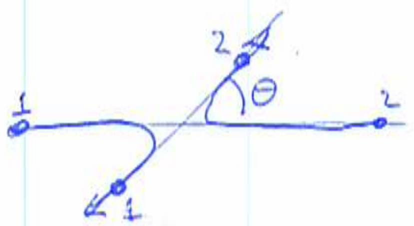
SYSTEM OF IDENTICAL PARTICLES

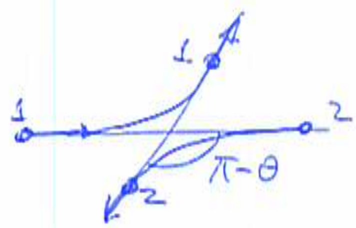
IN CLASSICAL MECHANICS, IDENTICAL PARTICLES ARE DISTINGUISHABLE. ONE CAN LABEL THEM AND FOLLOW THEIR TRAJECTORY

IN QUANTUM MECHANICS, THIS IS NO LONGER TRUE BECAUSE OF THE UNCERTAINTY PRINCIPLE

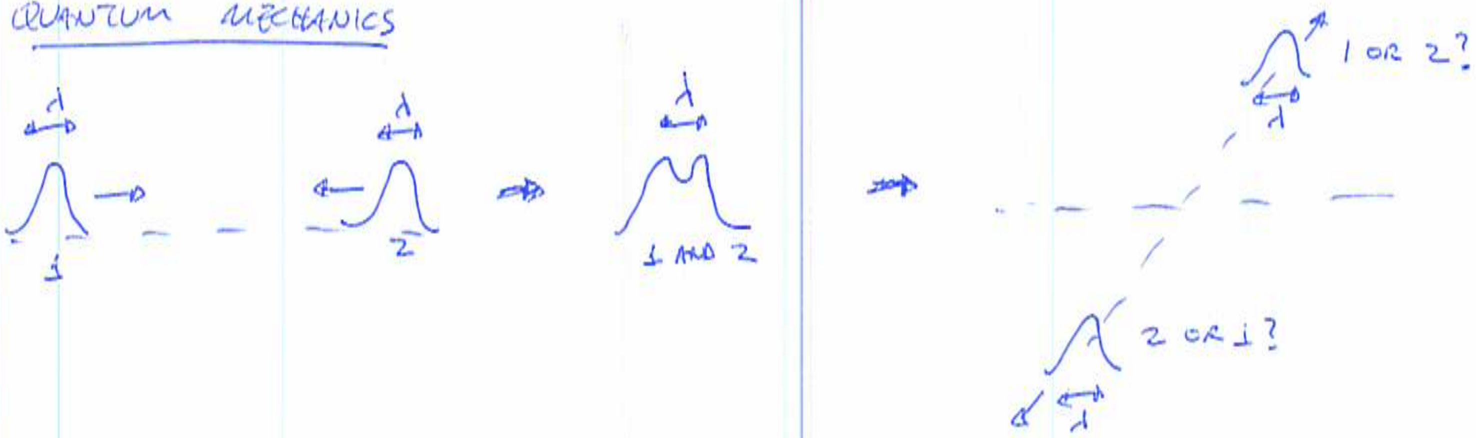
CLASSICAL MECHANICS



IS DIFFERENT FROM



QUANTUM MECHANICS



THUS, IN Q.M., IDENTICAL PARTICLES LOSE THEIR "INDIVIDUALITY" ENTIRELY AND ARE THUS, INDISTINGUISHABLE

CONSIDER A SYSTEM OF N IDENTICAL QUANTUM PARTICLES THE WAVEFUNCTION IS

$$\psi(q_1, \cdots q_i \cdots q_j \cdots q_N) \quad \text{where}$$

$q_i = (\vec{r}_i, s_i)$, where s_i REPRESENTS THE SPIN

PROJECTION OF THE i-th PARTICLE AND ALSO ANY OTHER DEGREE OF FREEDOM

LET P_{ij} BE THE PERMUTATOR OPERATOR WHICH EXCHANGE PARTICLES i AND j WITH EACH OTHER

$$\Rightarrow \quad P_{ij}\psi(\cdots q_i \cdots q_j \cdots) = e^{i\gamma}\psi(\cdots q_j \cdots q_i \cdots)$$

SYMMETRIZATION POSTULATE:

BY REPEATING THE EXCHANGE, WE RETURN TO THE ORIGINAL STATE. THUS

$$e^{2i\gamma} = 1 \quad \Rightarrow \quad e^{i\gamma} = \pm 1$$

$$\gamma = 0 \quad \text{OR} \quad \pi$$

IN SHORT, $\boxed{\psi(\cdots q_i \cdots q_j \cdots) = \pm\psi(\cdots q_j \cdots q_i \cdots)}$

THEREFORE, THE WAVE FUNCTIONS ARE EITHER SYMMETRIC ($\gamma = 0$) OR ANTI-SYMMETRIC ($\gamma = \pi$) WHEN A PAIR OF PARTICLES ARE INTERCHANGED

THE PROPERTY OF BEING DESCRIBED BY SYMMETRICAL OR ANTISYMMETRICAL WAVEFUNCTIONS DEPEND ON THE NATURE OF THE PARTICLES:

$\gamma = 0$ ARE BOSONS → OBEY BOSE-EINSTEIN STATISTICS

$\gamma = \pi$ ARE FERMIONS → " FERMI-DIRAC "

FOR NON-INTERACTING (RELATIVISTIC) QUANTUM FIELD THEORY, PAULI SHOWED THE SPIN-STATISTICS THEOREM [PHYS. REV. 58, 716 (1940)]

- PARTICLES OF INTEGER SPIN ARE BOSONS
- " " HALF-ODD INTEGER " " FERMIONS

ACCORDING TO FEYNMAN, THIS THEOREM CANNOT BE UNDERSTOOD FROM SIMPLE BASIC PRINCIPLES.

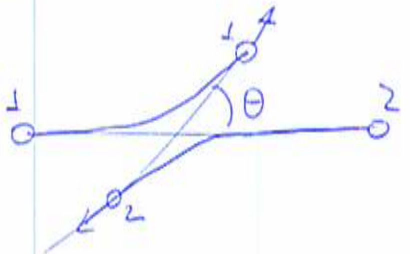
THUS, IT MEANS THAT OUR UNDERSTANDING ON THE SUBJECT NEEDS TO BE IMPROVED.

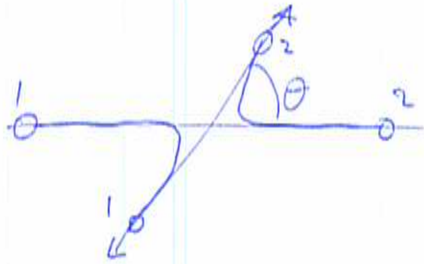
IT IS WORTHY NOTING THAT, SINCE FEYNMAN, THERE HAVE BEEN DEVELOPMENTS.

