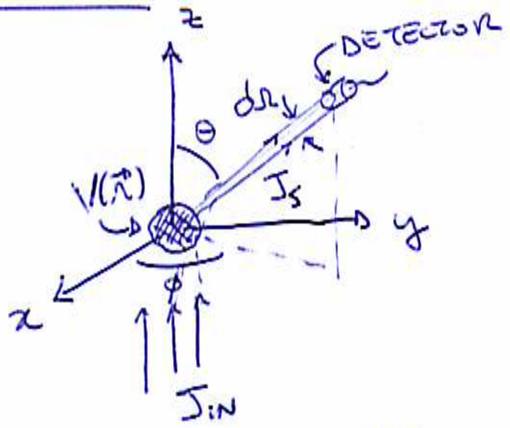
SCATTERING THEORY

1 - CROSS SECTION



I_0 = INTENSITY OF THE INCIDENT BEAM

$$= \frac{\text{NR. PARTICLES INCIDENT}}{\text{TIME} \times \text{AREA} \perp \hat{z}}$$

$$dn = \frac{\text{NR. PARTICLES HITTING THE DETECTOR}}{\text{TIME}}$$

DEFINITION OF DIFFERENTIAL CROSS SECTION: $dn = \sigma(\theta,\phi)\, I_0\, d\Omega$

OR $\sigma(\theta,\phi) = \dfrac{1}{I_0}\dfrac{dn}{d\Omega}$. NOTICE $[\sigma]$ = AREA

IN MANY CASES, THE DIFFERENTIAL CROSS SECTION IS DENOTED BY $\dfrac{d\sigma(\theta,\phi)}{d\Omega}$

TOTAL CROSS SECTION: $\sigma_{TOT} = \int \sigma(\theta,\phi)\, d\Omega = \int \sigma(\theta,\phi) \sin\theta\, d\theta\, d\phi$

FOR THE NOTATION $\dfrac{d\sigma}{d\Omega} \Rightarrow \sigma_{TOT} = \displaystyle\int \frac{d\sigma(\theta,\phi)}{d\Omega}\, d\Omega$

IN TERMS OF THE PROBABILITY FLUXES $\vec{J}_{IN}$ AND $\vec{J}_S$,

$$\sigma(\theta,\phi)\, d\Omega = \frac{J_S\, r^2 d\Omega}{J_{IN}} \rightarrow \sigma = \frac{r^2 J_S}{J_{IN}}$$

WITH $r^2 d\Omega$ = AREA OF THE DETECTOR

2 - HYPOTHESIS

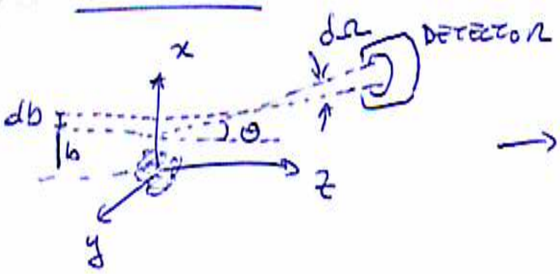
HERE, WE WILL BE INTERESTED IN THE CASE OF

a) LOW-INTENSITY BEAM. ACTUALLY, WE ARE INTERESTED IN THE CASE OF NON-INTERACTING PARTICLES.

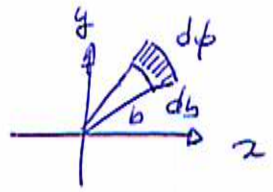
b) ELASTIC SCATTERING

c) FAR-FIELD AND STEADY-STATE REGIME: THE DETECTOR IS FAR FROM THE SCATTERING CENTER AND WE ARE NOT INTERESTED IN NON-TRANSIENT

ASSICAL CASE



$\longrightarrow$ IN THE xy PLANE



$$dA = b\,db\,d\phi$$

$$dn = I_0\,dA = I_0\,b\,db\,d\phi$$

$$\Rightarrow \sigma = \frac{1}{I_0} \times \frac{I_0\,dA}{d\Omega} = \frac{1}{I_0} \times \frac{I_0\,b\,db\,d\phi}{\sin\theta\,d\theta\,d\phi} \Rightarrow \boxed{\sigma(\theta,\phi) = \frac{b}{\sin\theta}\frac{db}{d\theta}}$$

- FOR THE COULOMB POTENTIAL $V(\vec{r}) = V(r) = \dfrac{Zz\,e^2}{4\pi\epsilon_0 r}$

$$\Rightarrow b(\theta) = \frac{D}{2}\cot g\left(\frac{\theta}{2}\right)$$

with $D = \dfrac{Zz\,e^2}{4\pi\epsilon_0 \frac{mv^2}{2}}$

$$\Rightarrow \boxed{\sigma \equiv \sigma(\theta) = \left(\frac{1}{4\pi\epsilon_0}\right)^2\left(\frac{Zz\,e^2}{2mv^2}\right)^2 \frac{1}{\sin^4(\theta/2)}}$$

$$= \frac{1}{16}\left(\frac{Zz\,e^2}{4\pi\epsilon_0 E_K}\right)\frac{1}{\sin^4(\theta/2)}$$

$E_K = \frac{1}{2}mv^2 = \frac{1}{2m}p^2$

4- QUANTUM TREATMENT

HERE, WE DO NOT HAVE A TRAJECTORY.

INSTEAD, EACH PARTICLE OF THE INCIDENT BEAM IS DESCRIBED AS A WAVEPACKAGE SUCH THAT, AT $t=0$, IT IS "BEHIND" THE SCATTERING POTENTI[AL]

$$\psi(\vec{r},0) = \int \varphi(\vec{k})\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}_0)}\, d^3k$$

WHERE $\varphi(\vec{k}) \equiv$ FUNCTION CENTERED AT $\vec{k}_0 = (0,0,k_0)$

$\vec{r}_0 \equiv$ CENTER OF THE WAVEPACKAGE $\equiv -r_0\hat{z}$

EXAMPLE: $\varphi(\vec{k}) = \dfrac{1}{\pi^{\frac{3}{4}}(\alpha\beta\gamma)^{1/2}} \exp\left\{-\dfrac{(k_z-k_0)^2}{2\alpha^3} - \dfrac{k_x^2}{2\beta^2} - \dfrac{k_y^2}{2\gamma^2}\right\}$

