List of exercises #7 (Rotation of a rigid body) - 7600040

1. Show the the scalar product is an invariant under orthogonal transformations.

2. Consider the three-dimensional unit vectors $\hat{e}_i$, $i = 1, 2, 3$, such that $\hat{e}_i \times \hat{e}_j = \sum_k \varepsilon_{ijk} \hat{e}_k$, where ε_{ijk} is the Levi-Civita symbol. Show that

   (a) $\sum_k \varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$, where δ_{ij} is the Kronecker delta,

   (b) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}\,(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}\,(\mathbf{a} \cdot \mathbf{b})$,

   (c) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$.

3. Let $\mathbb{M}$ be a real and symmetric matrix. Show that

   (a) the eigenvalues of $\mathbb{M}$ are real (what can be said about the eigenvectors?),

   (b) the eigenvectors of distinct eigenvalues are orthogonal to each other,

   (c) a superposition of degenerate eigenvectors is also an eigenvector with the same eigenvalue,

   (d) the determinant of $\mathbb{M}$ is invariant under orthogonal transformations,

   (e) the eigenvalues of $\mathbb{M}$ are invariant under orthogonal transformations.

4. Transform the tensor $\mathbb{M} = \mathbf{a}\mathbf{b}^T + \mathbf{b}\mathbf{a}^T$, where $\mathbf{a} = 5\hat{x} - 3\hat{y} + 2\hat{z}$ and $\mathbf{b} = 5\hat{y} + 10\hat{z}$ by applying a rotation of $\frac{\pi}{4}$ about the z-axis.

5. Compute the eigenvectors and eigenvalues of $\mathbb{M} = \begin{pmatrix} 7 & \sqrt{6} & -\sqrt{3} \\ \sqrt{6} & 2 & -5\sqrt{2} \\ -\sqrt{3} & -5\sqrt{2} & -3 \end{pmatrix}$ and $\mathbb{N} = \frac{1}{2}\begin{pmatrix} 3 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 3 \end{pmatrix}$.

6. Show that $(\mathbb{M}\mathbb{N})^T = \mathbb{N}^T\mathbb{M}^T$ and that $\left(\mathbf{a}^T\mathbb{M}\mathbf{b}\right)^T = \mathbf{b}^T\mathbb{M}^T\mathbf{a}$.

7. Consider a homogeneous cube of mass m and size a.

   (a) Compute the inertia tensor in a coordinate system whose axes coincides with the edges of the cube.

   (b) The cube is set to spin around the z-axis of that coordinate system with angular velocity ω. Compute the angular momentum. Is it parallel to the angular velocity? Explain.

   (c) Now, simply translate that coordinate system by $\frac{1}{2}a\hat{z}$. Thus, the new xy plane cuts the cube in half. Repeat the tasks (a) and (b) for this new coordinate system.

8. Correction to the spherical gravitational field. We want to compute the first correction to the gravitational field of a quasi-spherical celestial body. This correction is due to the non-spherical shape of this body (due to, e.g., its rotation).

   (a) The gravitational potential at the position $\mathbf{r}$ due to a distribution of mass is $V(\mathbf{r}) = -G\sum_j \frac{m_j}{|\mathbf{r}-\mathbf{r}_j|}$, where G is the gravitational constant, m_j is the mass of the jth point particle of that distribution, and $\mathbf{r}_j$ is its position. For $r \gg r_j$, show that $|\mathbf{r} - \mathbf{r}_j|^{-1} = \frac{1}{r} + \frac{\mathbf{r}\cdot\mathbf{r}_j}{r^3} + \frac{3(\mathbf{r}\cdot\mathbf{r}_j)^2 - r^2 r_j^2}{2r^5} + \ldots$.
   (Curiosity: The general term of this series is $\frac{r_j^\ell}{r^{\ell+1}} P_\ell\left(\hat{r} \cdot \hat{r}_j\right)$, where P_ℓ is the Legendre polynomial of order ℓ.)

   (b) Show that the first term of the series gives the monopole contribution $V_0(\mathbf{r}) = -\frac{GM}{r}$ where M is the total mass of the distribution.

   (c) Show that the second (dipole) term is $V_1(\mathbf{r}) = -\frac{GM}{r^3}\mathbf{r} \cdot \mathbf{R}$. What is the meaning of $\mathbf{R}$?

   (d) For a celestial body with a symmetry axis ($\rho(\mathbf{r}) = \rho(r, z)$), show that the third (quadrupole) term is $V_2(\mathbf{r}) = Gr^{-5}(I_z - I_x)\left(3z^2 - r^2\right) = Gr^{-3}(I_z - I_x)\left(3\cos^2\theta - 1\right)$, where $I_{x,z}$ are the principal moments of inertia, and θ is the angle between the body symmetry axis and the $\mathbf{r}$.

   (e) It is usually said that the quadrupole correction is the first non-trivial one, i.e., one can usually neglect the dipole correction. Why? (Why a finite electric dipole cannot be made null?)

9. (Optional... for relaxing) Tides. We want to compute the tides. For that, consider a simple model where the Earth is a perfect sphere with a liquid ocean. We also neglect the its rotation. (The effect of rotation is to drag the tides and displace it ahead the line the connects the two celestial bodies.) Other important effects are also neglected such as the gravitational ocean attraction, geography, currents, etc.

