List of exercises #4 (Stability theory & the restricted 3-body problem) - 7600040

1. Consider a system with one degree of freedom and Hamiltonian $H = \frac{1}{2}p^2 + q\left(1 - \frac{1}{3}\left(\frac{q}{a}\right)^2\right)$, where a is a constant.
   (a) Find the equilibrium points.
   (b) Discuss their behavior (existence and stability) with respect to the values of a. Consider the cases of (i) $a \in \mathbb{R}$ and of (ii) a being a pure imaginary.
   (c) Using the eigenvalues and eigenvectors of the Jacobian matrix $\mathbb{J}\mathbb{H}''$, compute $\delta q\,(t)$ and $\delta p\,(t)$, the small deviations from the fixed points. The initial conditions are $\delta q\,(0) = \delta q_0$ and $\delta p\,(0) = \delta p_0$.
   (d) Sketch the corresponding flow in the phase space for $a \in \mathbb{R}$. Zoom in the unstable fixed point and sketch carefully the trajectory of many initial conditions surrounding that point. Why the unstable fixed point is also called hyperbolic fixed point?

2. Consider a system with one degree of freedom and Hamiltonian $H = \frac{1}{2m}p^2 + V\,(q)$.
   (a) Show that the stability of the fixed points q^* depend only on $V''\,(q^*)$.
   (b) Show that the eigenvalues of the Jacobian matrix are $\pm\sqrt{-V''\,(q^*)\,/m}$.
   (c) If q^* is an stable fixed point, what is the corresponding frequency of small oscillations?

3. A system of predator-prey can be modeled by $\dot{x} = -\alpha x + \beta xy$ and $\dot{y} = \gamma y - \delta xy$, where x and y are the populations of predators and preys, respectively, and α, β, γ and δ are positive constants.
   (a) Find the fixed points and study their stability.
   (b) Is the system Hamiltonian (i.e., conservative) for certain values of α, β, γ and δ?

4. Consider the restricted circular three-body problem discussed in class.
   (a) Calculate the corresponding Hamiltonian in the rotating reference frame.
   (b) Show that the acceleration of the lightest body is $\ddot{\mathbf{r}} = \mathbf{g}\,(\mathbf{r}) - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2\boldsymbol{\omega} \times \dot{\mathbf{r}}$, where $\mathbf{g}$ is the gravitational field produced by the larger masses, and $\boldsymbol{\omega} = \omega\hat{z}$. The second term on the right-hand-side is called centrifugal acceleration. What is called that third term?
   (c) Notice that $\ddot{\mathbf{r}} + 2\boldsymbol{\omega} \times \dot{\mathbf{r}} = -\nabla U$. Calculate U.
   (d) Notice that, although U cannot be interpreted as an effective potential (since $\ddot{\mathbf{r}} \neq -\nabla U$), the 5 Lagrangian points are obtained from $\nabla U = 0$. For the Lagrangian points 4 and 5, show that U is a local maximum (not a minimum). Thus, the third term (the other fictitious acceleration) is responsible for the stability of those points.
   (e) For what ratio between the two largest masses (M_1/M_2) are the Lagrangian points 4 and 5 stable equilibrium points?
   (f) This stability condition is well satisfied for the case of the Sun and Jupiter, for which $M_{☼}/M_{♃} \approx 1047$. Indeed, in that case these positions are occupied by the so-called Trojan asteroids, whose orbital periods are the same as Jupiter's, 11.86 years. Find for this case the periods of small oscillations about the 'equilibrium' points (in the plane of the orbit).