List of exercises #1 - 7600040

1. Fermat's principle (also known as the principle of least time) states that the path taken by a ray of light between two given points is the path that can be traveled in the least time.

   (a) Show that this principle yields to straight rays in a medium of same refraction index.

   (b) Show that this principle yield to Snell's law when the ray travels through media of different refraction index.

   (c) Calculate the trajectory of a ray of light $y(x)$ (and discuss on the possibility of mirages) in a medium where the refraction index is

      i. $n = n_0 e^{\alpha y}$, with n_0 and α being constants (physically, this is only valid in the region $y \geq -\alpha^{-1} \ln n_0$), and

      ii. $n = c/\sqrt{v_0^2 - 2gy}$, with c being the speed of light in vacuum and v_0 and g are constants (mathematically, this is valid only for $y \leq v_0^2/(2g)$, physically, there is an additional constraint $y \geq -c^2/(2g)$). In this case, what is the relation of this problem with the brachistochrone?

2. The Lorentz force on a particle of mass m and charge Q is $\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. Although it depends on the particle velocity, it is still possible to write a Lagrangian $L = T - V$, with a suitable choice of V such that $F_\alpha = -\frac{\partial V}{\partial q_\alpha} + \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}_\alpha}\right)$. In other words, even though the force is non-conservative, sometimes it is possible to write an effective potential such that the equations of motion is obtained from the usual Lagrange's equation for conservative forces $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) = \frac{\partial L}{\partial q_\alpha}$. For the case under consideration, show that $T = \frac{1}{2} m \dot{q}^2$ is the kinetic energy and $V = Q(\phi - \dot{\mathbf{q}} \cdot \mathbf{A})$, with $\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$ and $\mathbf{B} = \nabla \times \mathbf{A}$, i.e., ϕ and $\mathbf{A}$ are, respectively, the scalar and vector potentials. (Hint: see Goldstein & Poole & Safko, Sec. 1.5 or Kibble & Berkshire, Sec. 10.5)

3. It is known (from experiments) that a certain particle free fell a height y_0 in the time interval $t_0 = \sqrt{2y_0/g}$. Assume that $y = at + bt^2$, with a and b being real constants.

   (a) What is the relation between a and b to ensure that the particle falls y_0 in t_0?

   (b) Compute the action $\int_0^{t0} L dt$ and show that it is an extremum only when $a = 0$ and $b = g/2$? Is it a minimum or maximum?

4. A cylinder of mass m, moment of inertia I, and radius r rolls without slipping on top of another cylinder which is fixed and has radius R. (The symmetry axes of these cylinders are parallel.) The only external force is that of gravity. If the top cylinder starts rolling from the rest on the top of the bottom cylinder, use the methods of Lagrange multiplier to find the point at which the top cylinder falls from the bottom one.

5. Sometimes it is possible to incorporate friction without introducing the dissipation function. For instance, consider the Lagrangian $L = \frac{1}{2} e^{\gamma t}\left(m\dot{q}^2 - kq^2\right)$.

   (a) How do you interpret this system?

   (b) Are there constants of motion?

   (c) Let $s = e^{\gamma t} q$. What is the Lagrangian for s, and what does it say about the conserved quantities for the system? (Hint: see Lemus Sec. 2.7.)

6. A massive point particle is constrained to move without friction on a hoop fixed in a vertical plane that rotates about its vertical symmetry axis with constant angular speed. Show and calculate the angular velocity above which the particle can remain at stable rest in a point of the hoop other than the bottom.

7. The electric and magnetic fields do not change under the gauge transformation $\mathbf{A} \to \mathbf{A} + \nabla\Gamma$ and $\phi \to \phi - \frac{\partial \Gamma}{\partial t}$, where $\Gamma \equiv \Gamma(\mathbf{r}, t)$ is an arbitrary differentiable function.

   (a) How the Lagrangian is changed by this transformation?

   (b) Do the equations of movement change?

8. The Lagrangian of a certain system is $L = \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right) - (\alpha x + \beta y)$, where α and β are constants.

   (a) How many degrees of freedom has the system?

(b) Prove that the Lagrangian (and, consequently, the action) is invariant under the infinitesimal transformation $x' = x + \epsilon\beta$ and $y' = y - \epsilon\alpha$.

(c) Using the Noether's theorem show that $A = m\left(\beta\dot{x} - \alpha\dot{y}\right)$ is a constant of the movement.

(d) Express x and y in terms of the generalized coordinates $\bar{x} = \alpha x + \beta y$ and $\bar{y} = \beta x - \alpha y$, and show that one of them is a cyclic coordinate.

(e) Show that A is proportional to the conjugate momentum of the cyclic coordinate.

(f) Give a geometric meaning of the whole procedure of this problem.

9. A particle of charge q and mass m moves in a constant magnetic field $\mathbf{B} = B\hat{z}$.

   (a) Show that the equations of movement are $\dot{\mathbf{v}} = -\boldsymbol{\omega} \times \mathbf{v}$. Find $\boldsymbol{\omega}$. (Hint: one possibility for the vector potential is $\mathbf{A} = \frac{1}{2}\mathbf{B} \times \mathbf{r}$.)

   (b) Write the Lagrangian in cylindrical coordinates (ρ, θ, z) and show that, although $\dot{\theta}$ is a cyclic coordinate, the angular momentum along the magnetic field $L_z = m\rho^2\dot{\theta}$ is not a constant of the movement. Explain.

10. A particle of mass m moves under the conservative central potential $V = -kr^{-1}$, where k is a constant.

    (a) Show that the Laplace-Runge-Lenz vector $\mathbf{A} = \mathbf{p} \times \mathbf{L} - mk\hat{r}$ is a constant of the movement, and that it is contained in the plane of the particle trajectory.

    (b) Solve for the trajectory equation by quadrature. (Hint: compute $\mathbf{A} \cdot \mathbf{r}$, and notice that $\mathbf{r} \cdot (\mathbf{p} \times \mathbf{L}) = L^2$.) How does the eccentricity of the orbit depends on A?

11. Reconsider the problem 4.

    (a) Show that the tangential constraint force on the cylinder is, in modulus, $f = \frac{\gamma m g}{1+\gamma}\sin\theta$, where θ is the angle between the vertical and the line the connects both cylinders, and $\gamma = I/\left(mr^2\right)$.

    (b) The non-slipping condition is $f \leq \mu N$, where N is the magnitude of the normal constraint force and μ is the static friction coefficient between the cylinders. Show that slipping starts at θ_s, where $\cos\theta_s = \frac{2\mu^2(\gamma+3)+\gamma\sqrt{\gamma^2+\mu^2(5+6\gamma+\gamma^4)}}{\gamma^2+\mu^2(\gamma+3)^2}$.

    (c) Show that the result of the problem 4 is only correct in the limit $\mu \to \infty$. (Intriguingly, none of these results depends on R.)

12. More later...