List of exercises #2 - 7600040

1. Let $S = S(q_1, t_1, q_2, t_2)$ be the action of a classical path from (q_1, t_1) to (q_2, t_2). Show that $\frac{\partial S}{\partial q_1} = -p_1$, $\frac{\partial S}{\partial q_2} = p_2$, $\frac{\partial S}{\partial t_1} = H(t_1)$ and $\frac{\partial S}{\partial t_2} = -H(t_2)$, where p_i $(i = 1, 2)$ is the conjugate momentum of q_i and $H = p\dot{q} - L$ is the energy function (Hamiltonian). With that, we conclude that the action $S(\mathbf{q}_i, \mathbf{q}_f, t)$ is the generating function of type $F_1 \equiv F_1(\mathbf{q} = \mathbf{q}_f, \mathbf{Q} = \mathbf{q}_i, t)$ for the time evolution from $\mathbf{q}_i$ to $\mathbf{q}_f$. (Hint: see Aguiar Sec. 3.7.)

2. Prove

   (a) the Jacobi identity $\{\{A, B\}, C\} + \{\{B, C\}, A\} + \{\{C, A\}, B\} = 0$, where $\{A, B\} \equiv \{A, B\}_{\mathbf{q},\mathbf{p}}$ are the Poisson brackets of A and B with respect to canonical coordinate system $(\mathbf{q}, \mathbf{p})$, and A, B, and C are functions of $\mathbf{q}$, $\mathbf{p}$, and t;

   (b) that $\{A, B\}_{\mathbf{q},\mathbf{p}} = \{A, B\}_{\mathbf{Q},\mathbf{P}}$, where $(\mathbf{Q}, \mathbf{P})$ is also a canonical coordinate system;

   (c) the Poisson's theorem which states that the Poisson brackets of two constants of movement is also a constant of movement.
   (Hint: see Lemos Sec. 8.5 or Aguiar Sec. 5.5.)

3. A particle of mass m moves under the conservative central potential $V = -kr^{-1}$, where k is a constant. Show that the Laplace-Runge-Lenz vector $\mathbf{A} = \mathbf{p} \times \mathbf{L} - mk\hat{r}$ is a constant of the movement by computing the Poisson bracket $\{\mathbf{A}, H\}$.

4. Verify if the following transformations are canonical. If yes, find the corresponding generating function.

   (a) $Q = q\cos\alpha - p\sin\alpha$, and $P = q\sin\alpha + p\cos\alpha$, where α is a constant.

   (b) $Q = \ln\left(1 + \sqrt{q}\cos p\right)$, and $P = 2(1 + \sqrt{q}\cos p)\sqrt{q}\sin p$.

   (c) $Q_1 = q_1 q_2$, $Q_2 = q_1 + q_2$, $P_1 = \frac{p_1 - p_2}{q_2 - q_1} - 1$, and $P_2 = \frac{q_2 p_2 - q_1 p_1}{q_2 - q_1} - (q_2 + q_1)$.

5. A particle of mass m and charge q moves in a region in space where there is a constant magnetic field $\mathbf{B} = B\hat{z}$.

   (a) Show that the following transformation is canonical: $x = \alpha^{-1}\left(\sqrt{2P_1}\sin Q_1 + P_2\right)$, $y = \alpha^{-1}\left(\sqrt{2P_1}\cos Q_1 + Q_2\right)$, $z = Q_3$, $p_x = \alpha\left(\sqrt{2P_1}\cos Q_1 - Q_2\right)/2$, $p_y = -\alpha\left(\sqrt{2P_1}\sin Q_1 - P_2\right)/2$, $p_z = P_3$.

   (b) Compute the new Hamiltonian $K(\mathbf{Q}, \mathbf{P})$, identify the constant α conveniently, and obtain the particle movement as a function of time.

6. A particle of mass m and charge q moves in a magnetic field $\mathbf{B}$.

   (a) Write the Hamiltonian of the problem and evaluate the following Poisson brackets $\{m\dot{r}_i, m\dot{r}_j\}$ and $\{m\dot{r}_i, r_j\}$, where $i, j = 1, 2, 3$ with $r_1 = x$, $r_2 = y$, and $r_3 = z$.

   (b) The field of a magnetic monopole is $\mathbf{B} = br^{-2}\hat{r}$, where b is the magnetic charge. Define a generalized angular momentum $\mathbf{D} = \mathbf{J} - qb\hat{r}$ and show that $\{H, D\} = 0$. Discuss your results.

