List of exercises #6 (Non-linear systems and chaos) - 7600040

1. A circuit with a nonlinear inductor can be modeled by the equations $\dot{x} = y$ and $\dot{y} = -ky - x^3 + B\cos t$.
   (a) Compare this system of equations with the Duffing equation.
   (b) Study numerically the dynamics for $k = 0.1$ and $9.8 \leq B \leq 13.4$.

2. Assume that $x(t) = b\cos(\omega_0 t) + u(t)$ is a solution of the van der Pol equation $\ddot{x} + \mu\left(x^2 - a^2\right)\dot{x} + \omega_0^2 x = 0$. Assume that the damping term μ is small and keep terms in $u(t)$ up to first order in μ. Show that $b = 2a$ and $u(t) = -\frac{\mu a^3}{4\omega_0}\sin(3\omega_0 t)$ is a solution. Produce a phase diagram of $\dot{x}$ versus x and plots of $x(t)$ and $\dot{x}(t)$ for values of $a = \omega_0 = 1$, and $\mu = 0.05$.

3. Consider the Standard map $J_{n+1} = J_n + K\sin\theta_n$ and $\theta_{n+1} = \theta_n + J_{n+1}$, with $0 \leq \theta_n \leq 2\pi$.
   (a) Study numerically the associated Poincaré section, i.e., for a fixed value of K, plot in the same graph J_n vs. θ_n for many different initial conditions $\{\theta_0, J_0\}$ and n ranging from 0 to 10^3. Consider the cases of $K = 0, 0.01, \ldots$. For what values of K the many tori start to be destructed, i.e., when does chaos appear?
   (b) Notice that as K increases there appear resonance zones, periodic orbits and bands of chaos as tori of $J = \text{const}$ undergo progressive breakdown.
   Study the Twist map and the Poincaré–Birkhoff theorem in Aguiar Sec. 10.2 and in Kibble & Berkshire Appendix D.3.

4. Consider the Logistic map $x_{i+1} = rx_i(1 - x_i) \equiv G(x_i)$, where $0 < x_0 < 1$ and r is a constant.
   (a) Show that there is a point attractor for $0 \leq r < 3$.
   (b) Show that there is a two-cycle attractor for $3 < r < 1 + \sqrt{6}$.
   (c) Show that α, β, s can be found such that y_n satisfies the logistic map with parameter s ($\neq r$) and $x_j = \alpha + \beta y_j$.
   [This is an example of the fact that all quadratic maps $y_{j+1} = A + By_j + Cy_j^2$ are essentially just the Logistic map, as it can be obtained by a suitably chosen linear relation between y_j and x_j.]
   (d) Let $x_j = G^{(j)}(x_0) \equiv G(G(\ldots G(x_0)\ldots))$. Show that $G^{(j)\prime}(x)\,|_{x=x_0} = \prod_{k=0}^{j-1} G'(x)\,|_{x=x_k}$.
   (e) Let $d_j \equiv \left|G^{(j)}(x_0 + \epsilon) - G^{(j)}(x_0)\right|$, i.e., d_j is the distance between two initial conditions after they evolved j steps. Assuming that $d_j = \epsilon e^{\lambda j}$ for $j \to \infty$, show that the Lyapunov exponent is $\lambda = \lim_{j\to\infty} \frac{1}{j}\sum_{k=0}^{j-1} \ln\left|G'(x)\,|_{x=x_k}\right|$.
   (f) Calculate the Lyapunov exponent for (i) $0 < r < 1$, (ii) $1 < r < 3$, (iii) $3 < r < 1 + \sqrt{6}$, and (iv) $r = 4$.
   (g) What is the behavior of the logistic map when $r > 4$?
   (h) Now let r and x_n be complex numbers. Find regions of the complex r plane for which the map has (i) a point attractor, (ii) a 2-cycle attractor.

5. The tent map is given by $x_{n+1} = 1 - 2\left|x_n - \frac{1}{2}\right|$ where $0 \leq x_n \leq 1$.
   (a) Find the fixed points.
   (b) Show that these fixed points are all unstable.
   (c) Show that this map exhibits extreme sensitivity to initial conditions, i.e., an uncertainty ϵ in x_0 is rapidly magnified. Estimate the number of iterations after which the range of uncertainty in the iterates is the complete interval $[0, 1]$.

6. For the cubic map $x_{n+1} = ax_n - x_n^3$, where a is real, show that, when $|a| < 1$, there is an asymptotically stable fixed point $x^* = 0$ and that, when $1 < a < 2$ there are two such fixed points at $x^* = \pm\sqrt{a-1}$. What happens when $a > 2$?

7. By calculating Lyapunov exponents examine sensitivity to initial conditions of the equal-area maps of the unit square $(0 \leq x, y \leq 1)$:
   (a) Arnold's cat map $x_{n+1} = x_n + y_n$, $y_{n+1} = x_n + 2y_n$ (each modulo 1).
   (b) The baker's transformation $(x_{n+1}, y_{n+1}) = \left(2x_n, \frac{1}{2}y_n\right)$ if $x_n < \frac{1}{2}$, and $(x_{n+1}, y_{n+1}) = \left(2 - 2x_n, \frac{1}{2}(2 - y_n)\right)$, if $x_n > \frac{1}{2}$.