List of exercises #6 - 7600037

1. **The propagator and the corresponding Green function.**

   (a) Show that the propagator defined as

   $$i\hbar G\left(x,t;x_0,t_0\right) \equiv \left\langle x\left|U(t,t_0)\right|x_0\right\rangle,$$

   with $U(t,t_0)$ being the time-evolution operator, obeys the Schrödinger equation. What is the boundary condition for G?

   (b) Show that the corresponding Green function, defined as $\mathcal{G}(x,t;x_0,t_0) = \Theta\left(t-t_0\right)G(x,t;x_0,t_0)$ obeys the equation

   $$\left(i\hbar\frac{\partial}{\partial t} - H\right)\mathcal{G}\left(x,t;x_0,t_0\right) = \delta\left(x-x_0\right)\delta\left(t-t_0\right).$$

   (In many books, the propagator is often mistaken with the Green function. Indeed, they are the same for $t > t_0$.)

   (c) Compute the free-particle Green function by solving the corresponding differential equation.

2. **Equivalence between the Schrödinger Equation and the Path Integral formalisms.**

   (a) Starting from the path integral formalism, show that $\psi(t,x)$ obeys the Schrödinger equation. [Notice the relation between theses formalisms is similar to that between the Newtonian and the Lagrangean ones of classical mechanics. The former approach is local in time (i.e., deals with the time evolution over infinitesimal periods) while the latter is global (i.e., deals with the evolution over finite time periods)].

   (b) Given the Eigenfunctions and Eigenvalues of a time-independent Hamiltonian, show that the propagator is given by

   $$i\hbar G\left(x,t;x_0,t_0\right) = \sum_n \psi_n\left(x\right)\psi_n^*\left(x'\right)e^{\left(\frac{t-t_0}{i\hbar}\right)E_n}. \tag{1}$$

   We now want to consider the reverse process in which we do not have to deal with the Schrödinger Equation formalism.

   i. Compute the propagator for the Harmonic Oscillator.

   ii. Set $x = x_0 = 0$. Expanding both sides of Eq. (1) in powers of $\epsilon = e^{i\omega(t-t_0)}$, show that $E_0 = \frac{1}{2}\hbar\omega$, $E_2 = \frac{5}{2}\hbar\omega$, $E_4 = \frac{9}{2}\hbar\omega$, ... What happened to the odd-n Eigenvalues? Find a physical interpretation for the choice $x = x_0 = 0$.

   iii. Now let us do a harder job. Set $x = x_0 \neq 0$. Repeating the same steps in item 2(b)ii, find E_n and $\left|\psi_n\left(x\right)\right|^2$ for $n = 0$ and 1. Find a physical interpretation for the choice $x = x_0 \neq 0$.

3. **Path integral for an Harmonic Oscillator.** Consider the 1D quantum Harmonic Oscillator given by the Hamiltonian

   $$H = \frac{1}{2m}P^2 + \frac{1}{2}m\omega^2 Q^2.$$

   (a) Derive the corresponding path integral and show that, in the continuum limit,

   $$Z = \mathrm{tr}e^{-\beta H} = \int \mathcal{D}[q(\tau)]e^{-S_E[q]}, \text{ with } S_E = \int_0^\beta \mathrm{d}\tau\left(\frac{m}{2}\dot{q}^2 + \frac{1}{2}m\omega^2 q^2\right).$$

   What is the exact expression for $\mathcal{D}[q(\tau)]$?

   (b) Now, let us step back and work on the driscret case. Defining the Fourier transform

   $$\tilde{q}_k = \frac{1}{\sqrt{M}}\sum_j e^{-2\pi ijk/M}q_j, \text{ and } q_j = \frac{1}{\sqrt{M}}\sum_k e^{2\pi ijk/M}q_j,$$

   show that

   $$S_E = \epsilon\sum_k \tilde{q}_k\left[\frac{1}{2}m\omega^2 + \frac{m}{\epsilon^2}\left(1 - e^{-2\pi ik/M}\right)\right]\tilde{q}_{-k}, \text{ with } \epsilon = \beta/M.$$

(c) Finally, show that

$$Z = \frac{1}{2\sinh\left(\frac{1}{2}\beta\omega\right)}.$$

(d) Performing an analytical continuation of the Euclidean time to the Minkowski time $\beta \to it$, show that

$$Z_M(t) = \frac{1}{2i\sin(\frac{1}{2}\omega t)} = \sum_n e^{-itE_n}, \text{ with } E_n = \left(n + \frac{1}{2}\right)\omega,$$

which is the Harmonic Oscillator spectrum. Why is that so?

4. Let the partition function be

$$Z = i\hbar \int \mathrm{d}\mathbf{r}\; G\left(\mathbf{r}, t; \mathbf{r}, 0\right)\big|_{t=-i\hbar\beta}.$$

Show that the ground-state energy is

$$\lim_{\beta\to\infty}\left(-\frac{\partial}{\partial\beta}\ln Z\right),$$

and illustrate this result explicitly for a particle in a one-dimentional box.

5. **Gaussian integral of complex variables.** Let $\mathbb{H}$ be an Hermitean positive definite $n \times n$ matrix, and $\mathbf{J} \in \mathbb{C}^n$ be a complex vector.

(a) Show that

$$\int \prod_{i=1}^{n} \frac{\mathrm{d}\left(\Re z_i\right)\mathrm{d}\left(\Im z_i\right)}{\pi} e^{-(\mathbf{z}^*)^T\cdot\mathbb{H}\cdot\mathbf{z}+(\mathbf{z}^*)^T\cdot\mathbf{J}+(\mathbf{J}^*)^T\cdot\mathbf{z}} = \left(\mathrm{Det}\,\mathbb{H}\right)^{-1} e^{(\mathbf{J}^*)^T\cdot\mathbb{H}^{-1}\cdot\mathbf{J}}.$$

(b) Use this result to show that

$$\left\langle z^*_{\alpha_1}\dots z^*_{\alpha_m} z_{\beta_1}\dots z_{\beta_m}\right\rangle = \sum_{\text{all pairings}} \mathbb{H}^{-1}_{\beta_1,\alpha_{P_1}}\dots\mathbb{H}^{-1}_{\beta_m,\alpha_{P_m}} = \sum_{\text{all pairings}} \left\langle z^*_{\beta_1} z_{\alpha_{P_1}}\right\rangle \dots \left\langle z^*_{\beta_m} z_{\alpha_{P_m}}\right\rangle$$

where $\{P_1, \dots, P_m\}$ is a permutation of $\{1, \dots, m\}$, and the average is defined as

$$\langle A\rangle = \frac{\int\prod_{i=1}^n \frac{\mathrm{d}(\Re z_i)\mathrm{d}(\Im z_i)}{\pi} e^{-(\mathbf{z}^*)^T\cdot\mathbb{H}\cdot\mathbf{z}} A}{\int\prod_{i=1}^n \frac{\mathrm{d}(\Re z_i)\mathrm{d}(\Im z_i)}{\pi} e^{-(\mathbf{z}^*)^T\cdot\mathbb{H}\cdot\mathbf{z}}}.$$

(This is the famous Wick theorem for bosonic variables.)