- SEE BERRY + ROBBINS, PROC. R. SOC. LONDON A, **453**, 1771 (1997)

- SEE NEW EXPERIMENTS ON DIRECT MEASUREMENT OF THE EXCHANGE PHASE, ROOS ET. AL PHYS. REV. LETT. **119**, 160401 (2017)

CONSEQUENCES FOR THE SCATTERING EXPERIMENT





$\frac{e^{ikr}}{r}\psi(\theta,\phi)$ REPRESENTS

PARTICLE 1 $\rightarrow$ θ AND ϕ

" 2 $\rightarrow$ $\pi-\theta$ AND $\phi+\pi$

$\frac{e^{ikr}}{r}\psi(\pi-\theta,\ \phi+\pi)$ REPRESENTS

PARTICLE 1 $\rightarrow$ $\pi-\theta$ AND $\phi+\pi$

" 2 $\rightarrow$ π AND ϕ

DUE TO CYLINDRICAL SYMMETRY $\psi(\theta,\phi)=\psi(\theta)$

$$\Rightarrow |\psi_{out}\rangle \propto \left(|\psi(\theta)\rangle + e^{i\gamma}|\psi(\pi-\theta)\rangle\right)\otimes\left|\frac{e^{ikr}}{r}\right\rangle$$

$\gamma=0$ BOSONS

$\gamma=\pi$ FERMIONS

PROB OF HITTING THE DETECTOR AT θ AND ANY ϕ IS

$$\sigma(\theta)\propto P(\theta)\propto|\psi_{out}|^2\propto|\psi(\theta)|^2+|\psi(\pi-\theta)|^2+\underbrace{e^{i\gamma}\langle\psi(\theta)|\psi(\pi-\theta)\rangle+h.c.}_{\text{INTERFERENCE}}$$

FOR $\theta=\pi/2$

$$\Rightarrow P(\theta)\rightarrow\left|\psi\left(\tfrac{\pi}{2}\right)\right|^2(1+\cos\gamma)$$

THUS, FOR $\begin{cases} \gamma = 0 \rightarrow \text{CONSTRUCTIVE INTERFERENCE} \\ \gamma = \pi \rightarrow \text{DESTRUCTIVE } '' \end{cases}$

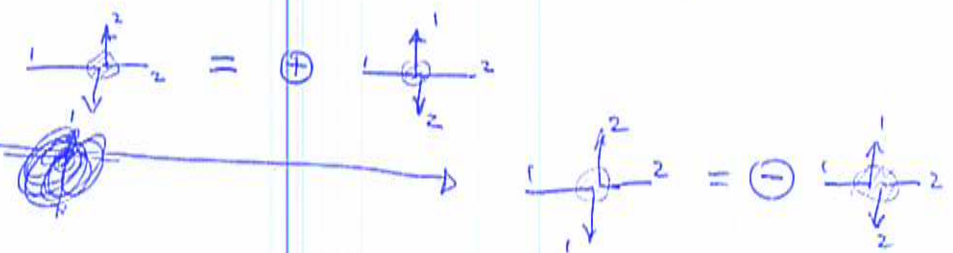
PHYSICAL INTERPRETATION

BECAUSE OF CYLINDRICAL SYMMETRY
$\phi \rightarrow \phi + \pi$

BUT THIS OPERATION IS EQUIVALENT OF EXCHANGING PART. 1 AND 2 IN THE FINAL STATE

BOSONS $(\gamma = 0)$ $\longrightarrow$

FERMIONS $(\gamma = \pi)$ $\longrightarrow$



THEREFORE, DESTRUCTIVE INTERFERENCE FOR FERMIONS

CONSTRUCTING SYMMETRICAL AND ANTISYMMETRICAL STATES

- THE 2-PARTICLES CASE

GROUP OF PERMUTATIONS $\equiv \{ P_{12}, P_{21} \} = \{ \mathbb{1}, P_{21} \}$

$\downarrow$ OPERATOR THAT PERMUTES PARTICLES 1 AND 2

$$P_{21} \, \psi(q_1, q_2) = \psi(q_2, q_1)$$

~~SUPPOSE~~ LET $\{ \phi_i \}$ BE A SET OF 1-PARTICLE ORBITAL WAVE FUNCTIONS WHICH IS COMPLETE AND ORTHONORMAL

WHAT ARE THE POSSIBLE SYMMETRIC AND ANTI-SYMMETRIC WAVE FUNCTIONS?

$$|\psi_{PHYS}(\vec{r}_1, \vec{r}_2)\rangle = \text{SYMMETRIZER}\left(|\psi_\alpha(\vec{r}_1)\rangle \otimes |\psi_\beta(\vec{r}_2)\rangle \right) = S\,|\psi_\alpha, \psi_\beta\rangle$$

OR

$$\text{ANTI-SYMMETRIZER}\left(|\psi_\alpha(\vec{r}_1)\rangle \otimes |\psi_\beta(\vec{r}_2)\rangle \right) = A\,|\psi_\alpha, \psi_\beta\rangle$$

where $|\psi_\alpha\rangle = \sum_i a_{\alpha_i} |\varphi_i\rangle$

$|\psi_\beta\rangle = \sum_i a_{\beta_i} |\varphi_i\rangle$

THUS, IN THE END, WE WILL HAVE TO SYMMETRIZE OR ANTISYMMETRIZE ORTHONORMAL ORBITS. FOR SIMPLICITY, LETS CONSIDER THAT $|\psi_\alpha\rangle$ AND $|\psi_\beta\rangle$ ARE EITHER ORTHOGONAL TO EACH OTHER (OR EQUAL) TO EACH OTHER.

THE OPERATION OF SYMMETRIZATION (S) AND THE OPERATION OF ANTISYMMETRIZATION (ALSO CALLED AS ALTERNATION) (A) ARE

$$S = \frac{\mathbb{1} + P_{21}}{2}, \quad A = \frac{\mathbb{1} - P_{21}}{2}$$

$$\Rightarrow \begin{cases} |\psi_{SYM}\rangle = \text{CONST} \times S|\psi_\alpha, \psi_\beta\rangle \\ |\psi_{ANTISYM}\rangle = \text{CONST} \times A|\psi_\alpha, \psi_\beta\rangle \end{cases}$$

FOR $|\psi_\alpha\rangle \neq |\psi_\beta\rangle \Rightarrow$

$$|\psi_S\rangle = \frac{1}{\sqrt{2}}\left(|\psi_\alpha \psi_\beta\rangle + |\psi_\beta \psi_\alpha\rangle\right)$$

$$|\psi_A\rangle = \frac{1}{\sqrt{2}}\left(|\psi_\alpha \psi_\beta\rangle - |\psi_\beta \psi_\alpha\rangle\right)$$

FOR $|\psi_\alpha\rangle = |\psi_\beta\rangle \Rightarrow \begin{cases} |\psi_S\rangle = |\psi_\alpha \psi_\alpha\rangle \\ |\psi_A\rangle = 0 \end{cases}$ (PAULI EXCLUSION PRINCIPLE)

HOW ABOUT THE SPINS? FOR $s = 1/2$ PARTICLES, THEN THE PHYSICAL SPINORIAL PART BECOMES

$$|\chi_S\rangle = \begin{cases} |++\rangle \\ \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle) \\ |--\rangle \end{cases} \quad \text{AND} \quad |\chi_A\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right)$$

(TRIPLET) $S=1$ (SINGLET) $S=0$

THUS, FOR 2 SPIN-1/2 PARTICLES (FERMIONS), THE PHYSICAL STATES ARE

$$|\psi_{PHYS}\rangle = |\psi_S\rangle \otimes |\chi_A\rangle \quad \text{AND} \quad |\psi_A\rangle \otimes |\chi_S\rangle$$

NOTICE THAT $P_{21}|\psi_{PHYS}\rangle = -|\psi_{PH}$

BECAUSE $S + A = \mathbb{1}$ AND $AS = 0 = SA$, AND $S^2 = S$, $A^2 = A$

THERE ARE ONLY 2 ~~TYPES~~ CLASSES OF SYMMETRIC STATES.

THIS IS NO LONGER TRUE FOR MORE THAN $N=2$ PARTICLES

- 3-PARTICLES CASE

GROUP OF PERMUTATIONS $= \{ \mathbb{1}, P_{213}, P_{321}, P_{132}, P_{312}, P_{231} \}$

P_{123} ...