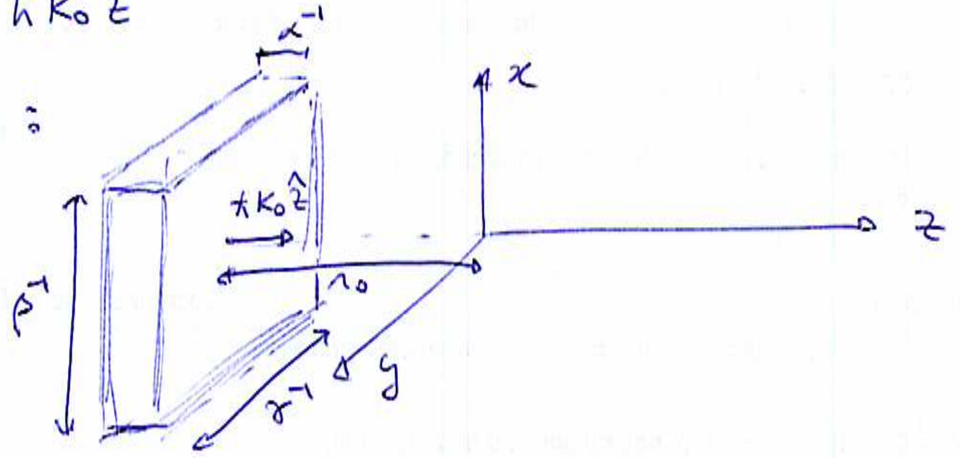
$$\Rightarrow \Delta k_z = \alpha, \quad \Delta k_x = \beta, \quad \Delta k_y = \gamma$$

$$\Rightarrow \psi(\vec{r},0) = \frac{\sqrt{\alpha\beta\gamma}}{\pi^{3/4}} \exp\left\{-\frac{\alpha^2}{2}(z+r_0)^2 - \frac{\beta^2}{2}x^2 - \frac{\gamma^2}{2}y^2 + i k_0 (z+r_0)\right\}$$

THUS, $\Delta z = \alpha^{-1}$ $\quad \Delta x = \beta^{-1}$ $\quad \Delta y = \gamma^{-1}$

$$\langle \vec{p} \rangle = \hbar k_0 \hat{z}$$

- FOR SMALL β AND γ :



TIME-EVOLUTION : NOW, WE NEED TO EVOLVE $\psi(\vec{r},0)$ FOR

$$H = \frac{P^2}{2m} + V(\vec{r})$$

IN THE PREVIOUS EXAMPLE, THEN

$$|\psi(\vec{r},t)\rangle = U(t,0)\,|\psi(\vec{r},0)\rangle$$

- FOR THE CASE $V(\vec{r})=0 \Rightarrow$ SIMPLE TIME EVOLUTION

$$\psi(\vec{r},t) = \frac{1}{(2\pi)^{3/2}} \int \varphi(\vec{k})\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}_0)}\, e^{-i\frac{E_k t}{\hbar}}\, d^3k$$

with $E_k = \dfrac{\hbar^2 k^2}{2m}$

- FOR $V(\vec{r}) \neq 0$, WE WILL ASSUME THAT

a) THE SOLUTIONS $H\psi = E\psi$ EXIST

b) THEY ARE OF TYPE

$$\psi^{(+)}_{\vec{k}}(\vec{r}) = \frac{1}{(2\pi)^{3/2}} \left(e^{i\vec{k}\cdot\vec{r}} + f_{\vec{k}}(\theta,\phi)\, \frac{e^{ikr}}{r} \right)$$

$$= \psi_{IN} + \psi_S$$

SCATTERING AMPLITUDE

c) $\psi(\vec{r},0) = \int \varphi(\vec{k})\, e^{-i\vec{k}\cdot\vec{r}_0}\, \psi^{(+)}_{\vec{k}}(\vec{r})\, d^3\vec{k}$

$$\psi(\vec{r},t) = \int \varphi(\vec{k})\, e^{-i\vec{k}\cdot\vec{r}_0}\, \psi^{(+)}_{k}(\vec{r})\, e^{-i\frac{E_k t}{\hbar}}\, d^3k$$

with $E_k = \dfrac{\hbar^2 k^2}{2m}$ FAR FROM THE SCATTERING POTENTIAL

HYPOTHESIS 5 IS VERY PLAUSIBLE. THE SCATTERED WAVE $\psi_s \propto f(\theta,\phi)\,\dfrac{e^{ikr}}{r}$ IS JUST AN OUTGOING SPHERICAL WAVE WEIGHTED BY $f(\theta,\phi)$ - THIS SHOULD BE OKAY FOR A FARAWAY DETECTOR.

EXAMPLE: 1D: $\psi_k^+ = \psi_{IN} + \psi_s = \begin{cases} A e^{ikx} + B e^{-ikx} & , x \to -\infty \\ C e^{ikx} & , x \to \infty \end{cases}$

where $\psi_{IN} = A e^{ikx}$

• CONNECTION ~~TO~~ BETWEEN THE SCATTERING AMPLITUDE AND THE DIFFERENTIAL CROSS SECTION

INCIDENT FLUX: $\vec{J}_{IN} = \dfrac{\hbar}{m}\, \mathrm{Im}\left(\psi_{IN}^* \nabla \psi_{IN}\right)$, $\psi_{IN} = A_k e^{i\vec{k}\cdot\vec{r}} = A_k e^{ikz}$

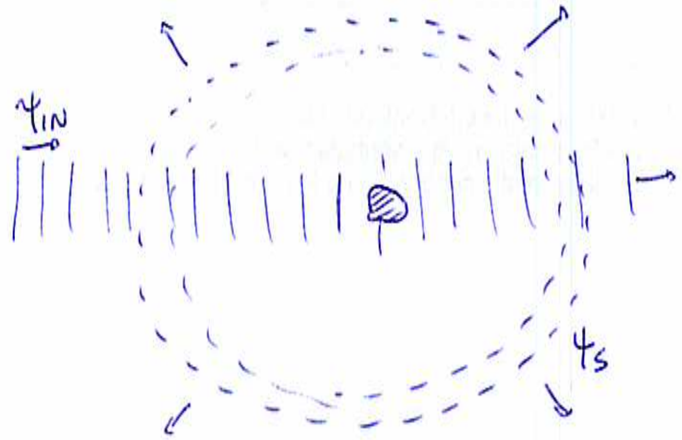
$$\boxed{\vec{J}_{IN} = |A_k|^2 \frac{\hbar}{m}\vec{k} \quad , \quad \vec{k} = k\hat{z}}$$

