List of exercises #3 - 7600040

1. Consider a one-dimensional Harmonic Oscillator the natural frequency of which is perturbed. Precisely, the Hamiltonian is $H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2$, and $\omega \equiv \omega(t) = \omega_0 (1 + \epsilon t)$. (This linear perturbation can be thought as the Taylor expansion of a more general perturbation.)

   (a) Solve exactly for $x(t)$ with the inicial conditions $x(0) = x_0$ and $p(0) = 0$. (Hint: see the parabolic cylinder functions.)

   (b) Plot $x(t)/x_0$ from $t = 0$ to $t = 10T_0$, where $T_0 = 2\pi\omega_0^{-1}$ for the values $\epsilon = 10^{-1}\omega_0$ and $10^{-2}\omega_0$. Compare with the nonperturbed case $\epsilon = 0$. (Hint: The software *Wolfram Mathematica* has the parabolic cylinder functions in its library.)

   (c) Repeat the same task for $H(t)/H_0$.

   (d) Repeat the same task for $J(t)/J_0$, where $J(t) = H(t)/\omega(t)$.

   (e) Interpret your results according to the adiabatic theory.

   (f) In this case, is the averaging principle useful even for times much greater than that of validity of the adiabatic approximation?

   (g) Repeat everything for $\omega(t) = \omega_0\sqrt{1 + \epsilon t}$. (Hint: now, use the Airy functions.)

2. A massive particle with negative total energy moves in one dimension where the potential is $V(x) = -V_0/\cosh^2(kx)$, where $k > 0$ is a constant. If V_0 varies slowly in time, show that $(V_0 + E)/\sqrt{V_0}$ is an adiabatic invariant.

3. A massive particle moves in one dimension where the potential is $V = kx^{-2}$, with $k > 0$ being a constant. Determine $x(t)$ by the Hamilton-Jacobi method if $x(0) = x_0$ and $\dot{x}(0) = 0$.

4. A massive particle with positive total energy moves in one dimension under the conservative potential $V(x) = F|x|$, where $F > 0$ is a constant.

   (a) Use action/angle variables to determine the period of the movement.

   (b) Apply the Bohr-Sommerfeld quantization rule $\oint pdq = \left(n + \frac{1}{2}\right)h$, where h is the Planck constant, and $n \in \mathbb{N}$. (In some text books, the $\frac{1}{2}$ additional factor is dropped out since the important quantity is the energy difference between the different energy levels. In the old quantum theory, this is also known as the Wilson-Sommerfeld quantization rule.) What is the resulting energy spectrum $E \equiv E(n)$?

5. A string of length ℓ with a mass m at each end passes through a hole in a frictionless horizontal plane. One mass moves horizontally on the plane and the other mass hangs vertically downwards.

   (a) Show that a suitable Hamiltonian for the system is $H = \frac{p_r^2}{4m} + \frac{p_\theta^2}{2mr^2} - mg(\ell - r)$, where (r, θ) are the polar co-ordinates of the particle on the plane and (p_r, p_θ) are the corresponding momenta.

   (b) Identify two constants of the motion.

   (c) Show that a steady motion with $r = r_0$ is possible (for any $r_0 > 0$) if p_θ is chosen suitably, and that the period of small oscillations about this motion is $2\pi\sqrt{\frac{2r_0}{3g}}$.

6. A particle of mass m constrained to move in one dimension is attached to the origin by a light elastic string of natural length ℓ, so that it is able to move freely along the x-axis if its distance from the origin is less than ℓ, but otherwise moves in a potential $V(x) = \frac{1}{2}k(|x| - \ell)^2$ for $|x| > \ell$.

   (a) Sketch the potential and the phase-plane trajectories for different values of the energy E.

   (b) Show that $E = \left(\sqrt{\Omega J + \beta^2} - \beta\right)^2$ where J is the action, $\beta = \pi^{-1}\sqrt{2k}\ell$, and $\Omega = \sqrt{\frac{k}{m}}$.

   (c) Explain briefly (without detailed calculation) how the angle variable ϕ conjugate to the action may be found in the form $\phi(x)$ and how x may be found as a function of the time t.

   (d) What happens in the limits of (i) small and (ii) large energies E?

7. The isotropic Harmonic Oscillator is a particular case of the 2-D Harmonic Oscillator where $w_1 = \omega_2 = \omega$. In this case, the force is central and the angular momentum is conserved. Defining $L_z = (q_1 p_2 - q_2 p_1)$, it is easy to show that $\{L_z, H_1\} = p_1 p_2 + m^2 \omega_1^2 q_1 q_2$, $\{L_z, H_2\} = -p_1 p_2 - m^2 \omega_2^2 q_1 q_2$, $\{L_z, H\} = m^2 \left(\omega_1^2 - \omega_2^2\right) q_1 q_2$. Thus, $\{L_z, H\} = 0$ for $\omega_1 = \omega_2$. Instead of using the three constants of the motion H_1, H_2, and L_z, it is interesting to use $K_1 = (2\omega)^{-1} (H_1 - H_2)$, $K_2 = (2m\omega)^{-1} \left(p_1 p_2 + m^2 \omega^2 q_1 q_2\right)$, and $K_3 = \frac{1}{2} L_z$. Then, $H = 2\omega \sqrt{K_1^2 + K_2^2 + K_3^2}$, where $\{K_i, H\} = 0$, and $\mathbf{G}_i = \mathbb{J} \nabla K_i$ are independent vectors, with $\mathbb{J} = \begin{pmatrix} \mathbf{0} & \mathbb{I} \\ -\mathbb{I} & \mathbf{0} \end{pmatrix}$ being the fundamental symplectic matrix. Notice that, although there are three constants of the motion, they are not in involution.

   (a) Calculate the vectors $\mathbf{G}_i$.

   (b) Show that ∇H is orthogonal to to all $\mathbf{G}_i$. Thus, ∇H and $\mathbf{G}_i$ form a basis to $\mathcal{F}^4$.

   (c) Show that $\{K_i, K_j\} = \epsilon_{ijk} K_k$. (This is the angular momentum algebra, and shows that the symmetry of the Hamiltonian is not SO(2), but SU(2) or SO(3).)

8. A particle of mass m and charge q moves in the xy-plane where there is a constant magnetic field $\mathbf{B} = B\hat{z}$.

   (a) Show that one possible Hamiltonian describing the system is $H = \frac{1}{2m} p_x^2 + \frac{1}{2m} \left(p_y - qBx\right)^2$.

   (b) Using the Hamilton-Jacobi method, obtain $\mathbf{r}(t)$ and $\mathbf{p}(t)$.

   (c) Show that another possible Hamiltonian describing the system is $H = \frac{1}{2m} \left(p_x + \frac{1}{2} qBy\right)^2 + \frac{1}{2m} \left(p_y - \frac{1}{2} qBx\right)^2$.

   (d) In this case, verify that the associated Hamilton-Jacobi equation is non-separable in the usual form. This shows that separability in Hamilton-Jacobi equation depends not only on the choice of the co-ordinates, but also on the chosen gauge. To separate the variables in this case, let $W = Kxy + \alpha_y y + X(x)$, where α_y is an arbitrary constant and K is another constant whose value is chosen conveniently. Finally, solve the equations of motion and compare with your previous results.