7. Let the infinitesimal canonical transformation $\delta\mathbf{q} = \mathbf{Q} - \mathbf{q} = \epsilon\frac{\partial G}{\partial \mathbf{p}} = \epsilon\{\mathbf{q}, G\}$ and $\delta\mathbf{p} = \mathbf{P} - \mathbf{p} = -\epsilon\frac{\partial G}{\partial \mathbf{q}} = \epsilon\{\mathbf{p}, G\}$.

   (a) If $\delta\mathbf{q} = dt\dot{\mathbf{q}}$ and $\delta\mathbf{p} = dt\dot{\mathbf{p}}$, what is the generating function G?

   (b) If $\delta\mathbf{q} = \epsilon\hat{\mathbf{n}}$ and $\delta\mathbf{p} = 0$, with $\hat{\mathbf{n}}$ being a constant unit vector, what is the generating function G?

   (c) If $\delta\mathbf{q} = \delta\phi\hat{\mathbf{n}} \times \mathbf{q}$ and $\delta\mathbf{p} = \delta\phi\hat{\mathbf{n}} \times \mathbf{p}$, with $\hat{\mathbf{n}}$ being a constant unit vector and $\delta\phi \ll 1$, what is the generating function G?

   (d) What is the physical interpretation of the above transformations?

8. By calculating the Poisson parentheses directly, verify that

   (a) $\{x, J_z\} = -y$, $\{y, J_z\} = x$, and $\{z, J_z\} = 0$, where $\mathbf{J} = \mathbf{r} \times \mathbf{p}$, and

   (b) $\{J_x, J_y\}$, $\{J_y, J_z\}$, $\{J_z, J_x\}$, $\{J^2, J_x\}$, $\{J^2, J_y\}$, and $\{J^2, J_z\}$, where $J^2 = J_x^2 + J_y^2 + J_z^2$.

9. In class, we showed that the volume in the phase $V = \int d\boldsymbol{\eta}$ space is conserved under the Hamiltonian dynamics (Liouville's theorem).

(a) Show that V is conserved not only in the Hamiltonian dynamics but also for any canonical transformation, i.e., show that $V' \equiv \int d\boldsymbol{\zeta} = V$, where $\boldsymbol{\zeta}$ is related to $\boldsymbol{\eta}$ via a canonical transformation.

(b) An ensemble of particles, all of the same mass m, has initial heights z_i and initial moments p_i contained in the intervals $-a \leq z_i \leq a$ and $-b \leq p_i \leq b$. These particles fall in free fall for a time t under the action of gravity $\mathbf{g} = -g\hat{z}$. Find the region in phase space they occupy at time t and show by a direct calculation that this volume is still $4ab$ as expected by Liouville's theorem.

10. A particle of mass m moves along a straight line under the action of the potential $V(x) = mkx^{-2}$, where k is a constant. Given the initial conditions $x\,(0) = x_0$ and $\dot{x}\,(0) = 0$, obtain $x(t)$ by means of the series solution involving successive Poisson parentheses (the Liouville's operator) for $u = x^2$.

11. Let the Hamiltonian of a point particle be $H = \frac{1}{2}\left(\frac{p}{f(pq)}\right)^2$,where $f(x) \neq 0$ is a well behaved function.

    (a) Show that the transformation $Q = qf(qp)$ and $P = p/f(qp)$ is canonical and, with that,

    (b) show that the equations of the trajectory are $q(t) = (At + B)\,/f(A^2t + AB)$ and $p(t) = Af\left(A^2t + AB\right)$, where A and B are constants.

    (c) Find $f(x)$ to solve problem 10 and compare the solutions. (Hint: take the limit $p/f\,(qp)$ when $p \to 0$ very carefully.)

12. The Poincaré-Cartan invariant is $\mathcal{S} = \oint_\Gamma \mathbf{p} \cdot d\mathbf{q} - Hdt$, where the integral is performed in a closed curve Γ in the phase space. Compute $\mathcal{S}$ for the case of a one-dimensional harmonic oscillator where Γ is a classical trajectory. How $\mathcal{S}$ depends on Γ?

13. State and prove the Poincaré's recurrence theorem. Read Lemos Sec. 8.7 and Aguiar Sec. 5.11.