SCATTERED FLUX: $\vec{J}_s = \dfrac{\hbar}{m}\, \mathrm{Im}\left(\psi_s^* \nabla \psi_s\right)$, $\psi_s = A_k f(\theta,\phi)\,\dfrac{e^{ikr}}{r}$

$$\nabla\psi_s = \hat{r}\frac{\partial}{\partial r}\psi_s + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\psi_s + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\psi_s = A_k\left(f(\theta,\phi)\, ik\frac{e^{ikr}}{r}\hat{r} + O(r^{-2})\right)$$

$$\Rightarrow \boxed{\vec{J}_s \approx |A_k|^2 \,|f_{\vec{k}}(\theta,\phi)|^2 \frac{\hbar k}{m}\frac{\hat{r}}{r^2}}$$

$$\Rightarrow \boxed{\sigma(\theta,\phi) = |f_k(\theta,\phi)|^2}$$



ALTERNATIVELY, WE COULD ASK ABOUT THE PROBABILITY OF FINDING THE PARTICLE AT THE DETECTOR BETWEEN t AND $t+dt$

$$P(t) = |\psi_s|^2 dV = |A_k|^2 |f_{\vec{k}}(\theta,\phi)|^2 \times \frac{1}{r^2} dV, \quad dV = r^2 d\Omega\, dr \text{ with } dr = v\,dt$$

$$= |A_k|^2 |f_k(\theta,\phi)|^2 v\,dt\, d\Omega$$

$$dn = \frac{\text{NR. OF PARTICLES}}{\text{TIME}} \times P(t)$$ ($\cancel{\text{DENSITY}}$)

$$I_0 = \frac{\text{NR. OF PARTICLES}}{\text{TIME}} \times |\psi_{IN}|^2 \frac{dV}{dA} = \frac{\text{NR. PARTICLES}}{\text{TIME}} \times |A_k|^2 dz, \quad \text{BUT } dz = v\,dt$$ ($\cancel{\text{DENSITY}}$)

AGAIN $\sigma(\theta,\phi) = \frac{1}{I_0}\frac{dn}{d\Omega} = |f(\theta,\phi)|^2$

SOLUTION OF THE TIME-INDEPENDENT SCHRÖDINGER EQUATION

$$H\psi = E\psi \rightarrow \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r})\right)\psi = E\psi$$

DEFINING $k^2 = \frac{2m}{\hbar^2}E$ AND $U = \frac{2m}{\hbar^2}V$

$$\Rightarrow \boxed{(\nabla^2 + k^2)\psi = U\psi}$$

THE GENERAL SOLUTION IS OBTAINED VIA GREEN'S FUNCTION

$$\psi_{\vec{k}}(\vec{r}) = \psi^{(0)}_{\vec{k}}(\vec{r}) - \frac{1}{4\pi}\int G(\vec{r},\vec{r}')\, U(\vec{r}')\, \psi_k(\vec{r}')\, d^3r'$$

where $(\nabla^2 + k^2)\, G(\vec{r},\vec{r}') = -4\pi\,\delta(\vec{r}-\vec{r}')$ AND $(\nabla^2+k^2)\psi^{(0)} = 0$

PROOF: $$\nabla^2\psi = \nabla^2\psi^{(0)} - \frac{1}{4\pi}\int (\nabla^2 G)\, U(\vec{r}')\psi(\vec{r}')\, d^3r'$$

$$= \nabla^2\psi^{(0)} - \frac{1}{4\pi}\int \left(-4\pi\delta(\vec{r}-\vec{r}') - k^2 G\right) U\psi\, d^3r$$

$$= -k^2\psi^{(0)} + U(\vec{r})\psi(\vec{r}) + k^2\left[\underbrace{\frac{1}{4\pi}\int G U\psi\, d^3r}\right]$$

THUS, $\boxed{\nabla^2 \psi + k^2 \psi = U\psi}$

- COMPUTING G

a) ~~ASSUME~~ NOTICE $G(\vec{r}, \vec{r}') = G(\vec{r} - \vec{r}')$, BECAUSE $\nabla^2 + k^2$ AND $\delta(\vec{r} - \vec{r}')$ ARE INVARIANT UNDER TRANSLATIONS OF $\vec{r}$ AND $\vec{r}'$

b) FOURIER TRANSFORM: $G(\vec{r}) = \int g(\vec{k}) e^{i\vec{k}\cdot\vec{r}} \, d\vec{k}$

$$\delta(\vec{r}) = \frac{1}{(2\pi)^3} \int e^{i\vec{k}\cdot\vec{r}} \, d\vec{k}$$

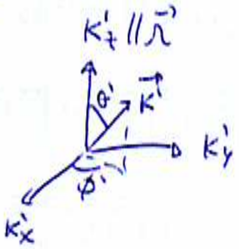
c) SOLVE FOR THE FREE PARTICLE: $(\nabla^2 + k^2) G(\vec{r} - \vec{r}') = -4\pi \delta(\vec{r} - \vec{r}')$

$$\Rightarrow \int (\nabla^2 + k^2) e^{i\vec{k}'\cdot\vec{r}} g(\vec{k}') \, d\vec{k}' = -\frac{4\pi}{(2\pi)^3} \int e^{i\vec{k}'\cdot\vec{r}} \, d\vec{k}'$$

$$\Rightarrow \int (k^2 - k'^2) e^{i\vec{k}'\cdot\vec{r}} g(\vec{k}') \, d\vec{k}' = -\frac{1}{2\pi^2} \int e^{i\vec{k}'\cdot\vec{r}} \, d\vec{k}'$$

$$\boxed{\begin{aligned} &\Rightarrow \quad g(\vec{k}') = \frac{1}{2\pi^2} \times \frac{1}{k'^2 - k^2} \\ &\Rightarrow \quad G(\vec{r}) = \frac{1}{2\pi^2} \int \frac{e^{i\vec{k}'\cdot\vec{r}}}{k'^2 - k^2} \, d\vec{k}' \end{aligned}}$$

