}$ to the spin-down band $\varepsilon_{\mathbf{k}\downarrow}$. Measuring all energies with respect to the bottom of the spin-up band, we have $\varepsilon_{\mathbf{k}\uparrow} = \hbar^2 k^2 / (2m)$ and

$$\varepsilon_{\mathbf{k}\downarrow} = \frac{\hbar^2 k^2}{2m} + \Delta,$$

where $\Delta = 4UM > 0$ is the gap. Consider that this process involves a change in energy of $\hbar\omega$ and a change in momentum of $\mathbf{q}$, i.e. $\varepsilon_{\mathbf{k}_2\downarrow} - \varepsilon_{\mathbf{k}_1\uparrow} = \hbar\omega$ and $\mathbf{k}_2 - \mathbf{k}_1 = \mathbf{q}$.

   (a) *(7 points)* The region in the $(\hbar\omega, q)$ plane where this type of excitation is allowed is called the Stoner continuum. Find the equations $\hbar\omega(q)$ describing the boundaries of the Stoner continuum. (*Hint:* in the boundaries the vectors $\mathbf{k}_1$ and $\mathbf{k}_2$ are either parallel or anti-parallel).

   (b) *(3 points)* Using the previous result, sketch the Stoner continuum in the $(\hbar\omega, q)$ plane. What happens (qualitatively) to the spin-wave excitations when they meet the Stoner continuum? (Recall that the spin waves dispersion is $\omega \propto q^2$ for small q)

8. **Spin-waves for an anisotropic ferromagnet**
Consider the anisotropic Heisenberg ferromagnet:

$$H = -\sum_{\langle ij \rangle} \left[J_z \hat{S}_i^z \hat{S}_j^z + J \left(\hat{S}_i^x \hat{S}_j^x + \hat{S}_i^y \hat{S}_j^y \right) \right]$$

where $J_z > J > 0$. This Hamiltonian interpolates between the Heisenberg model ($J = J_z$, studied in class) and the Ising model ($J = 0$).

   (a) *(8 points)* Repeat the Holstein-Primakoff calculation we did for the isotropic case, which assumes that the ordered state corresponds to all spins pointing to the $\hat{\mathbf{z}}$ direction. Show that the spin-wave dispersion in the anisotropic case is given by

$$\hbar\omega_{\mathbf{k}} = 2Sz \left(J_z - J\gamma_{\mathbf{k}} \right),$$

   where z and $\gamma_{\mathbf{k}}$ are the same as in the isotropic case.

   (b) *(2 points)* Show that $\omega_{\mathbf{k}\to 0}$ remain finite in the anisotropic case. The spin-wave excitation spectrum is said to be gapped, since there is a minimum energy necessary to excite spin waves. Why is it different than the isotropic case, where $\omega_{\mathbf{k}\to 0}$ goes to zero?

9. **Spin-wave dispersion for an antiferromagnet**
Apply the Holstein-Primakoff transformation to an antiferromagnet in a hyper-cubic lattice. This lattice can be subdivided into two other hyper-cubic sub-lattices A and B. In the Néel state, all the spins in A point up and all the spins in B point down. The Heisenberg Hamiltonian is thus

$$H = J \sum_{\langle i,j \rangle} \left[\hat{S}_{A,i}^z \hat{S}_{B,j}^z + \frac{1}{2} \left(\hat{S}_{A,i}^+ \hat{S}_{B,j}^- + \hat{S}_{A,i}^- \hat{S}_{B,j}^+ \right) \right]$$

where $J > 0$, i is a site of sublattice A, and j is a site of sublattice B. By definition, i and j are nearest-neighbors of the original lattice, i.e. $\mathbf{R}_j = \mathbf{R}_i + \boldsymbol{\delta}$.