P_{213}: $2 \leftrightarrow 1$, $3 \leftrightarrow 3$

P_{321}: $1 \leftrightarrow 3$, $2 \leftrightarrow 2$

P_{132}: $1 \leftrightarrow 1$, $2 \leftrightarrow 3$

1 PAIR OF PARTICLES BEING PERMUTED (SIMPLE PERMUTATION) = TRANSPOSITION

P_{312}: $1 \to 2$, $2 \to 3$, $3 \to 1$ $= P_{321} P_{213}$

P_{231}: $2 \to 1$, $1 \to 3$, $3 \to 2$ $= P_{213} P_{32}$

2 SIMPLE PERMUTATIONS (OR 2 TRANSPOSITIONS)

NOTICE THAT THE HAMILTONIAN COMMUTES WITH ANY OPERATOR OF THE GROUP

$$P^{+}_{ijk} H P_{ijk} = H$$

BUT THIS IS A NON ABELIAN GROUP $P_{ijk} P_{\alpha\beta\gamma} \neq P_{\alpha\beta\gamma} P_{ijk}$

$\Rightarrow$ IMPOSSIBLE TO CONSTRUCT STATES THAT ARE EIGENSTATES OF H AND OF ALL OPERATORS OF THE GROUP

ANTI-SYMMETRIC STATE

$$|\psi_A\rangle = \text{CONST} * \sum_{\{P\}} (-1)^p \, P \, |\psi\rangle$$

FOR 3-PARTICLES, LET $|\psi\rangle = |R,G,B\rangle$; WITH $R \neq G$, $R \neq B$, $G \neq B$

$\{P\} \equiv$ SET OF ALL POSSIBLE PERMUTATIONS
/ THE IRREDUCIBLE REPRESENTATION OF THE GROUP

$p =$ PARITY OF THE PERMUTATION

$= \begin{cases} -1, & \text{IF THE NO. OF SIMPLE PERMUTATION IS ODD} \\ +1, & \text{" " " " " " EVEN} \end{cases}$

AND $\text{CONST} = \frac{1}{\sqrt{3!}}$ (FOR N FERMIONIC PARTICLES $\text{CONST} = \frac{1}{\sqrt{N!}}$)

$$\Rightarrow \boxed{|\psi_A\rangle = \frac{1}{\sqrt{6}}\Big(|RGB\rangle + |GBR\rangle + |BRG\rangle - |GRB\rangle - |BGR\rangle - |RBG\rangle \Big)}$$

THIS IS THE BARIONIC COLOR SINGLET

SYMMETRIC STATE

$$|\psi_S\rangle = \text{CONST} * \sum_{\{P\}} P |\psi\rangle \quad , \quad \text{FOR } R \neq G,\ R \neq B,\ B \neq G$$

$$\Rightarrow \boxed{|\psi_S\rangle = \frac{1}{\sqrt{6}}\Big(|RGB\rangle + |GBR\rangle + |BRG\rangle + |GRB\rangle + |BGR\rangle + |RBG\rangle \Big)}$$

FOR $R = G \neq B \Rightarrow |\psi_S\rangle = \sqrt{\frac{2!}{3!}} \times \frac{1}{2} \sum_{\{P\}} P |RRB\rangle = \frac{1}{\sqrt{3}}\Big(|RRB\rangle + |RBR\rangle + |BRR\rangle\Big)$

FOR $R = G = B \Rightarrow |\psi_S\rangle = \sqrt{\frac{3!}{3!}} \times \frac{1}{6} \sum_{\{P\}} P \{RRR\} = |RRR\rangle$

IN GENERAL, FOR A SYSTEM OF N PARTICLES THE SYMMETRIC AND ANTI-SYMMETRIC WAVE FUNCTIONS ARE OBTAINED VIA THE FORMULAS

$$\psi_A(q_1 \cdots q_N) = \frac{1}{\sqrt{N!}} \begin{vmatrix} \psi_1(q_1) & \psi_1(q_2) & \cdots & \psi_1(q_N) \\ \psi_2(q_1) & \psi_2(q_2) & \cdots & \psi_2(q_N) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_N(q_1) & \psi_N(q_2) & \cdots & \psi_N(q_N) \end{vmatrix}$$

WHICH IS A SLATER DETERMINANT

FOR BOSONIC PARTICLES, IT IS NOT A DETERMINANT, BUT A PERMANENT. THE NORMALIZATION IS NOT SO SIMPLE DUE TO ~~REPEA~~ POSSIBLE REPEATED STATES

$$\psi_S(q_1 \cdots q_N) = \cancel{\frac{N_1! N_2! \cdots N_N!}{N!}} \begin{bmatrix} \psi_1(q_1) & \cdots & \psi_1(q_N) \\ \vdots & \ddots & \vdots \\ \psi_N(q_1) & \cdots & \psi_N(q_N) \end{bmatrix} * \text{CONST}$$

WHERE N_i IS THE NO. OF TIMES A STATE IS OCCUPIED BY A BOSONIC PARTICLE

AND CONST IS THE NORMALIZATION CONSTANT

$$= \sqrt{\frac{1}{N_0! \, N_1! \cdots N_N!}}$$

FOCK SPACE AND SECOND QUANTIZATION

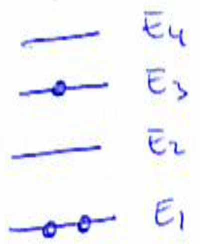
WE NOW KNOW HOW TO SYMMETRIZE OR ANTISYMMETRIZE. IT IS CUMBERSOME. IN ORDER TO MAKE THINGS SIMPLER, WE DO NOT WORK ANYMORE IN THE HILBERT SPACE WHICH ARE NOT "NATURALLY" SYMMETRIZED. WE RATHER WORK WITH STATES THAT GIVES THE OCCUPATION NUMBER OF A GIVEN PHYSICAL STATE OF THE HILBERT SPACE.

THIS OCCUPATIONAL STATE LIVES IN THE FOCK SPACE

*OF COURSE, WE WILL HAVE TO LEARN HOW OPERATORS ACT ON THOSE STATES

EXAMPLE

BOSONS :



$$|\psi\rangle \equiv |m_1, m_2, m_3, m_4, m_5 \cdots\rangle$$

STATE OF THE FOCK SPACE

m_i = NO. OF PARTICLES IN THE i-th STATE

IN THE ABOVE CONFIGURATION

$$|\psi\rangle = |2, 0, 1, 0, \cdots\rangle \quad \text{(FOCK SPACE)}$$

IN THE HILBERT SPACE, WE WOULD HAVE TO SYMMETRIZE $|\psi\rangle = |E_1, E_1, E_3\rangle$

$$\longrightarrow \frac{1}{\sqrt{3}}\left(|E_1 E_1 E_3\rangle + |E_1 E_3 E_1\rangle + |E_3 E_1 E_1\rangle \right.$$

IN GENERAL,

$$|\psi\rangle = |m_1 m_2 m_3 \cdots\rangle = \text{CONST} * \overbrace{\sum_{\{P\}} |E_{P(1)} E_{P(2)} \cdots\rangle}^{= \sum_{\{P\}} P|E_1 E_2 \cdots\rangle}$$

$\downarrow$ FOCK SPACE

ALL POSSIBLE PERMUTATIONS (CAN HAVE REPEATED INDICES FOR THE CASE OF STATES WITH $m_i > 1$)

$\downarrow$ HILBERT SPACE OF N PARTICLES $\equiv \mathcal{H}_{(N)}$

$$\text{CONST} = \frac{1}{\sqrt{N!\, m_1!\, m_2! \cdots}}$$

IMPORTANT: $|n_1, n_2, \dots\rangle$ REPRESENTS A STATE WITH A NUMBER OF PARTICLES $N = \sum_i n_i$ WHICH DOES NOT NEED TO BE CONSTANT

$\Rightarrow$ FOCK SPACE $= \mathcal{H}_{0} \oplus \mathcal{H}_{1} \oplus \mathcal{H}_{2} \oplus \dots$

FOR FERMIONS, THINGS ARE SIMPLER BECAUSE $n_i = 0$ OR 1 DUE TO THE EXCLUSION PRINCIPLE. HOWEVER, WE HAVE TO CONSIDER THE PARITY OF THE PERMUTATION

$$|n_1, n_2, \dots\rangle = \frac{1}{\sqrt{N!}} \sum_{\{P\}} (-1)^P \, |\varepsilon_{P(1)}, \varepsilon_{P(2)}, \dots\rangle$$

OPERATORS IN THE FOCK SPACE

WE NOW WANT TO OPERATE WITH THE SYMMETRIC OR ANTISYMMETR STATES OF THE FOCK SPACE. IT IS THEN CONVENIENT TO FORMULATE THE OPERATORS IN TERMS OF **THEIR** ACTION **ON** THE OCCUPATION NUMBER (FOCK) STATES.