INTEGRATING OVER $\vec{k}'$: SET $k'_z \parallel \vec{r}$



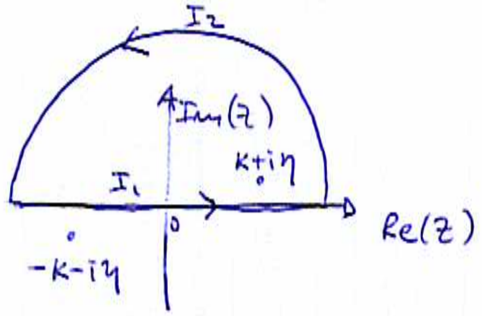
$$G(\vec{r}) = \frac{2\pi}{2\pi^2} \int \frac{e^{ik'r\cos\theta'}}{k'^2 - k^2} k'^2 \sin\theta' \, dk' \, d\theta'$$

$$\underset{(-\sin\theta' d\theta' = d\mu)}{\overset{(\cos\theta' = \mu)}{=}} \frac{1}{\pi} \int_0^\infty \int_{-1}^{1} \frac{e^{ik'r\mu}}{k'^2 - k^2} \, d\mu \, k'^2 dk' = \frac{1}{i\pi r} \int_0^\infty \frac{e^{ik'r} - e^{-ik'r}}{k'(k'^2 - k^2)} k'^2 dk'$$

$$= \frac{-1}{\pi r} \frac{d}{dr} \left(\int_{-\infty}^{\infty} \frac{e^{ik'r}}{k'^2 - k^2} \, dk' \right)$$

COMPUTING THE INTEGRAL VIA RESIDUES

$$I = \int_{-\infty}^{\infty} \frac{e^{ik'r}\, dk'}{\left[k'-(k+i\eta)\right]\left[k'+(k+i\eta)\right]} \qquad \text{with } \eta \to 0^+$$



$$I_1 + I_2 = 2\pi i \sum_j R(z_j)$$

$$I_1 = \int_{-\infty}^{\infty} \frac{e^{ik'r}}{k'^2-k^2}\, dk' = I$$

$$I_2 = \int_0^{\pi} \frac{e^{i|z|(\cos\theta + i\sin\theta)r}}{e^{2i\theta}z^2-k^2}\, |z|\, d\theta \xrightarrow{|z|\to\infty} 0$$

$$\Rightarrow \quad I_1 = 2\pi i * \frac{e^{i(k+i\eta)r}}{2(k+i\eta)} \longrightarrow \pi i \frac{e^{ikr}}{k}$$

FINALLY, $\quad G(\vec{r}) = -\dfrac{1}{\pi r}\dfrac{d}{dr}\left(\pi i \dfrac{e^{ikr}}{k}\right) = \dfrac{e^{ikr}}{r}$

USUALLY, THIS GREEN'S FUNCTION IS DENOTED BY

$$G^{(+)} = \frac{e^{ikr}}{r} \quad \to \text{OUTGOING WAVE}$$

IF WE HAD CHOSEN $\eta \to 0^-$, THEN THE POLE WOULD HAPPEN FOR $-k$

AND WE WOULD HAVE $\quad G^{(-)} = \dfrac{e^{-ikr}}{r} \quad \to$ INCOMING WAVE

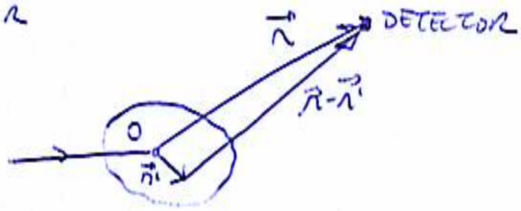
BACK TO ψ: $\quad \psi^{(+)}_{\vec{k}}(\vec{r}) = \dfrac{1}{(2\pi)^{3/2}}\left(e^{i\vec{k}\cdot\vec{r}} - \dfrac{(2\pi)^{3/2}}{4\pi}\displaystyle\int \dfrac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}\, U(\vec{r}')\, \psi_{\vec{k}}(\vec{r}')\, d\vec{r}'\right)$

FOR $r\to\infty$, ONE COULD THINK IN DOING $\dfrac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \approx \dfrac{e^{ikr}}{r}$, BUT THIS YIELDS ψ_s WITH NO DEPENDENCE ON θ AND ϕ.
BETTER JOB:
EXPANDING TO $r \gg r'$

- $|\vec{r}-\vec{r}'| = \sqrt{r^2 - 2\vec{r}\cdot\vec{r}' + r'^2} \approx r\left(1 - \dfrac{\vec{r}\cdot\vec{r}'}{r^2}\right) = r - \vec{r}'\cdot\hat{r}$

THUS, $\dfrac{1}{|\vec{r}-\vec{r}\,'|} \approx \dfrac{1}{r}\left(1 + \dfrac{\vec{r}\,'\cdot\hat{r}}{r}\right) \rightarrow \dfrac{1}{r}$

BECAUSE IT INTRODUCES ERRORS THAT VANISH FOR $r \rightarrow \infty$. THIS IS NOT SO FOR THE EXPONENTIAL FACTOR



$$e^{ik|\vec{r}-\vec{r}\,'|} \approx e^{ikr} \times e^{-ik\hat{r}\cdot\vec{r}\,'}$$

BUT $k\hat{r}$ IS THE SCATTERED WAVE VECTOR, THUS, LETS CALL $\vec{k}_s = k\hat{r}$

FINALLY

$$\psi^{(+)}_{\vec{k}}(\vec{r}) \cong \frac{e^{i\vec{k}\cdot\vec{r}}}{(2\pi)^{3/2}} - \frac{1}{4\pi}\int \frac{e^{ikr}}{r}\, e^{-i\vec{k}_s\cdot\vec{r}\,'}\, U(\vec{r}\,')\, \psi^{(+)}_{\vec{k}}(\vec{r}\,')\, d\vec{r}\,'$$

THUS,

$$\boxed{f_{\vec{k}}(\theta,\phi) = f_{\vec{k}}(\hat{r}) = -\frac{(2\pi)^{3/2}}{4\pi}\int e^{-ik\hat{r}\cdot\vec{r}\,'}\, U(\vec{r}\,')\, \psi^{(+)}_{\vec{k}}(\vec{r}\,')\, d\vec{r}\,'}$$

BORN SERIES (OR BORN APPROXIMATION)

SO FAR, $\psi_{\vec{k}} = \psi_{IN} + \psi_S = \psi^{(0)}_{\vec{k}} - \dfrac{1}{4\pi}\displaystyle\int G(\vec{r}-\vec{r}\,')\, U(\vec{r}\,')\, \psi_{\vec{k}}(\vec{r}\,')\, d\vec{r}\,'$

IN KET FORM,

$$\langle \vec{r}\,|\,\psi_{\vec{k}}\rangle = \langle \vec{r}\,|\,\psi^{(0)}_{\vec{k}}\rangle - \frac{1}{4\pi}\int \langle \vec{r}\,|\,G\,|\,\vec{r}\,'\rangle\langle \vec{r}\,'\,|\,U\,|\,\psi_{\vec{k}}\rangle\, d\vec{r}\,'$$

$$= \langle \vec{r}\,|\,\psi^{(0)}_{\vec{k}}\rangle - \frac{1}{4\pi}\langle \vec{r}\,|\,G\,U\,|\psi_{k}\rangle$$

$$\Rightarrow\ |\psi_{\vec{k}}\rangle = |\psi^{(0)}_{\vec{k}}\rangle - \frac{1}{4\pi}G\,U\,|\psi_{\vec{k}}\rangle$$

