PICTURES OF QUANTUM MECHANICS

- **SCHRÖDINGER PICTURE**

- BASE VECTORS OF OPERATORS ARE FIXED IN TIME (EXCEPT WHEN THE OPERATOR HAS EXPLICIT TIME DEPENDENCE)
- STATE VECTORS (DESCRIBING PHYSICAL SYSTEMS) HAVE EXPLICIT TIME DEPENDENCE $\Rightarrow$ CHANGE IN TIME

SCHRÖDINGER EQUATION: $i\hbar \frac{d}{dt} |\psi_s(t)\rangle = H_s(t) |\psi_s(t)\rangle$

SOLUTION: $|\psi_s(t)\rangle = U_s(t,t_0) |\psi_s(t_0)\rangle$

U_s ↓ TIME EVOLUTION OPERATOR

$U_s(t_0,t_0) = \mathbb{1}$

$U_s^\dagger(t,t_0) = U_s^{-1}(t,t_0) = U_s(t_0,t)$

↑ UNITARY

$$|\psi_s(t+dt)\rangle = |\psi_s(t)\rangle + \frac{dt}{i\hbar} H_s(t) |\psi_s(t)\rangle$$

$$= \underbrace{\left(\mathbb{1} + \frac{dt}{i\hbar} H_s(t)\right)}_{U_s(t+dt,t)} |\psi_s(t)\rangle$$

BUT $\frac{dU_s(t,t_0)}{dt} \overset{dt \to 0}{=} \frac{U_s(t+dt,t_0) - U_s(t,t_0)}{dt}$

$$= \left(\frac{U_s(t+dt,t_0)U_s^{-1}(t,t_0) - \mathbb{1}}{dt}\right) U_s(t,t_0)$$

$$= \left(\frac{U_s(t+dt,t) - \mathbb{1}}{dt}\right) U_s(t,t_0)$$

$$\Rightarrow \frac{d}{dt} U_s(t,t_0) = \left(\frac{\mathbb{1} + \frac{dt}{i\hbar} H_s(t) - \mathbb{1}}{dt}\right) U_s(t,t_0)$$

$$\boxed{i\hbar \frac{d}{dt} U_s(t,t_0) = H_s(t) U_s(t,t_0)}$$

ALTERNATIVELY,

$i\hbar \frac{d}{dt}\left(U_s |\psi_s(t_0)\rangle\right) = H_s(t) U_s |\psi_s(t_0)\rangle$

SINCE $\frac{d}{dt} |\psi_s(t_0)\rangle = 0$

$\Rightarrow \boxed{i\hbar \frac{d}{dt} U_s = H_s U_s}$

②

NOTICE THAT $U(t,t_0) = U(t_0+ndt,\, t_0+(n-1)dt) \cdots U(t_0+2dt,\, t_0+dt)\, U(t_0+dt,\, t_0)$

$$= \left[\mathbb{1} + \frac{dt}{i\hbar} H_s\big(t_0+(n-1)dt\big) \right] \cdots \left[\mathbb{1} + \frac{dt}{i\hbar} H_s(t_0) \right]$$

$$\boxed{\text{FOR } \frac{\partial H_s}{\partial t} = 0 \Rightarrow U(t,t_0) = \lim_{n\to\infty} \left(\mathbb{1} + \left(\frac{t-t_0}{i\hbar}\right) H_s \right)^n = e^{-\frac{i}{\hbar}(t-t_0) H_s}}$$

FOR $H_s(t_1) \neq H_s(t_2) \Rightarrow$ IN GENERAL $[H_s(t_1), H_s(t_2)] \neq 0$

$$\Rightarrow e^{\frac{dt}{i\hbar} H_s(t_1)}\, e^{\frac{dt}{i\hbar} H_s(t_2)} \neq e^{\frac{dt}{i\hbar}\left(H_s(t_1) + H_s(t_2)\right)}$$

– AVERAGES : $\langle O_s \rangle = \langle \psi_s(t) | O_s | \psi_s(t) \rangle$

$$\Rightarrow i\hbar \frac{d}{dt} \langle O_s \rangle = \underbrace{\left(i\hbar \frac{\partial}{\partial t} \langle \psi_s | \right)}_{-\langle \psi_s | H_s} O_s | \psi_s \rangle + \langle \psi_s | O_s \underbrace{\left(i\hbar \frac{\partial}{\partial t} | \psi_s \rangle \right)}_{H_s | \psi_s \rangle} + \langle \psi_s | i\hbar \frac{\partial O_s}{\partial t} | \psi_s \rangle$$

$$\Rightarrow \boxed{i\hbar \frac{d}{dt} \langle O_s \rangle = \langle [O_s, H_s] \rangle + i\hbar \left\langle \frac{\partial}{\partial t} O_s \right\rangle}$$