WHAT KIND OF OPERATORS DO WE HAVE IN MIND?

$$H = \underbrace{\underbrace{\sum_i \frac{P_i^2}{2m_i}}_{\text{KINETIC ENERGY}} + \underbrace{\sum_i U(\vec{x}_i)}_{\text{EXTERNAL POTENTIAL ENERGY}}}_{\text{1-BODY OPERATORS (OR 1-PARTICLE OPERATORS)}} + \underbrace{\frac{1}{2} \sum_{i \neq j} V(\vec{x}_i - \vec{x}_j)}_{\text{INTERACTION POTENTIAL ENERGY} \to \text{2-BODY OPERATOR}} + \underbrace{\dots}_{\text{OTHER TYPES OF OPERATORS (3-BODY, N-BODY...)}}$$

- SINCE ALL PARTICLES ARE INDISTINGUISHABLE, THERE IS NO MEANING IN COMPUTING $\vec{P}_4$, FOR INSTANCE (UNLESS $N=1$).
- ~~A~~ SOMEWHAT "INDIVIDUAL" OPERATOR THAT IS MEANINGFUL IS A LOCAL OPERATOR. FOR INSTANCE, WE COULD INQUIRE ABOUT THE Z-PROJECTION OF THE SPIN AT SOME POINT IN SPACE.

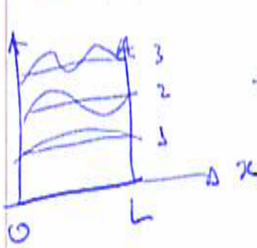
WHAT DO WE EXPECT FROM THE ACTION OF THE OPERATORS ON THE FOCK STATES?

1- WE EXPECT COUNTINGS
AND
2- WE EXPECT TRANSITIONS

1- COUNTINGS: TAKE THE MATRIX ELEMENT $\langle m_1 m_2 \cdots | O | m_1 m_2 \cdots \rangle$
$$= O_{m_1 m_2 \cdots}$$

NOTICE THAT IT DEPENDS ON $\{m_1, m_2 \cdots\}$. THUS, IT DEPENDS ON HOW MANY PARTICLES THERE ARE ON EACH STATE. IN OTHER WORDS, THIS MATRIX ELEMENT DEPENDS ON THE COUNTING OF THE PARTICLES IN THOSE STATES.

EXAMPLE: N (NON-INTERACTING) IDENTICAL PARTICLES IN AN INFINITE 1D SQUARE POTENT.



$$\rightarrow \psi_m(x) = \sqrt{\frac{2}{L}} \sin k_m x \,, \quad \text{with} \quad k_m = \frac{\pi}{L} m \,, \quad m \in \mathbb{N}^*$$

N-PARTICLES WAVE FUNCTION

$$\Psi_{m_1 m_2 \cdots}(x_1, x_2 \cdots, x_N) = \frac{1}{\sqrt{N!\, m_1! \cdots m_N!}} \sum_{\{P\}} \zeta^P \left(\frac{2}{L}\right)^{\frac{N}{2}} (\sin k_{P(1)} x_1) \cdots (\sin k_{P(N)} x_N)$$

where $\begin{cases} \xi = +1 & \text{FOR BOSONS} \\ \xi = -1 & \text{FOR FÉRMIONS} \end{cases}$

LET US ~~…~~ COMPUTE $T\psi = \sum_i \frac{p_i^2}{2m}\psi = \frac{-\hbar^2}{2m}\left(\sum_i \frac{\partial^2}{\partial x_i^2}\right)\psi$

NOTICE THAT IN ANY PARCEL OF THE SUM

$$\sum_{\{P\}} \xi^P \sin(k_{P(1)} x_1) \sin(k_{P(2)} x_2) \cdots \sin(k_{P(N)} x_N)$$

THE ACTION OF T YIELDS THE SAME RESULT WHICH IS THE TOTAL KINETIC ENERGY

$$\Rightarrow T\psi = \frac{\hbar^2}{2m}\underbrace{\left(n_1 k_{n_1}^2 + n_2 k_{n_2}^2 + \cdots + n_N k_{n_N}^2\right)}_{= \text{SUM OVER THE OCCUPIED LEVELS}}\psi$$

THUS, WE EXPECT THAT

$$\hat{T} = \underbrace{\sum_{i=1}^{\infty}}_{\text{SUM OVER ALL STATES}} \frac{\hbar^2 k_i^2}{2m}\hat{n}_i ,$$

where $\hat{n}_i$ IS THE OPERATOR THAT COUNTS THE OCCUPATION OF STATE i

$$\Rightarrow \hat{n}_i |n_1 n_2 \cdots\rangle = n_i |n_1 n_2 \cdots\rangle$$

AND WE HAVE ADDED THE HAT IN ORDER TO AVOID CONFUSION BETWEEN OPERATOR AND EIGENVALUE

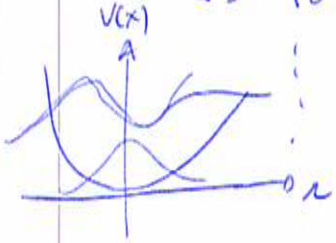
$\Rightarrow \hat{T}$ IS AN OPERATOR THAT ~~COUNTS~~ SUMS THE ~~…~~ KINETIC ENERGY OF ALL STATES WEIGHTED WITH ITS OCCUPATION.

2 - TRANSITIONS

IN THE PREVIOUS EXAMPLE, WE HAVE USED AN OPERATOR THAT IS DIAGONAL IN THE FOCK SPACE.

LET'S NOT DO THIS IN THE NEXT EXAMPLE

EX: N-NONINTERACTING IDENTICAL PARTICLES IN AN HARMONIC 1D POTENTIAL



FOR SIMPLICITY, LETS CONSIDER ONLY 2 BOSONIC PARTICLES IN THE GROUND STATE

THUS, $$\psi(x_1,x_2) = \left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}} e^{-y_1^2/2} * \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-y_2^2/2} = \phi_0(x_1)\phi_0(x_2)$$

where $\alpha = \frac{m\omega}{\hbar}$ AND $y_i = \sqrt{\alpha}\, x_i$

THE TOTAL KINETIC ENERGY IS

$$T = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) = -\frac{\hbar^2\alpha}{2m}\left(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2}\right)$$

$$\Rightarrow T\psi = -\frac{\hbar^2\alpha}{2m} * \left(\frac{\alpha}{\pi}\right)^{\frac{1}{2}}\left[\left(-e^{-y_1^2/2} + y_1^2 e^{-y_1^2/2}\right)e^{-y_2^2/2} + 1 \leftrightarrow 2\right]$$

SINCE THE 2ND EXCITED STATE OF A SINGLE PARTICLE IS

$$\phi_2(x) = \left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2}}(2y^2-1)e^{-y^2/2}$$

THEN $$T\psi = -\frac{\hbar^2\alpha}{2m}\left[\left(\frac{1}{\sqrt{2}}\phi_2(x_1) - \frac{1}{2}\phi_0(x_1)\right)\phi_0(x_2) + 1 \leftrightarrow 2\right]$$

$$= -\frac{\hbar^2\alpha}{2m}\left\{\frac{1}{\sqrt{2}}\left(\phi_2(x_1)\phi_0(x_2) + \phi_2(x_2)\phi_0(x_1)\right) - \phi_0(x_1)\phi_0(x_2)\right\}$$

$$= -\frac{\hbar^2\alpha}{2m}\left\{\underbrace{|1,0,1,0,0\ldots\rangle}_{\text{NON-DIAGONAL (TRANSITIONAL) PART}} - \underbrace{|2,0,0,0\ldots\rangle}_{\text{DIAGONAL PART}}\right\}$$

