List of exercises #5 (Time-independent perturbation theory) - 7600040

1. Consider a system with one degree of freedom and Hamiltonian $H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 + \epsilon\left(aq + \frac{b}{2}q^2\right)$, where a, b, ω, and ϵ are constants. Given that $q(0) = q_0$ and $p(0) = 0$,
   (a) Solve the problem exactly.
   (b) Assuming $\epsilon \ll 1$, solve the problem using perturbation theory.
   (c) Expand the exact result up to first order in ϵ and compare with your perturbative result.

2. Consider the perturbed one-dimensional Harmonic Oscillator $H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2 + \frac{1}{2n}\epsilon x^{2n}$, where $n = 2, 3, 4, \ldots$.
   (a) Write H in terms of the action/angle variables (J_0, ϕ_0) of the unperturbed system.
   (b) Find the generating function $S(\phi_0, J_1)$ of the canonical transformation $(\phi_0, J_0) \to (\phi_1, J_1)$ that eliminates the angular dependence of the Hamiltonian up to 1st order in ϵ.
   (c) Compute the perturbed frequency of oscillation as a function of the system energy.

3. Show that the period of a simple pendulum is $T = 4\sqrt{\frac{\ell}{g}}\int_0^{\frac{\pi}{2}} \frac{du}{\sqrt{1-k^2\sin^2 u}}$, where $k = \sin\left(\frac{\theta_0}{2}\right)$, and $0 < \theta_0 < \pi$ is the amplitude of the oscillatory movement. Expand T up to 1st order in θ_0 and compare with the result obtained in class.

4. Consider a system with one degree of freedom and Hamiltonian $H = \frac{1}{2m}p^2\left(1 - \frac{1}{4}\epsilon p^2\right) + \frac{1}{2}m\omega^2 x^2$. (For $\epsilon = \left(\frac{1}{mc}\right)^2$, H is the Hamiltonian of Harmonic Oscillator with the leading relativistic correction.) Compute the frequency of oscillation up to 1st order of perturbation theory in ϵ.

5. The potential energy of a particle of mass m is $V(\mathbf{r}) = V(r) = -kr^{-1} + \frac{1}{3}\epsilon r^{-3}$, where $k > 0$ and ϵ is a small constant. Find the angular velocity ω in a circular orbit of radius a, and the angular frequency ω' of small radial oscillations about this circular orbit. Hence show that a nearly circular orbit is approximately an ellipse whose axes precession at an angular velocity $\Omega \approx \frac{\epsilon}{ka^2}\omega$.

6. Consider a system with two degrees of freedom and Hamiltonian $H = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \frac{1}{2}m\left(\omega_x^2 x^2 + \omega_y^2 y^2\right) + \epsilon x^2 y^2$.
   (a) Study the case of non-resonance and compute the natural frequencies up to 1st order of perturbation theory in ϵ.
   (b) Study the case of resonance. What is the effective theory for the non-trivial degree of freedom?

7. Consider the perturbed one-dimensional Harmonic Oscillator $H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 x^2 + \frac{1}{3}\epsilon x^3$, where ϵ is the perturbative parameter.
   (a) Show that the first order correction to the Hamiltonian is vanishing.
   (b) Calculate the correction to the natural frequency in second order of perturbation theory. (Hint: see Sec. 12.4 of Goldstein, Poole, and Safko.)