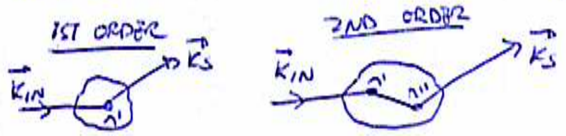
$$\Rightarrow\ \boxed{|\psi_{\vec{k}}\rangle = \frac{1}{1+\frac{1}{4\pi}GU}\,|\psi^{(0)}_{\vec{k}}\rangle}$$

RECALLING THAT $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

$$\Rightarrow \boxed{|\psi_{\vec{k}}\rangle = \left(1 - \frac{GU}{4\pi} + \frac{1}{4\pi} GUGU - \cdots\right) |\psi_{\vec{k}}^{(0)}\rangle}$$

IN THE SPACE REPRESENTATION

$$\psi_{\vec{k}}(\vec{r}) = \psi_{k}^{(0)}(\vec{r}) - \frac{1}{4\pi}\int G(\vec{r}-\vec{r}')U(\vec{r}')\psi_{\vec{k}}^{(0)}(\vec{r}')\,d\vec{r}' + \frac{1}{(4\pi)^2}\iint G(\vec{r}-\vec{r}')U(\vec{r}')G(\vec{r}'-\vec{r}'')U(\vec{r}'')\psi_{\vec{k}}^{(0)}(\vec{r}'')\,d\vec{r}'\,d\vec{r}'' + \cdots$$



ALTERNATIVELY, WE COULD START FROM

$$\psi_k = \psi_k^{(0)} - \frac{1}{4\pi}\int GU\psi_k$$

AND DOING THE APPROXIMATION THAT $\psi_k \approx \psi_k^{(0)}$ IN THE INTEGRAL. THEN

$$\psi_k \approx \psi_k^{(0)} - \frac{1}{4\pi}\int GU\psi_k^{(0)}$$

WHICH IS THE BORN APPROXIMATION IN 1st ORDER IN U

IF WE WISH THE SECOND ORDER APPROXIMATION, THEN WE FEED THE 1st-ORDER ONE ON THE ORIGINAL EQUATION, YIELDING

$$\psi_k \cong \psi_k^{(0)} - \frac{1}{4\pi}\int GU\psi_k^{(0)} + \frac{1}{(4\pi)^2}\iint GUGU\psi_k^{(0)}$$

NOTICE $f(\vec{k}_{IN}, \vec{k}_S) \equiv f(\underbrace{\vec{k}_{IN} - \vec{k}_S}_{\text{MOMENTUM TRANSFER}})$

REPEATING THIS PROCEDURE WE RECOVER THE BORN SERIES

TO <u>1st ORDER</u> $\boxed{f_{\vec{k}}(\vec{r}) = f_{\vec{k}_{IN}}(\vec{k}_S) \approx -\frac{1}{4\pi}\int e^{i(\vec{k}_{IN}-\vec{k}_S)\cdot\vec{r}'}\,U(\vec{r}')\,d\vec{r}'}$ WHICH IS THE FOURIER TRANSFORM OF THE POTENTIAL $\downarrow$ T MATRIX

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VALIDITY OF THE BORN APPROXIMATION

UP TO 1st ORDER IN V,

$$\psi_k(\vec{r}) = \psi_k^{(0)}(\vec{r}) - \frac{1}{4\pi}\int G(\vec{r}-\vec{r}')U(\vec{r}')\psi_{\vec{k}}^{(0)}(\vec{r}')\,d\vec{r}'$$

MEANING THAT WE SUBSTITUTED $\psi = \psi_{IN} + \psi_S$ BY ψ_{IN} IN THE INTEGRAND

THUS, IN A ROUGH SENSE, WE COULD SAY THAT

$$|\psi_S| \ll |\psi_{\vec{k}}^{(0)}| \quad \text{IN THE REGION OF THE POTENTIAL}$$

THUS , $f^{(1)}(\vec{k}_{IN}, \vec{k}_S) = -\frac{1}{4\pi} \times (2\pi)^3 \times \frac{2m}{\hbar^2} \underbrace{\int \frac{e^{-i\vec{k}_S\cdot\vec{r}}}{(2\pi)^{3/2}} V(\vec{r}) \frac{e^{i\vec{k}_{IN}\cdot\vec{r}}}{(2\pi)^{3/2}} d\vec{r}}_{\langle \vec{k}_S | \vec{r} \rangle \langle \vec{r} | V | \vec{k}_{IN} \rangle}$ (9a)

$$\Rightarrow f^{(1)}(\vec{k}_{IN}, \vec{k}_S) = -\frac{1}{4\pi}(2\pi)^3 \frac{2m}{\hbar^2} \langle \vec{k}_S | T^{(1)} | \vec{k}_{IN} \rangle$$

with $T^{(1)} = V$ BEING THE TRANSFER MATRIX OPERATOR (OR THE TRANSITION OPERATOR) IN 1st ORDER IN V

USING ALL ORDERS , THEN

$$T = V + VGU + VGUGU + VGUGUGU + \cdots$$

IT IS POSSIBLE TO SHOW THAT THIS IS EQUAL TO

$$\boxed{T = V + V \frac{1}{E - H_0 + i\eta} V + V \frac{1}{E - H_0 + i\eta} V \frac{1}{E - H_0 + i\eta} V + \cdots}$$

$$= \frac{V}{1 - \frac{1}{E - H_0 + i\eta} V} = V + V \frac{1}{E - H_0 + i\eta} T$$

ROUGHLY, $\dfrac{|\psi_s(0)|}{|\psi^{(0)}_{\vec{k}}(0)|} \ll 1 \Rightarrow \left| \int e^{i(\vec{k}_{IN} - \vec{k}_S)\cdot \vec{r}'} \dfrac{U(\vec{r}')}{r'} d\vec{r}' \right| \ll 1$

$\boxed{\text{THE "DISTORTION" OF THE INCIDENT WAVE IS SMALL}}$

with $\vec{k}_{IN} = k\hat{z}$ AND $\vec{k}_S = k\hat{r}$. CHOOSING $\hat{z}' \parallel \hat{z}$