EXERCISE : SHOW THAT THE DENSITY OPERATOR $\rho_s = |\psi_s(t)\rangle\langle \psi_s(t)|$ HAS, ON AVERAGE, NO EXPLICIT TIME DEPENDENCE

ANSWER:
- $\langle \rho_s \rangle = \langle \psi(t) | \rho_s | \psi(t) \rangle = 1 \Rightarrow \frac{d}{dt} \langle \rho_s \rangle = 0$
- $\langle [\rho_s, H_s] \rangle = \langle \psi_s | \psi_s \rangle \langle \psi_s | H_s | \psi_s \rangle - \langle \psi_s | H_s | \psi_s \rangle \langle \psi_s | \psi_s \rangle = 0$

NOW USING THAT $i\hbar \frac{d}{dt} \langle \rho_s \rangle = \langle [\rho_s, H_s] \rangle + i\hbar \langle \frac{\partial}{\partial t} \rho_s \rangle$

$$\Rightarrow \boxed{\left\langle \frac{\partial}{\partial t} \rho_s \right\rangle = 0}$$

- HEISENBERG PICTURE

  - BASE VECTORS OF OPERATORS ARE TIME DEPENDENT
    $\Rightarrow$ OPERATORS ARE TIME DEPENDENT
  - STATE VECTORS OF PHYSICAL SYSTEMS ARE FIXED IN TIME

DICTIONARY BETWEEN THE REPRESENTATIONS

$$\begin{cases} |\psi_S(t)\rangle = U_S(t,t_0)\,|\psi_S(t_0)\rangle = U_S(t,t_0)\,|\psi_H\rangle \\ \mathcal{O}_H(t) = U_S^\dagger(t,t_0)\,\mathcal{O}_S\,U_S(t,t_0) \end{cases}$$

$\rightarrow$ TIME-DEPENDENT CANONICAL TRANSFORMATION

IMPORTANT (a) OPERATORS HAVE THE SAME SPECTRUM IN BOTH REPRESENTATIONS
(b) INNER PRODUCT BETWEEN 2 STATES ARE THE SAME
(c) AVERAGES OF OBSERVABLES ARE THE SAME

$$\langle \mathcal{O}_S\rangle = \langle\psi_S(t)|\,\mathcal{O}_S\,|\psi_S(t)\rangle = \langle\psi_S(t_0)|\,U^\dagger(t,t_0)\,\mathcal{O}_S\,U(t,t_0)\,|\psi_S(t_0)\rangle$$

$$= \langle\psi_H|\,\mathcal{O}_H(t)\,|\psi_H\rangle = \langle\mathcal{O}_H\rangle$$

"EQUATION OF MOTION" FOR THE OPERATORS

$$i\hbar\frac{d}{dt}\mathcal{O}_H = \underbrace{\left(i\hbar\frac{\partial}{\partial t}U_S^\dagger\right)}_{-U_S^\dagger H_S}\mathcal{O}_S U_S + U_S^\dagger\mathcal{O}_S\underbrace{\left(i\hbar\frac{\partial}{\partial t}U_S\right)}_{H_S U_S} + \underbrace{i\hbar\, U^\dagger\left(\frac{\partial}{\partial t}\mathcal{O}_S\right)U}_{i\hbar\frac{\partial}{\partial t}\mathcal{O}_H \text{ (DEFINITION)}}$$

$$i\hbar\frac{d}{dt}\mathcal{O}_H = U_S^\dagger\mathcal{O}_S U_S U_S^\dagger H_S U_S - U_S^\dagger H_S U_S U_S^\dagger \mathcal{O}_S U_S + i\hbar\frac{\partial\mathcal{O}_H}{\partial t}$$

$$\boxed{i\hbar\frac{d}{dt}\mathcal{O}_H = [\mathcal{O}_H, H_H] + i\hbar\frac{\partial}{\partial t}\mathcal{O}_H}$$

RESEMBLES THE EQUATION FOR $\langle\mathcal{O}_S\rangle$

FOR THE SPECIAL CASE $\frac{\partial}{\partial t} H_S = 0 \Rightarrow U_S = e^{-\frac{i}{\hbar}(t-t_0)H_S} \Rightarrow [U_S, H_S] = 0$ (4)