THUS, THIS OPERATOR CAN BE SEEN AS AN OPERATOR THAT COUNTS SOMETHING (THE DIAGONAL PART) AND INDUCES TRANSITIONS (THE NON-DIAGONAL PART). SINCE THIS IS A 1-BODY OPERATOR, THE TRANSITION INDUCED IS A SIMPLE PARTICLE CHANGING OF STATE, i.e.

$$\hat{T} \propto \sum_i T_i \hat{n}_i + \lambda |10000\cdots\rangle\langle 2000\cdots|$$

WHICH WE EXPECT TO GENERALIZE TO

$$\sum_{i \neq k} \lambda_{ik} |n_1 \cdots n_i - 1 \cdots n_k + 1 \cdots\rangle\langle n_1 \cdots n_i \cdots n_k|$$

THIS LAST TERM MEANS THAT 1 PARTICLE TRANSITIONED FROM STATE $|\phi_i\rangle$ TO STATE $|\phi_k\rangle$ WITH AMPLITUDE PROBABILITY $\lambda_{ik} = \langle \phi_k | \hat{T}_1 | \phi_i \rangle$

SINCE ALL PARTICLES ARE INDISTINGUISHABLE WE JUST TAKE $\hat{T}_1$ INSTEAD OF $\sum_i \hat{T}_i$

NOW, THE OPERATOR

$$|n_1 \cdots n_i - 1 \cdots n_k + 1 \cdots\rangle\langle n_1 \cdots n_i \cdots n_k \cdots|$$

IS CONVENIENTLY REPRESENTED VIA CREATION AND ANNIHILATION OPERATORS a_i^+ AND a_i, i.e.

$$|n_1 \cdots n_i - 1 \cdots n_k + 1 \cdots\rangle\langle n_1 \cdots n_i \cdots n_k \cdots| \propto a_k^+ a_i$$

CREATES A PARTICLE IN STATE $|\phi_k\rangle$

ANNIHILATES A PARTICLE IN STATE $|\phi_i\rangle$

FOR THIS REASON, WE NOW CONSTRUCT THESE OPERATORS:

$$a^\dagger : \mathcal{H}_{N} \longrightarrow \mathcal{H}_{N+1} \quad \text{(CREATION)}$$

$$a : \mathcal{H}_{N} \longrightarrow \mathcal{H}_{N-1} \quad \text{(ANNIHILATION)}$$

IT IS INTERESTING TO NOTICE THAT THESE OPERATORS CONNECT HILBERT SPACES OF DIFFERENT NUMBERS OF PARTICLES. IT IS <u>NOT</u> SURPRISING THAT THEY APPEARED SINCE THE

$$\text{FOCK STATE} = \mathcal{H}_{0} \oplus \mathcal{H}_{1} \oplus \mathcal{H}_{2} \oplus \cdots$$

<u>HOW DO WE CONSTRUCT THESE OPERATORS?</u>

WE WILL NOT DO THIS HERE SINCE IT IS THE HOMEWORK (LIST OF EXERCISES).

INSTEAD, WE WILL SIMPLY GIVE THEIR DEFINITION AND THEIR PHYSICAL CONTENTS.

SINCE THEY CREATE AND ANNIHILATE STATES WITH DIFFERENT SYMMETRY PROPERTIES (BOSONS OR FERMIONS) i.e.,

$$a_i^\dagger |m_1 \cdots m_i \cdots\rangle \propto |m_1 \cdots m_i + 1 \cdots\rangle$$

$$a_i |m_1 \cdots m_i \cdots\rangle \propto |m_1 \cdots m_i - 1 \cdots\rangle$$

WE EXPECT THAT THESE OPERATORS OBEY DIFFERENT ALGEBRA DEPENDING ON WHETHER THEY ARE BOSONIC OR FERMIONIC CREATION AND ANNIHILATION OPERATORS.

IT HAPPENS THAT

- FOR BOSONS, THEY OBEY THE BOSE-EINSTEIN ~~STATISTICS~~ ALGEBRA

$$[a_i^+, a_j^+] = [a_i, a_j] = 0$$

$$[a_i, a_j^+] = \delta_{ij}$$

- FOR FERMIONS, THEY OBEY THE FERMI-DIRAC ALGEBRA

$$\{a_i^+, a_j^+\} = \{a_i, a_j\} = 0$$

$$\{a_i, a_j^+\} = \delta_{k\ell}$$

- IN BOTH CASES,

$$[n_k, a_\ell^+] = \delta_{k\ell}\, a_\ell^+$$

$$[n_k, a_\ell] = -\delta_{k\ell}\, a_\ell$$

where $n_k = a_k^+ a_k$ IS THE <u>NUMBER OPERATOR</u> IN THE 1-PARTICLE STATE $|k\rangle$

(NOTICE WE HAVE DROPED THE HAT)

HERE $[A, B] = AB - BA$ IS THE COMMUTATOR

$\{A, B\} = AB + BA$ IS THE ANTI-COMMUTATOR

USUALLY ONE SIMPLIFIES THE NOTATION TO

$$[A, B]_\zeta = AB - \zeta BA \quad \text{WITH} \quad \begin{cases} \zeta = +1 & \text{FOR BOSONS} \\ \zeta = -1 & \text{FOR FERMION} \end{cases}$$

THEN $[A, B]_\zeta$ REPRESENTS THE COMMUTATOR FOR BOSONS AND THE ANTI-COMMUTATOR FOR FERMIONS.

<u>HOW THESE OPERATORS ACT ON THE FOCK STATES?</u>

- BOSONS

$$\boxed{\begin{aligned} a_\ell^+ |\cdots, n_\ell, \cdots\rangle &= \sqrt{n_\ell + 1}\, |\cdots, n_\ell + 1, \cdots\rangle \\ a_\ell |\cdots, n_\ell, \cdots\rangle &= \sqrt{n_\ell}\, |\cdots, n_\ell - 1, \cdots\rangle \end{aligned}}$$

AND $n_\ell \geq 0$

> * NOTICE THIS STRUCTURE IS ISOMORPHIC TO THOSE OF THE HARMONIC OSCILLATOR

- FERMIONS $n_\ell = 0$ OR 1

$$a_\ell^+ |\cdots, n_\ell, \cdots\rangle = \begin{cases} 0\,, & \text{IF } n_\ell = 1 \quad \text{(EXCLUSION PRINCIPLE)} \\ (\text{PHASE})\, |\cdots, 1_\ell, \cdots\rangle\,, & \text{IF } n_\ell = 0 \end{cases}$$

$$= \text{PHASE} * (1 - n_\ell)\, |\cdots, n_\ell + 1, \cdots\rangle$$

HOW DO WE DETERMINE THE PHASE?