THEN, $\left| \int e^{ikr'} e^{-ikr'\cos\theta'} \dfrac{U(\vec{r}')}{r'} d\vec{r}' \right| \ll 1$ CHOOSING $U(\vec{r}') = U(r')$

$$\Rightarrow 2\pi \int e^{ikr'} \frac{e^{-ikr'} - e^{ikr'}}{-ikr'} \frac{U(r')\, r'^2 dr'}{r'} = \frac{4\pi}{k} \int e^{ikr'} \sin kr'\, U(r')\, dr'$$

THUS, $$\frac{2m}{\hbar^2 k} \left| \int e^{ikr'} \sin(kr')\, V(r')\, dr' \right| \ll 1$$

- LOW-ENERGY ANALYSIS: $(kr' \to 0) \Rightarrow e^{ikr'} \sin kr' \to kr'$

$$\Rightarrow \frac{2m}{\hbar^2} \left| \int r' V(r')\, dr' \right| \ll 1 \Rightarrow \frac{2m}{\hbar^2} \frac{|V_0| R_0^2}{2} \ll 1$$

(MAKE $V = V_0$ AND OF SIZE R_0)

$$\boxed{\text{THUS, } V_0 \ll \frac{\hbar^2}{m R_0^2}}$$

PHYSICAL MEANING $\rightarrow V_0 \ll \dfrac{\hbar^2 \tilde{k}^2}{2m}$, $\tilde{k} = \dfrac{\sqrt{2}}{R_0}$

WHICH IS THE WAVE NUMBER OF A PARTICLE CONFINED IN A REGION OF ORDER R_0

$\Rightarrow$ THE POTENTIAL HAS TO BE TOO SHALLOW IN ORDER TO BIND A PARTICLE

- HIGH-ENERGY ANALYSIS: $(kR_0 \gg 1) \Rightarrow e^{ikr'} \sin kr' = \dfrac{e^{2ikr'} - 1}{2i} \to \dfrac{-1}{2i}$

SINCE $e^{2ikr'}$ OSCILLATES RAPIDLY

$$\Rightarrow \frac{2m}{\hbar^2 k} \cdot \frac{1}{2} V_0 R_0 \ll 1 \Rightarrow \boxed{V_0 \ll \frac{\hbar^2}{m R_0^2} * (kR_0)}$$

THUS, IF THE BORN APPROXIMATION WORKS IN THE LOW-ENERGY REGIME, IT ALSO WORKS IN THE HIGH-ENERGY ONE.

EXAMPLES:

1) YUKAWA POTENTIAL: $V(\vec{r}) = V(r) = \dfrac{V_0 e^{-\alpha r}}{\alpha r}$, $\alpha > 0$

2) COULOMB POTENTIAL: $V(r) = \lim_{\substack{\alpha \to 0 \\ V_0\alpha \to cte}} \dfrac{V_0 e^{-\alpha r}}{\alpha r} = \left(\dfrac{V_0}{\alpha}\right) \dfrac{e^{-\alpha r}}{r} \to \dfrac{V_0}{\alpha r}$

3) SPHERICAL BARRIER: $V(\vec{r}) = V(r) = \begin{cases} V_0, & \text{IF } r \leq a \\ 0, & \text{IF } r > a \end{cases}$

BORN APPROXIMATION IN 1st-ORDER TO SPHERICAL POTENTIALS

$$f_{\vec{k}_{IN}}(\vec{k}_S) = \frac{-1}{4\pi} \int e^{i(\vec{k}_{IN} - \vec{k}_S)\cdot\vec{r}\,'} \frac{2m}{\hbar^2} V(\vec{r}\,')\, d\vec{r}\,'$$

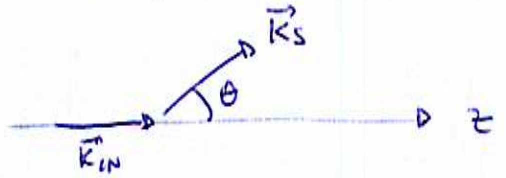
LET US TAKE $\hat{z}' \parallel \Delta\vec{k} = \vec{k}_{IN} - \vec{k}_S$

$$= \frac{-2\pi}{4\pi} \times \frac{2m}{\hbar^2} \iint e^{i\Delta k r' \cos\theta'} V(r')\, r'^2 \sin\theta\, dr'\, d\theta'$$

$$\boxed{f_{\vec{k}_{IN}}(\vec{k}_S) = \frac{-2m}{\Delta k \hbar^2} \int_0^\infty \sin(\Delta k r')\, V(r')\, r'\, dr'}$$

NOTICE $f_{\vec{k}_{IN}}(\vec{k}_S) = f(\theta)$ AND DOES NOT DEPEND ON ϕ

AND $f(\theta) \in \mathbb{R}$



$\vec{k}_{IN} = k\hat{z}$
$\vec{k}_S = k\hat{r}$

$$\Delta k = \sqrt{k_{IN}^2 - 2\vec{k}_{IN}\cdot\vec{k}_S + k_S^2} = k\sqrt{2 - 2\cos\theta} = k\sqrt{4\sin^2\frac{\theta}{2}} = 2k\left|\sin\frac{\theta}{2}\right|$$

1) YUKAWA POTENTIAL

$$f(\theta) = \frac{-2m}{\hbar^2 2k\left|\sin\frac{\theta}{2}\right|} \int_0^\infty x \sin\left(2k\left|\sin\frac{\theta}{2}\right| x\right) \frac{V_0}{\alpha x} e^{-\alpha x}\, dx$$

$$= \frac{-2m}{\hbar^2 \Delta k}\left(\frac{V_0}{\alpha}\right) \times \frac{1}{2i} \int_0^\infty \left(e^{(i\Delta k - \alpha)x} - e^{-(i\Delta k + \alpha)x}\right) dx$$

$$= \frac{-2m}{\hbar^2 \Delta k}\left(\frac{V_0}{\alpha}\right) \times \frac{1}{2i}\left[\frac{1}{\alpha - i\Delta k} - \frac{1}{\alpha + i\Delta k}\right] = \frac{-2m}{\hbar^2 \Delta k} \frac{V_0}{\alpha} \frac{\Delta k}{\alpha^2 + \Delta k^2}$$

FINALLY, $f(\theta) = -\frac{2m}{\hbar^2}\left(\frac{V_0}{\alpha}\right)\frac{1}{\alpha^2+\Delta K^2}$

$$\Rightarrow \boxed{\sigma = \sigma(\theta) = \frac{4m^2}{\hbar^4}\left(\frac{V_0}{\alpha}\right)^2\left(\frac{1}{\alpha^2+4K^2\sin^2(\theta/2)}\right)^2} \rightarrow$$