$\Rightarrow \underline{H_S = H_H}$

CURIOSITY: LET $|v_S\rangle$ BE AN EIGENVECTOR OF $\mathcal{O}_S$

$$\Rightarrow |v_S\rangle = U_S(t,t_0)\,|v_H\rangle$$

$$\Rightarrow \underbrace{i\hbar \frac{d}{dt}|v_S\rangle}_{\Rightarrow \text{ (BY DEFINITION)}} = \underbrace{\left(i\hbar \frac{\partial}{\partial t} U_S\right)}_{H_S U_S |v_H\rangle} |v_H\rangle + i\hbar\, U_S \frac{d}{dt}|v_H\rangle$$

$$\Rightarrow i\hbar \frac{d}{dt}|v_H\rangle = -U_S^{-1} H_S U_S\, |v_H\rangle$$

$$\boxed{i\hbar \frac{d}{dt}|v_H\rangle = -H_H\,|v_H\rangle}$$

→ SCHRÖDINGER EQUATION WITH A MINUS SIGN. THE NEGATIVE SIGNAL IS BECAUSE $|v_H\rangle$ "ROTATES" BACKWARDS

EXERCISE: COMPUTE $i\hbar \frac{\partial}{\partial t}\rho_H$

$$i\hbar \frac{\partial}{\partial t}\rho_H \equiv i\hbar\, U_S^{+}\left(\frac{\partial}{\partial t}\rho_S\right)U_S = -U_S^{+}[\rho_S, H_S]\,U_S = [H_H, \rho_H]$$

EXERCISE: SHOW THAT $\frac{d}{dt}|\psi_H\rangle = 0$

$$i\hbar \frac{d}{dt}|\psi_H\rangle = \left(i\hbar \frac{d}{dt}U_S^{+}\right)|\psi_S\rangle + U_S^{+}\left(i\hbar \frac{d}{dt}|\psi_S\rangle\right) = -U_S^{+}H_S\,|\psi_S\rangle + U_S^{+}H_S\,|\psi_S\rangle = 0$$

- <u>INTERACTION PICTURE</u>

LET $H_S(t) = H_0 + V_S(t)$ where $\frac{\partial}{\partial t}H_0 = 0$

$$\boxed{\text{DICTIONARY: } \mathcal{O}_I = U_0^{+}(t,t_0)\,\mathcal{O}_S\,U_0(t,t_0) = e^{\frac{i}{\hbar}(t-t_0)H_0}\,\mathcal{O}_S\,e^{-\frac{i}{\hbar}(t-t_0)H_0}}$$

NOTICE THAT $H_{0,S} \equiv H_0 = H_{0,I}$, SINCE $[H_0, U_0] = 0$

* THE DYNAMICS OF $\mathcal{O}_I$ IS DICTATED BY H_0 (OR $H_{0,S}$) (WHICH MAY BE SIMPLE)

$$i\hbar \frac{d}{dt}\mathcal{O}_I = \left(-U_0^{+}H_0\right)\mathcal{O}_S U_0 + U_0^{+}\mathcal{O}_S\left(H_0 U_0\right) + i\hbar\, \underbrace{U_0^{+}\left(\frac{\partial}{\partial t}\mathcal{O}_S\right)U_0}_{\equiv \frac{\partial}{\partial t}\mathcal{O}_I}$$

$$\Rightarrow \boxed{i\hbar \frac{d}{dt} O_I = [O_I, H_0] + i\hbar \frac{\partial}{\partial t} O_I}$$

$$\boxed{\text{DICTIONARY:} \quad |\psi_I(t)\rangle = U_0^\dagger(t,t_0)\, |\psi_s(t)\rangle}$$

THEREFORE, $\langle O_I \rangle = \langle \psi_I | O_I | \psi_I \rangle = \langle \psi_s | U_0 U_0^\dagger O_s U_0 U_0^\dagger | \psi_s \rangle = \langle \psi_s | O_s | \psi_s \rangle = \langle O_s \rangle$

TIME DEPENDENCE OF $|\psi_I\rangle$:

$$i\hbar \frac{d}{dt} |\psi_I\rangle = \left(i\hbar \frac{d}{dt} U_0^\dagger \right) |\psi_s\rangle + U_0^\dagger \left(i\hbar \frac{d}{dt} |\psi_s\rangle \right) = -U_0^\dagger H_0 |\psi_s\rangle + U_0^\dagger H_s |\psi_s\rangle$$

$$= U_0^\dagger V_s |\psi_s\rangle = U_0^\dagger V_s U_0 U_0^\dagger |\psi_s\rangle$$

$$\Rightarrow \boxed{i\hbar \frac{d}{dt} |\psi_I\rangle = V_I(t)\, |\psi_I\rangle}$$

DYNAMICS IS DICTATED BY THE INTERACTION $V_I(t)$

* FOR $V(t) = 0 \Rightarrow$ RECOVERS THE HEISENBERG PICTURE
* FOR $H_0 = 0 \Rightarrow$ " " SCHRÖDINGER "