   (a) Adopt the origin of a coordinate system the center of Earth. A point on the Earth's surface is at position $\mathbf{r}$. The center of the Moon be at position $a\hat{z}$. Assuming that $a \gg r$, show that the gravitational potential at $\mathbf{r}$ due to the Moon is $V(\mathbf{r}) \approx V_0 + V_1 + V_2$ where $V_0 = -\frac{GM_{☾}}{a}$, $V_1 = -\frac{GM_{☾}}{a^2}z$, $V_2 = -\frac{GM_{☾}}{2a^3}\left(3z^2 - r^2\right) = -\frac{GM_{☾}}{2a^3}r^2\left(3\cos^2\theta - 1\right)$, and $M_{☾}$ is the Moon mass.
   (V_0 is a constant and does not produce any force. V_1 corresponds to a constant force towards the Moon. This is compensated by the non-inertial force at the Earth reference frame. The interesting term is the quadrupole one V_2.)

   (b) The profile of the tide is given by $R + h(\theta)$ where R is the Earth's radius and $h(\theta)$ is the height of the ocean's surface above the Earth solid core. Show that the gravitational potential due to the Earth at the ocean's surface is $U(\mathbf{r}) \approx -\frac{GM_{♁}}{R} + \frac{GM_{♁}}{R^2}h(\theta)$.

   (c) As the surface of the liquid ocean is a gravitational equipotential, show that $h(\theta) \approx \frac{M_{☾}}{M_{♁}}\left(\frac{R}{a}\right)^3\left(\frac{3\cos^2\theta - 1}{2}\right)R$, and thus, the height difference between the high and low tides is $\frac{3}{2}\frac{M_{☾}}{M_{♁}}\left(\frac{R}{a}\right)^3 R$.

   (d) Which one contributes the most to the tide: the Moon or the Sun?
   Curiosity: Due to the rotation, the bulge tide is $\approx 10^\circ$ ahead the line connecting the Earth and the Moon. Due to friction, it is estimate that the duration of the day increases 2.3 ms/century. Although small, it corresponds to a dissipation rate of the rotational kinetic energy of 4 TW. Due to this bulge, Earth accelerates the Moon's rotation. The corresponding rate of energy transfer from the Earth's rotation to the Moon's translation is estimated in 0.1 TW, which corresponds to the an increase of the distance between the Earth and the Moon of 3.8 cm/year. Finally, paleontologists estimate that the duration of the day was 23.5 hours 70My ago and 21.9 hours 620 My ago. Also, there where 13 lunations/year 620 My ago.

10. Let $\{I_1, I_2, I_3\}$ be the principal moments of inertia of a solid computed in a certain system of coordinates. Show that the moment of inertia I with respect to any axis passing through the origin of the same system of coordinates is such that $\min\{I_i\} \le I \le \max\{I_i\}$.

11. Let $\mathbb{I}_{\text{CM}}$ be the inertia tensor of a body of mass M with respect to a system of coordinates the origin of which is the center of mass of that body.

   (a) Show that $\mathbb{I} = \mathbb{I}_{\text{CM}} + M\left(R_{\text{CM}}^2\mathbb{1} - \mathbf{R}_{\text{CM}}\mathbf{R}_{\text{CM}}^T\right)$, where $\mathbb{I}$ and $\mathbf{R}_{\text{CM}}$ are, respectively, the inertia tensor and the position of the center of mass of the body with respect to a generic system of coordinates.

   (b) Recover the theorem of parallel axes from this result.

12. A very thin, homogeneous and perfectly rigid rod of mass m and length L is kept spinning with angular velocity ω through an axis that makes an angle α with the rod and contains the rod's center of mass. Let $\hat{e}_{x'}$ be the x-axis attached to the body which coincides with the rod symmetry axis. In addition let $\hat{e}_{z'}$ be an axis perpendicular to the rod which contains its center of mass and that the spinning axis is in the $x'z'$-plane.

   (a) Compute the inertia tensor with respect to the rod's coordinate system.

   (b) Compute the angular momentum $\mathbf{L}$ with respect to the rod's coordinate system.

   (c) Compute the corresponding torque via (i) direct differentiation of $\mathbf{L}$ and via (ii) the Euler equations.

13. Consider the frictionless symmetric spinning top with a fixed point (discussed in class) with the initial condition $\beta(0) = \beta_0$, $\dot{\beta}(0) = \dot{\alpha}(0) = 0$ and $\dot{\gamma}(0) = \omega_0$, where α, β, and γ are the Euler angles.

   (a) Show that the maximum value attained by the polar angle is $\beta_{\max} = \arccos\left(\frac{1 - \sqrt{1 + 4a^2 - 4a\cos\beta_0}}{2a}\right)$, where $a = 2\frac{I_x mg\ell}{p_\gamma^2}$.
   Hint: Recast the effective potential $V_{\text{eff}}(\beta)$ in terms of the constants p_γ, I_x, a, β_0.

   (b) When $p_\gamma^2 \gg I_x mg\ell$ (fast spinning), the wobble amplitude $\delta\beta_0$ is small. In this case, show that $\beta_{\max/\min} = \beta^* \pm \delta\beta_0$, where $\beta_{\min} = \beta_0$ and $\delta\beta_0 \approx \frac{1}{2}a\sin\beta_0$.

(c) Rewrite $\beta = \beta^* + \delta\beta\,(t) = \beta_0 + \delta\beta_0 + \delta\beta\,(t)$. Then, expanding the effective potential up to second order in $\delta\beta_0$ and $\delta\beta\,(t)$, show that $V_{\text{eff}} \approx V_{\text{eff}}\,(\beta^*) + \frac{1}{2}\frac{p_\gamma^2}{I_x}\delta\beta^2$, and, thus, $\delta\beta = -\delta\beta_0 \cos\left(\Omega t\right)$. Compute the wobble angular velocity Ω.

(d) Show that the precession angular velocity is $\dot{\alpha} \approx \frac{I_z \omega_0 a}{2 I_x}\left(1 - \cos\left(\Omega t\right)\right)$, and, thus, that the average angular velocity of precession is $\frac{mg\ell}{I_z \omega_0}$.

14. Study the Poinsot construction at Lemos Sec. 4.8.