PROBABLY MISLEADING BECAUSE THE BORN APPROXIMATION IS NOT GOOD FOR LOW ENERGIES

2) COULOMB POTENTIAL, TAKING $\alpha \to 0$ AND $\frac{V_0}{\alpha} \to \frac{Zze^2}{4\pi\epsilon_0}$

$$\Rightarrow \sigma(\theta) = \frac{4m^2}{\hbar^4}\left(\frac{Zze^2}{4\pi\epsilon_0}\right)^2\frac{1}{(4K^2)^2\sin^4\left(\frac{\theta}{2}\right)}.$$

IDENTIFYING $\frac{\hbar^2K^2}{2m} = E_K$ AS THE KINETIC ENERGY OF THE INCOMING PARTICLE BEAM

$$\Rightarrow \boxed{\sigma(\theta) = \frac{1}{16}\left(\frac{Zze^2}{4\pi\epsilon_0 E_K}\right)^2\frac{1}{\sin^4(\theta/2)}}$$

WHICH IS EXACTLY THE CLASSICAL RESULT

3) $f(\theta) = \frac{-2m}{\hbar^2\Delta K}\int_0^a x\sin(\Delta K x)\,V_0\,dx = \frac{-2mV_0}{\hbar^2\Delta K}\left[-x\frac{\cos(\Delta K x)}{\Delta K}\Big|_0^a + \frac{1}{\Delta K}\int_0^a\cos(\Delta K x)\,dx\right]$

$$= \frac{-2mV_0}{\hbar^2\Delta K}\left[\frac{-a}{\Delta K}\cos(\Delta K a) + \frac{1}{\Delta K^2}\sin(\Delta K a)\right]$$

$$\Rightarrow \boxed{f(\theta) = \frac{2mV_0}{\hbar^2\Delta K^3}\left[\Delta K a\cos(\Delta K a) - \sin(\Delta K a)\right]}$$

$$\sigma(\theta) = |f(\theta)|^2$$

METHOD OF PARTIAL WAVES

- SCATTERING BY A CENTRAL POTENTIAL $V(\vec{r}) = V(r)$
- $V(r)$ IS LOCALIZED $\Rightarrow$ $V(r) = 0$ FOR $r > a$

IDEA OF THE METHOD

$$H|\psi\rangle = \left(H_0 + V(r)\right)|\psi\rangle = E|\psi\rangle$$

- FOR $V=0$ $\Rightarrow$ $|\psi\rangle = |\vec{k}\rangle = \sum_{k\ell m} a_{k\ell m}|k,\ell,m\rangle$

where $|k\ell m\rangle = |k\rangle \otimes \underbrace{|\ell,m\rangle}_{\text{SPHERICAL HARMONICS}}$

$$\langle \vec{r}|k\ell m\rangle = R^{(0)}_{k\ell}(r)\, Y_{\ell,m}(\theta,\phi)$$

$\Rightarrow$ PLANE WAVE = SUPERPOSITION OF FREE SPHERICAL WAVES

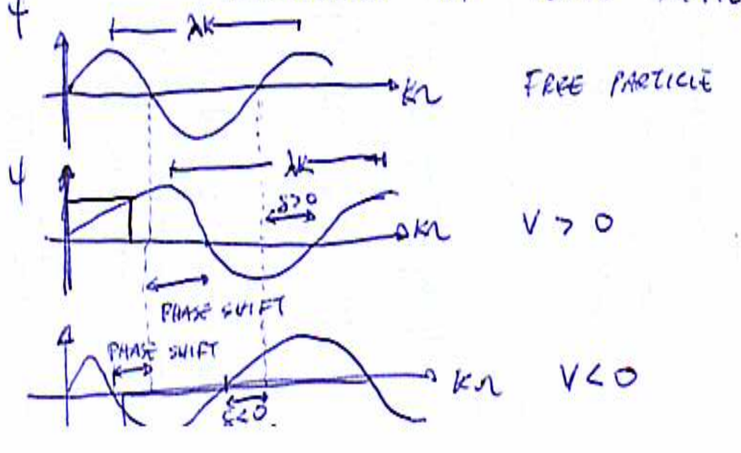
= SUPERPOSITION OF PARTIAL WAVES

- FOR $V \neq 0$ $\Rightarrow$ $|\psi\rangle = \sum_{k\ell m} a_{k\ell m}\,|k\ell m\rangle$

where $\langle \vec{r}|k\ell m\rangle = R_{k\ell}(r)\, Y_{\ell m}(\theta,\phi)$

$V(r)$ INFLUENCES ONLY $R_{k\ell}(r)$, THE RADIAL PART, NOT THE ANGULAR.

IN ADDITION, FOR $r \to \infty$, THE FREE-PARTICLE WAVE SHOULD BE RECOVERED BUT WITH SOME PHASE SHIFT WHICH IS DETERMINED BY SOME MATCHING CONSTRAINT



- PLANE WAVES IN TERMS OF FREE SPHERICAL WAVES

(THE CASE OF $V=0$)

SOLVING $H_0|\psi\rangle = E|\psi\rangle$, $H_0 = \frac{p^2}{2m}$

$$\psi^{(0)}_{k\ell m} = R^{(0)}_{k\ell}(r)\, Y_{\ell m}(\theta,\phi) = \frac{u_{k\ell}(r)}{r}\, Y_{\ell m}(\theta,\phi)$$

$$\Rightarrow \Big(\underbrace{-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{\hbar^2 \ell(\ell+1)}{2mr^2}}_{\text{CENTRIFUGAL POTENTIAL}}\Big) u_{k\ell}(r) = E\, u_{k\ell}(r) = \frac{\hbar^2 k^2}{2m} u^{(0)}_{k\ell}(r)$$

GENERAL SOLUTION: $$R^{(0)}_{k\ell} = \sqrt{\frac{2k^2}{\pi}}\Big(A_\ell \underbrace{j_\ell(kr)}_{\text{SPHERICAL BESSEL}} + B_\ell \underbrace{n_\ell(kr)}_{\text{SPHERICAL NEUMANN}}\Big)$$