IT COMES FROM THE DEFINITION OF THE GENERIC FOCK STATE:

$$|n_1\, n_2 \cdots n_\ell \cdots\rangle \equiv \frac{(a_1^+)^{n_1}}{\sqrt{n_1!}} \frac{(a_2^+)^{n_2}}{\sqrt{n_2!}} \cdots \frac{(a_\ell^+)^{n_\ell}}{\sqrt{n_\ell!}} \cdots \underbrace{|0\rangle}_{\text{VACUUM } \in \mathcal{H}_0}$$

WORKS FOR BOSONS AND FERMIONS

SINCE FOR BOSONS $a_\ell^+ a_k^+ = a_k^+ a_\ell^+$, THEN THE ORDER OF INDICES DOES NOT MATTER

THUS, $|1\,2\,0\cdots\rangle = \frac{a_1^+ (a_2^+)^2}{\sqrt{2!}}|0\rangle = \frac{(a_2^+)^2 a_1^+}{\sqrt{2!}}|0\rangle = \frac{a_2^+ a_1^+ a_2^+}{\sqrt{2!}}|0\rangle$

HOWEVER, FOR FERMIONS, THE ORDER OF INDICES DO MATTER !!!

$$a_1^+ a_2^+ |0\rangle = |1,1,0,0\ldots\rangle \quad \text{(BY DEFINITION)}$$

HOWEVER $a_2^+ a_1^+ |0\rangle = -a_1^+ a_2^+ |0\rangle = -|1,1,0,0\ldots\rangle$

$$\Downarrow$$

$$\{a_1^+, a_2^+\} = 0$$

THEREFORE, THE PHASE IS DETERMINED IN THE FOLLOWING MANNER

$$\boxed{a_\ell^+ |n_1 n_2 \ldots, n_{\ell-1}, 0_\ell, n_{\ell+1}, \ldots\rangle =}$$

$$= a_\ell^+ (a_1^+)^{n_1} (a_2^+)^{n_2} \cdots (a_{\ell-1}^+)^{n_{\ell-1}} (a_{\ell+1}^+)^{n_{\ell+1}} \cdots |0\rangle$$

MOVE TO HERE $\Rightarrow$ ANTI-COMMUTES $(n_1 + n_2 + \cdots + n_{\ell-1})$ TIME

$$= (a_1^+)^{n_1} (a_2^+)^{n_2} \cdots (a_{\ell-1}^+)^{n_{\ell-1}} a_\ell (a_{\ell+1}^+)^{n_{\ell+1}} \cdots |0\rangle * (-1)^{\sum_{i=1}^{\ell-1} n_i}$$

$$\boxed{= (-1)^{\sum_{i=1}^{\ell-1} n_i} |n_1, n_2, \ldots n_{\ell-1}, 1_\ell, n_{\ell+1}, \ldots\rangle}$$

EXCLUSION PRINCIPLE

$$a_\ell^+ |\ldots, 1_\ell, \ldots\rangle = a_\ell^+ \left[\prod_{k=1}^{\ell-1} (a_k^+)^{n_k}\right] a_\ell^+ \left[\prod_{k=\ell+1}^{\infty} (a_k^+)^{n_k}\right] |0\rangle$$

$$= (-1)^{\sum_{i=1}^{\ell-1} n_i} \left[\prod_{k=1}^{\ell-1} (a_k^+)^{n_k}\right] a_\ell^+ a_\ell^+ \left[\prod_{k=\ell+1}^{\infty} (a_k^+)^{n_k}\right] |0\rangle$$

NOW, REMEMBER THAT $\{a_i^+, a_j^+\} = 0$

FOR $i = j = \ell \Rightarrow \{a_\ell^+, a_\ell^+\} = 2 a_\ell^+ a_\ell^+ = 2(a_\ell^+)^2 = 0$

CANNOT CREATE 2 FERMIONS IN STATE $|\phi_\ell\rangle$

LIKEWISE, FOR THE ANNIHILATION OPERATOR

$$a_\ell |m_1 \cdots m_\ell \cdots\rangle \equiv a_\ell \left[\prod_{k=1}^{\ell-1} (a_k^+)^{m_k}\right] (a_\ell^+)^{m_\ell} \left[\prod_{k=\ell+1}^{\infty} (a_k^+)^{m_k}\right] |0\rangle$$

$$= (-1)^{\sum_{j=1}^{\ell-1} m_j} \left[\prod_{k=1}^{\ell-1} (a_k^+)^{m_k}\right] a_\ell (a_\ell^+)^{m_\ell} \left[\prod_{k=\ell+1}^{\infty} (a_k^+)^{m_k}\right] |0\rangle$$

where WE HAVE USED THAT

$$\{a_k^+, a_\ell\} = 0 \quad \text{FOR } k \neq \ell$$

NOW, LET'S WORK OUT $a_\ell (a_\ell^+)^{m_\ell}$

FOR $m_\ell = 0 \Rightarrow a_\ell (a_\ell^+)^{m_\ell} = a_\ell$

AND THE ANNIHILATION OPERATOR MUST DESTROY THE VACUUM (OTHERWISE WE WOULD HAVE NEGATIVE OCCUPATION), i.e.

$$a_\ell |\cdots 0_\ell \cdots\rangle = 0$$

$\{a_\ell, a_\ell^+\} = 1$

FOR $m_\ell = 1 \Rightarrow a_\ell (a_\ell^+)^{m_\ell} = a_\ell a_\ell^+ = 1 - a_\ell^+ a_\ell$

AND $(1 - a_\ell^+ a_\ell)|\cdots 0_\ell \cdots\rangle = |\cdots 0_\ell \cdots\rangle - 0$

$$= |\cdots 0_\ell \cdots\rangle$$

FINALLY,

$$\boxed{\begin{aligned} a_\ell |m_1 \cdots m_\ell \cdots\rangle &= (-1)^{\sum_{k=1}^{\ell-1} m_k} m_\ell |m_1 \cdots 0_\ell \cdots\rangle \\ &= \begin{cases} (-1)^{\sum_{k=1}^{\ell-1} m_k} |m_1 \cdots 0_\ell \cdots\rangle, & \text{IF } m_\ell = 1 \\ 0, & \text{IF } m_\ell = 0 \end{cases} \end{aligned}}$$

1-BODY OPERATORS

NOW THAT WE HAVE DEFINED THE OPERATORS a_ℓ^+ AND a_ℓ, WE ARE IN POSITION OF REWRITING THE OPERATORS IN TERMS OF THEM. LET'S START WITH THE 1-BODY OPERATORS

$$\hat{O}_1 = \sum_{\alpha=1}^{N} \hat{\theta}_\alpha$$, where $\hat{\theta}_\alpha$ ACTS ON THE α-th PARTICLE

NOW, WE WANT TO WRITE IN TERMS OF a_ℓ AND a_ℓ^+ WHICH ARE RELATED TO THE SET OF 1-PARTICLE STATES $\{|\phi_\ell\rangle\}$

FIRSTLY, WE DIAGONALIZE THE 1-PARTICLE PROBLEM, i.e., WE DIAGONALIZE $\hat{\theta}_\alpha$, FOR ANY α, ~~FOR~~ THIS LABEL DOES NOT MATTER. HAVING DIAGONALIZED $\hat{\theta}$, WE NOW HAVE THE SET OF 1-PARTICLE STATES $\{|\varphi_k\rangle\}$ WHICH ARE ORTHONORMAL. RELATED TO THESE STATES, THERE ARE THE CREATION AND ANNIHILATION OPERATOR b_k^+ AND b_k SUCH THAT $b_k^+|0\rangle = |\cdots 1_k \cdots\rangle = |\varphi_k\rangle$

IT IS EASY TO WRITE $\hat{O}_1$ IN TERMS OF THESE OPERATORS (SINCE IT IS DIAGONAL)

$$\hat{\theta} = \sum_k \theta_k |\varphi_k\rangle\langle\varphi_k| = \sum_k \theta_k b_k^+ b_k$$, where θ_k IS THE CORRESPONDING EIGENVALUE.

FOR $\hat{O}_1 = \sum_\alpha \hat{\theta}_\alpha$, THERE IS NO DIFFERENCE SINCE WE WILL ONLY NEED TO COUNT FOR MORE PARTICLES,

$$\Rightarrow \hat{O}_1 = \sum_{k=1}^{\infty} \theta_k b_k^+ b_k = \sum_{k=1}^{\infty} \theta_k \hat{n}_k$$

FOR INSTANCE, WE COULD HAVE CHOSEN

$$\hat{O}_1 = \hat{T} = \sum_{\alpha=1}^{N} \frac{1}{2m} P_\alpha^2 = \sum_{i=1}^{\infty} T_i \hat{m}_i = \sum_{i=1}^{\infty} T_i b_i^+ b_i$$

$T_i \to$ KINETIC ENERGY OF THE i-th STATE

HERE b_j^+ CREATES A PARTICLE ~~IN~~ THE PLANE-WAVE STATE $|e^{i\vec{k}_j\cdot\vec{r}}\rangle \Rightarrow T_j = \dfrac{\hbar^2 k_j^2}{2m}$

HOWEVER, WE WANT TO WRITE $\hat{O}_1$ IN TERMS OF a_ℓ AND a_ℓ^+.