   (a) *(5 points)* Use the Holstein-Primakoff transformation

$$\hat{S}_{A,i}^z = S - c_i^\dagger c_i, \quad \hat{S}_{B,j}^z = -S + d_j^\dagger d_j,$$

$$\hat{S}_{A,i}^+ = \sqrt{2S} \left(1 - \frac{c_i^\dagger c_i}{2S} \right)^{1/2} c_i, \quad \hat{S}_{B,j}^+ = \sqrt{2S}\, d_j^\dagger \left(1 - \frac{d_j^\dagger d_j}{2S} \right)^{1/2},$$

$$\hat{S}_{A,i}^- \sqrt{2S}\, c_i^\dagger \left(1 - \frac{c_i^\dagger c_i}{2S} \right)^{1/2}, \quad \hat{S}_{B,j}^- = \sqrt{2S} \left(1 - \frac{d_j^\dagger d_j}{2S} \right)^{1/2} d_j,$$

in the large-S limit to show that the Hamiltonian can be recast in the form

$$H = JSz \sum_{\mathbf{k}} \left[\gamma_{\mathbf{k}} \left(c_{\mathbf{k}} d_{-\mathbf{k}} + d^{\dagger}_{-\mathbf{k}} c^{\dagger}_{\mathbf{k}} \right) + \left(c^{\dagger}_{\mathbf{k}} c_{\mathbf{k}} + d^{\dagger}_{-\mathbf{k}} d_{-\mathbf{k}} \right) \right], \tag{1}$$

where $c^{\dagger}_{\mathbf{k}}$ and $d^{\dagger}_{\mathbf{k}}$ are the appropriate Fourier-transformed bosonic operators and $\gamma_{\mathbf{k}}$ is the same as in the ferromagnetic case.

(b) *(5 points)* The bosonic Hamiltonian (1) can be solved via a Bogliubov transformation

$$\begin{aligned} c_{\mathbf{k}} &= u_{\mathbf{k}} \alpha_{\mathbf{k}} - v_{\mathbf{k}} \beta^{\dagger}_{\mathbf{k}}, \\ d_{-\mathbf{k}} &= u_{\mathbf{k}} \beta_{\mathbf{k}} - v_{\mathbf{k}} \alpha^{\dagger}_{\mathbf{k}}. \end{aligned}$$

Show that

$$H = \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} \left(\alpha^{\dagger}_{\mathbf{k}} \alpha_{\mathbf{k}} + \beta^{\dagger}_{\mathbf{k}} \beta_{\mathbf{k}} \right),$$

with the spin-wave dispersion

$$\hbar\omega_{\mathbf{k}} = JSz\sqrt{1-\gamma^2_{\mathbf{k}}}.$$

Show that $\omega_{\mathbf{k}} \propto k$ for small k.

(c) *(5 points)* The deviation of the staggered magnetization $\bar{M}$ from the saturation value S is given by

$$\Delta\bar{M} = \left\langle \sum_i \left(\hat{S}^z_{A,i} - S \right) - \sum_j \left(\hat{S}^z_{B,j} + S \right) \right\rangle = -\sum_{\mathbf{k}} \left(\left\langle c^{\dagger}_{\mathbf{k}} c_{\mathbf{k}} \right\rangle + \left\langle d^{\dagger}_{-\mathbf{k}} d_{-\mathbf{k}} \right\rangle \right).$$

Express $c^{\dagger}_{\mathbf{k}} c_{\mathbf{k}}$ and $d^{\dagger}_{\mathbf{k}} d_{\mathbf{k}}$ in terms of the Bogoliubov operators $\alpha_{\mathbf{k}}$ and $\beta_{\mathbf{k}}$. Using the fact that in the ground state there are no spin waves, i.e., $\left\langle \alpha^{\dagger}_{\mathbf{k}} \alpha_{\mathbf{k}} \right\rangle = \left\langle \beta^{\dagger}_{\mathbf{k}} \beta_{\mathbf{k}} \right\rangle = 0$, show that

$$\Delta\bar{M} = -\sum_{\mathbf{k}} \left(\frac{1}{\sqrt{1-\gamma^2_{\mathbf{k}}}} - 1 \right).$$

Show that $\Delta\bar{M}$ diverges for the one-dimensional case. For simplicity, use the small k expansion. What does the divergence of $\Delta\bar{M}$ imply?