EXERCISE: SHOW THAT $U_H(t, t_0) = \mathbb{1}$

FOR $V_s = 0 \Rightarrow U_I = U_H$, MOREOVER $U_0 = U_I = e^{-\frac{i}{\hbar}(t-t_0) H_0}$

TIME-DEPENDENT EQUATION FOR U_I:

$$i\hbar \frac{d}{dt} U_I = \left(-H_0 U_0^\dagger \right) U_s U_0 + U_0^\dagger U_s \left(H_0 U_0 \right) + U_0^\dagger \left(H_s U_s \right) U_0$$

<u>TIME-EVOLUTION OPERATOR</u>: WE WANT $|\psi_I(t)\rangle = U_I(t,t_0)\, |\psi_I(t_0)\rangle$

WE KNOW THAT $|\psi_I(t)\rangle = U_0^\dagger(t,t_0)\, |\psi_s(t)\rangle = U_0^\dagger(t,t_0)\, U_s(t,t_0)\, |\psi_s(t_0)\rangle$

$$= U_0^\dagger(t,t_0)\, U_s(t,t_0)\, U_0(t_0,t_0)\, |\psi_I(t_0)\rangle$$

$$\Rightarrow \boxed{U_I(t,t_0) = U_0^\dagger(t,t_0)\, U_s(t,t_0) = U_0^\dagger(t,t_0)\, U_s(t,t_0)\, U_0(t_0,t_0)}$$

NOTICE IT IS SLIGHTLY DIFFERENT FROM $O_I = U_0^\dagger O_s U_0(t,t_0)$

$$\Rightarrow i\hbar \frac{d}{dt} U_I = -H_0 U_0^\dagger(t,t_0)\, U_s(t,t_0) + U_0^\dagger(t,t_0)\, H_s\, U_s(t,t_0)$$

$$\Rightarrow \boxed{i\hbar \frac{d}{dt} U_I(t,t_0) = V_I(t)\, U_I(t,t_0)}$$

IN SOME BOOKS, THE INTERACTION PICTURE MAY APPEAR DEFINED IN A SLIGHTLY DIFFERENT MANNER:

THE DICTIONARY IS
$$\begin{cases} \mathcal{O}_I = T_0^+(t)\, \mathcal{O}_S\, T_0(t) \\ |\psi_I(t)\rangle = T_0^+(t)\, |\psi_S(t)\rangle \end{cases}$$

with $\boxed{T_0(t) = e^{-i\frac{H_0}{\hbar}t}}$

with THIS MODIFICATION, WE THEN HAVE THAT

- $\boxed{i\hbar \frac{d}{dt}\mathcal{O}_I = [\mathcal{O}_I, H_0] + i\hbar \frac{\partial}{\partial t}\mathcal{O}_I}$ (STILL THE SAME)
- $\boxed{i\hbar \frac{d}{dt}|\psi_I\rangle = V_I\, |\psi_I\rangle}$ (STILL THE SAME)
- $|\psi_I(t)\rangle = U_I(t,t_0)|\psi_I(t_0)\rangle = T_0^+(t)\,|\psi_S(t)\rangle = T_0^+(t)\, U_S(t,t_0)\,|\psi_S(t_0)\rangle$
$$= T_0^+(t)\, U_S(t,t_0)\, T_0(t_0)\,|\psi_I(t_0)\rangle$$

$\Rightarrow \boxed{U_I(t,t_0) = T_0^+(t)\, U_S(t,t_0)\, T_0(t_0)}$ (SLIGHTLY DIFFERENT)

THIS SEEMS A MORE APPEALING WAY OF DEFINING THE INTERACTION PICTURE BECAUSE THE TIME EVOLUTION OPERATOR TRANSFORMS MORE LIKELY THE USUAL OPERATORS. BUT EITHER WAY OF DEFINING THE PICTURE YIELDS TO THE SAME RESULTS, AS SHOULD BE.

- FINALLY, $i\hbar \frac{d}{dt} U_I = -H_0\, T_0^+(t) U_S\, T_0(t_0) + T_0^+ H_S\, U_S\, T_0(t_0)$

$\Rightarrow \boxed{i\hbar \frac{d}{dt} U_I = V_I(t)\, U_I(t,t_0)}$ (STILL THE SAME)

FOR THE PURPOSES OF BUILDING A PERTURBATION THEORY FRAMEWORK, BOTH DEFINITIONS OF THE INTERACTING PICTURE GIVE THE SAME RESULTS. THE ONE IN THIS PAGE MAY BE A LITTLE MORE CONVENIENT BECAUSE IT HELPS WITH THE ALGEBRA.