FOR THAT, A SIMPLE CHANGE OF BASIS IS SUFFICIENT

$$\hat{O}_1 = \sum_{k=1}^{\infty} o_k b_k^+ b_k = \sum_k \langle\varphi_k|\hat{o}|\varphi_k\rangle b_k^+ b_k$$

$$\cancel{\mathbb{1} = \sum_m |\phi_m\rangle\langle\phi_m| \quad \mathbb{1} = \sum_n |\phi_n\rangle\langle\phi_n|}$$

$$\cancel{= \sum_{mnk} \langle\varphi_k|\phi_m\rangle\langle\phi_m|\hat{o}|\phi_n\rangle\langle\phi_n|\varphi_k\rangle b_k^+ b_k}$$

HOW DO WE RELATE b_k^+ WITH a_ℓ^+?

NOTICE THAT $b_k^+|0\rangle = |\varphi_k\rangle = \sum_m |\phi_m\rangle\langle\phi_m|\varphi_k\rangle$

BUT $|\phi_m\rangle = a_m^+|0\rangle$

$$\Rightarrow \boxed{b_k^+ = \sum_m \langle\phi_m|\varphi_k\rangle a_m^+ \Rightarrow b_k = \sum_m \langle\varphi_k|\phi_m\rangle a_m}$$

$$\Rightarrow \hat{O}_1 = \sum_k o_k b_k^+ b_k = \sum_{kmn} o_k \langle\phi_m|\varphi_k\rangle\langle\varphi_k|\phi_n\rangle a_m^+ a_n$$

$$= \sum_{mn} \langle\phi_m| \Big(\sum_k o_k |\varphi_k\rangle\langle\varphi_k| \Big) |\phi_n\rangle a_m^+ a_n$$

SINCE $o_k |\varphi_k\rangle = \hat{o} |\varphi_k\rangle$

$$\Rightarrow \hat{O}_1 = \sum_{m n} \langle \phi_m | \Big(\hat{o} \underbrace{\sum_k |\varphi_k\rangle\langle\varphi_k|}_{= \mathbb{1}} \Big) | \phi_n \rangle a_m^+ a_n$$

$$\Rightarrow \boxed{\hat{O}_1 = \sum_{m,n} \langle \phi_m | \hat{o} | \phi_n \rangle a_m^+ a_n}$$

EX: TOTAL NUMBER OF PARTICLES

$$N = \sum_i n_i = \sum_i a_i^+ a_i$$

$$= \sum_i \sum_{jk} \langle \varphi_j | \phi_i \rangle b_j^+ \langle \phi_i | \varphi_k \rangle b_k = \sum_{jk} \underbrace{\langle \varphi_j | \Big(\overbrace{\sum_i |\phi_i\rangle\langle\phi_i|}^{= \mathbb{1}} \Big) | \varphi_k \rangle}_{\delta_{jk}} b_j^+ b_k$$

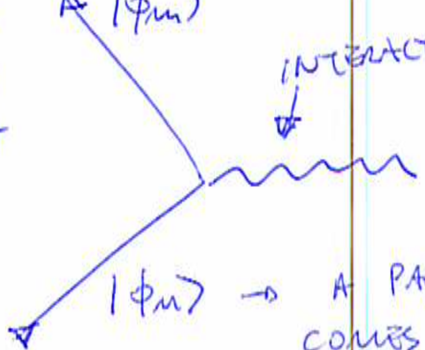
$$= \sum_j b_j^+ b_j$$

THUS, N <u>DOES NOT</u> DEPEND ON THE CHOICE OF BASIS (AS EXPECTED).

DIAGRAMATIC REPRESENTATION

EXTERNAL POTENTIAL (FOR INSTANCE): $\hat{U} = \sum_{\alpha=1}^{N} \hat{u}(\vec{x}_\alpha) = \sum_{m n} \langle \phi_m | \hat{u} | \phi_n \rangle a_m^+ a_n$

AND IS SCATTERED TO STATE $|\phi_m\rangle$ WITH AMPLITUDE PROBABILITY $\langle \phi_m | \hat{u} | \phi_n \rangle$



$|\phi_m\rangle$

INTERACTS WITH THE POTENTIAL

$|\phi_n\rangle \rightarrow$ A PARTICLES COMES IN STATE $|\phi_n\rangle$

2-BODY OPERATORS

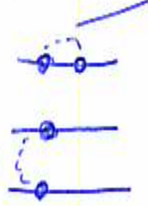
LET'S CONSIDER A 2-BODY INTERACTION ENERGY (BUT IT IS MORE GENERAL THAN THAT)

$$V_{TOT} = \frac{1}{2} \sum_{\substack{\alpha,\beta \\ (\alpha\neq\beta)}} V_{\alpha\beta}(\vec{r}_\alpha, \vec{r}_\beta) = \frac{1}{2} \sum_{i\neq j} n_i n_j V_{ij} + \frac{1}{2} \sum_i n_i (n_i - 1) V_{ii}$$

SUM OVER THE PARTICLES

SUM OVER THE STATES i AND j WITH $i \neq j$ — INTERACTION BETWEEN PARTICLES IN DIFFERENT 1-PARTICLE STATES

SUM OVER THE STATES i — INTERACTION BETWEEN PARTICLES IN THE SAME 1-PARTICLE STATE



$$\Rightarrow V_{TOT} = \frac{1}{2} \sum_{i,j} (n_i n_j - n_i \delta_{ij}) V_{ij}$$

BUT NOTICE THAT

$$n_i n_j - n_i \delta_{ij} = b_i^\dagger b_j^\dagger b_j b_i$$

PROOF: $i \neq j \Rightarrow n_i n_j - n_i \delta_{ij} = n_i n_j = b_i^\dagger b_i b_j^\dagger b_j = b_i^\dagger b_j^\dagger b_j b_i$

FOR EITHER BOSONS OR FERMIONS

$i = j \Rightarrow n_i n_j - n_i \delta_{ij}$

$= n_i (n_i - 1)$, FOR FERMIONS, SINCE $n_i = 0$ OR 1

$\Rightarrow n_i(n_i - 1) = 0$ BUT $b_i^\dagger b_i^\dagger b_i b_i$ IS ALSO $= 0$ FOR FERMIONS

FOR BOSONS $\Rightarrow b_i^\dagger b_i b_i^\dagger b_i - b_i^\dagger b_i = b_i^\dagger (1 + b_i^\dagger b_i) b_i - b_i^\dagger b_i$

$= b_i^\dagger b_i^\dagger b_i b_i$. END OF PROOF

THEREFORE,

$$V_{tot} = \frac{1}{2} \sum_{i,j} b_i^\dagger b_j^\dagger b_j b_i \, V_{ij}$$

AGAIN, WE HAVE USED A SPECIAL BASIS IN WHICH WE CAN SIMPLY COUNT THE INTERACTION ENERGY, i.e., FOR INSTANCE, 2 PARTICLES IN THE SAME STATES INTERACT WITH EACH OTHER AND ARE NOT SCATTERED TO OTHER STATES: $\hat{V}|\varphi_i \varphi_j\rangle = V_{ij}|\varphi_i \varphi_j\rangle$

FOR THE GENERIC CASE, WE APPLY AGAIN A BASIS TRANSFORMATION (UNITARY TRANSFORMATION)

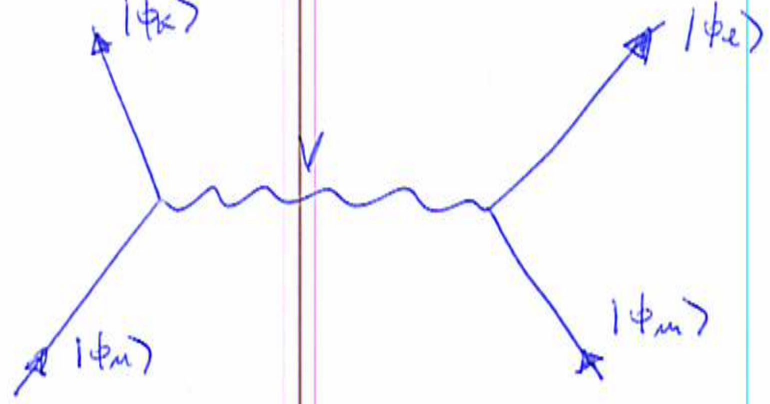
$$V_{tot} = \frac{1}{2} \sum_{i,j} \sum_{klmn} \langle \phi_k|\varphi_i\rangle \langle \phi_l|\varphi_j\rangle \langle \varphi_j|\phi_m\rangle \langle \varphi_i|\phi_n\rangle V_{ij} \, a_k^\dagger a_l^\dagger a_m a_n$$

$$= \frac{1}{2} \sum_{ij} \sum_{klmn} \langle \phi_k, \phi_l | \varphi_i \varphi_j\rangle \langle \varphi_i, \varphi_j | \phi_n, \phi_m\rangle V_{ij} \, a_k^\dagger a_l^\dagger a_m a_n$$

$$= \frac{1}{2} \sum_{klmn} \langle \phi_k, \phi_l | \underbrace{\left(\sum_{ij} V_{ij} |\varphi_i \varphi_j\rangle\langle \varphi_i \varphi_j| \right)}_{\hat{V}} |\phi_n, \phi_m\rangle \, a_k^\dagger a_l^\dagger a_m a_n$$

$$\boxed{V_{tot} = \frac{1}{2} \sum_{klmn} \langle \phi_k, \phi_l | \hat{V} | \phi_n \phi_m\rangle \, a_k^\dagger a_l^\dagger a_m a_n}$$

DIAGRAMATICALLY,



NON-RELATIVISTIC FIELD OPERATORS

(NOT A TRANSITION TO QUANTUM FIELD THEORY)

LET'S MAKE A CHANGE OF BASIS TO THE POSITION EIGEN STATES $\{|\vec{r}\rangle\}$

IF ONE WANTS TO ATTACH THE SPIN

THE FIELD OPERATOR IS $\psi^+_\sigma(\vec{r})\,|0\rangle = |\vec{r},\sigma\rangle$

THUS,
$$\begin{cases} \psi^+_\sigma(\vec{r}) = \sum_l \langle \phi_{l,\sigma}|\vec{r}\rangle\, a^+_{l,\sigma} = \sum_l \phi^*_{l,\sigma}(\vec{r})\, a^+_{l,\sigma} \\ \psi_\sigma(\vec{r}) = \sum_l \phi_{l,\sigma}(\vec{r})\, a_{l,\sigma} \end{cases}$$

WE THEN HAVE THAT $[\psi_\sigma(\vec{r}), \psi_{\sigma'}(\vec{r}')]_\zeta = [\psi^+, \psi^+] = 0$

$$[\psi_\sigma(\vec{r}), \psi^+_{\sigma'}(\vec{r}')]_\zeta = \delta_{\sigma\sigma'}\,\delta(\vec{r}-\vec{r}')$$

$$N = \sum_\sigma \int d\vec{r}\, \underbrace{\psi^+_\sigma(\vec{r})\,\psi_\sigma(\vec{r})}_{\text{NUMBER DENSITY OPERATOR}}$$

KINETIC + EXTERNAL POTENTIAL ENERGY:
$$T+U = \sum_{\sigma\sigma'} \int d\vec{r}\, d\vec{r}'\, \psi^+_\sigma(\vec{r})\,\langle \vec{r}\sigma|\, K+U\, |\vec{r}'\sigma'\rangle\, \psi_{\sigma'}(\vec{r}')$$
$$= \sum_\sigma \int d\vec{r}\, \psi^+_\sigma(\vec{r}) \left(\frac{-\hbar^2}{2m}\nabla^2 + U(\vec{r}) \right) \psi_\sigma(\vec{r})$$

2-BODY INTERACTION:
$$V = \frac{1}{2} \sum_{\substack{\sigma_1\sigma_2 \\ \sigma_3\sigma_4}} \int d\vec{r}_1 \cdots d\vec{r}_4\, \psi^+_{\sigma_1}(\vec{r}_1)\,\psi^+_{\sigma_2}(\vec{r}_2)\,\langle r_1\sigma_1\, r_2\sigma_2|\, V\, |r_4\sigma_4\, r_3\sigma_3\rangle\, \psi_{\sigma_3}(\vec{r}_3)\,\psi_{\sigma_4}$$
$$= \frac{1}{2} \sum_{\sigma\sigma'} \int d\vec{r}\, d\vec{r}'\, \psi^+_\sigma(\vec{r})\,\psi^+_{\sigma'}(\vec{r}')\, V(\vec{r}-\vec{r}')\,\psi_{\sigma'}(\vec{r}')\,\psi_\sigma(\vec{r})$$

IN MOMENTUM SPACE:

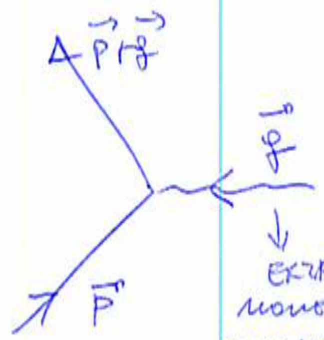
$$\begin{cases} \pi^{+}_{\sigma}(\vec{p}) = \int d\vec{r}\; \psi^{+}_{\sigma}(\vec{r})\, e^{i\frac{\vec{p}\cdot\vec{r}}{\hbar}} \\ \psi^{+}(\vec{r}) = \int \frac{d\vec{p}}{(2\pi\hbar)^{d}}\; \pi^{+}(\vec{p})\, e^{-i\frac{\vec{p}\cdot\vec{r}}{\hbar}} \end{cases}$$

THEN

$$T+U = \sum_{\sigma}\int \frac{d\vec{p}}{(2\pi\hbar)^{d}}\, \frac{p^{2}}{2m}\, \pi^{+}_{\sigma}(\vec{p})\,\pi_{\sigma}(\vec{p}) \; +$$

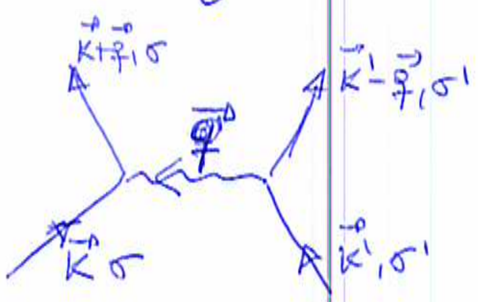
$$\sum_{\sigma}\int \frac{d\vec{p}\,d\vec{q}}{(2\pi\hbar)^{2d}}\; \tilde{U}(\vec{q})\, \pi^{+}_{\sigma}(\vec{p}+\vec{q})\,\pi_{\sigma}(\vec{p})$$

where $\tilde{U}(\vec{q}) = \int d\vec{r}\, U(\vec{r})\, e^{-i\vec{q}\cdot\frac{\vec{r}}{\hbar}}$



EXTRA MOMENTUM COMING FROM THE EXTERNAL POTENTIAL

$$V = \frac{1}{2}\int \frac{d\vec{k}\,d\vec{k}'\,d\vec{q}}{(2\pi\hbar)^{3d}} \sum_{\sigma\sigma'} \tilde{V}(q)\, \pi^{+}_{\sigma}(\vec{k}+\vec{q})\,\pi^{+}_{\sigma'}(\vec{k}'-\vec{q})\,\pi_{\sigma'}(\vec{k}')\,\pi_{\sigma}(\vec{k})$$



$\vec{q}$ = MOMENTUM EXCHANGED BY THE PARTICLES