NORMALIZATION: $|A_\ell|^2 + |B_\ell|^2 = 1$

$$j_\ell(z) = (-1)^\ell z^\ell \left(\frac{1}{z}\frac{d}{dz}\right)^\ell \left(\frac{\sin z}{z}\right)$$

$$n_\ell(z) = (-1)^{\ell+1} z^\ell \left(\frac{1}{z}\frac{d}{dz}\right)^\ell \left(\frac{\cos z}{z}\right)$$

Ex: $j_0 = \frac{\sin z}{z}$, $j_1 = -\frac{\cos z}{z} + \frac{\sin z}{z^2}$

$n_0 = -\frac{\cos z}{z}$, $n_1 = -\frac{\sin z}{z} + \frac{\cos z}{z^2}$

ASYMPTOTIC BEHAVIOR ($r \to \infty$)

$$j_\ell(z) = (-1)^\ell z^\ell \left(\frac{1}{z}\frac{d}{dz}\right)^{\ell-1}\left(+\frac{\cos z}{z^2} \mp \cancel{\frac{\sin z}{z^3}}^{\,\to 0}\right) = (-1)^\ell z^\ell \left(\frac{1}{z}\frac{d}{dz}\right)^{\ell-2}\left(-\frac{\sin z}{z^3} - \cancel{\frac{\cos z}{z^4}}^{\,\to 0}\right)$$

$$= (-1)^\ell z^\ell \times \frac{1}{z^\ell} \times \frac{1}{z}\left(\frac{d^\ell}{dz^\ell}\sin z\right) = \frac{(-1)^\ell}{z}\frac{d^\ell}{dz^\ell}\left(\frac{e^{iz}-e^{-iz}}{2i}\right) = \frac{(-1)^\ell (i)^\ell}{z}\left(\frac{e^{iz}-(-1)^\ell e^{-iz}}{2i}\right)$$

$$= \frac{1}{z}\left(\frac{(-i)^\ell e^{iz} - i^\ell e^{-iz}}{2i}\right) = \frac{1}{z}\left(\frac{e^{-i\frac{\pi}{2}\ell} e^{iz} - e^{i\frac{\pi}{2}\ell} e^{-iz}}{2i}\right)$$

$$\boxed{j_\ell(z) \to \frac{1}{z}\sin\left(z - \frac{\pi}{2}\ell\right)}$$

ANALOGOUSLY, $\boxed{\eta_\ell \xrightarrow{z\to\infty} -\frac{\cos(z-\frac{\pi}{2}\ell)}{z}}$

FINALLY, BECAUSE $\psi_{k\ell m}$ DOES NOT DIVERGE AT $r=0$

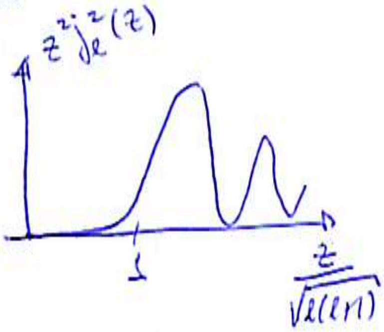
$\Rightarrow B_\ell = 0$ (NO ~~NE~~ SPHERICAL NEUMANN)

THUS THE BASIS IS $\{\psi_{k\ell m}\}$

$$\boxed{\psi^{(0)}_{k\ell m}(r,\theta,\phi) = \sqrt{\frac{2k^2}{\pi}}\, j_\ell(kr)\, Y_{\ell m}(\theta,\phi)}$$

- SOME PHYSICAL PROPERTIES OF THE FREE SPHERICAL WAVES
  - a) WELL DEFINED ANGULAR MOMENTUM
  - b) BEHAVIOR NEAR THE ORIGIN

$$P(\vec{r})\,d\vec{r} \propto j_\ell^2(kr)\,|Y_{\ell m}(\theta,\phi)|^2\, r^2\,dr\,d\Omega$$



~~FOR~~ SMALL r, $j_\ell(kr) \sim \frac{(kr)^\ell}{(2\ell+1)!!}$

$$\Rightarrow P(\vec{r}) \propto r^{2\ell+2} = r^{2(\ell+1)}$$

IN FACT $z^2 j_\ell^2(z)$ IS SMALL FOR $z < \sqrt{\ell(\ell+1)}$

$\Rightarrow$ PROB. IS SMALL FOR $kr < \sqrt{\ell(\ell+1)}$

CONSEQUENCE:

A PARTICLE IN STATE $\psi_{k\ell m}$ IS WEAKLY AFFECTED BY A POTENTIAL OF ~~[illegible]~~ WIDTH

$$b_\ell(k) = \frac{1}{k}\sqrt{\ell(\ell+1)}$$

IN CLASSICAL MECHANICS, THE IMPACT PARAMETER IS OBTAINED FROM $L = mvb$

$\Rightarrow b = \frac{L}{p}$ $\Rightarrow$ IF THE POTENTIAL WIDTH IS $a < b \Rightarrow$ NOTHING HAPPENS ~~NOTICE~~ $b_\ell(k) = \frac{\hbar\sqrt{\ell(\ell+1)}}{\hbar k} = \frac{L}{p}$

c) ASYMPTOTIC BEHAVIOR FOR $r \to \infty$

$$\psi^{(0)}_{k\ell m} \xrightarrow[r\to\infty]{} +\sqrt{\frac{2k^2}{\pi}}\, Y_{\ell m}(\theta,\phi)\, \frac{e^{+i(kr-\frac{\ell\pi}{2})} - e^{-i(kr-\frac{\ell\pi}{2})}}{2ikr}$$

$$\propto Y_{\ell m}(\theta,\phi)\, \frac{e^{ikr} - e^{-i(kr-\ell\pi)}}{2ikr}$$

- WHICH IS ~~A~~ SUPERPOSITION OF AN INCOMING WAVE $\frac{e^{-ikr}}{r}$ ~~WHICH~~ WITH AN OUTGOING ONE $\frac{e^{ikr}}{r}$.

- SAME AMPLITUDE
- PHASE DIFFERENCE $= \pi\ell$

  IF WE CONSTRUCT A ~~WAVE~~ SPHERICAL WAVE PACKET, LOCATED AT $r \to \infty$ AT $t \to -\infty$, AND LET IT EVOLVE, THEN IT FOCUS AT THE ORIGIN AND BECOME AN OUTGOING PACKET AT $t \to +\infty$ WITH A PHASE DIFFERENCE OF $\pi\ell$

FINALLY, PLANE WAVES

WE ARE NOW ABLE TO EXPAND THE PLANE WAVE $\langle \vec{r}|\vec{k}\rangle = \frac{e^{i\vec{k}\cdot\vec{r}}}{(2\pi)^{3/2}}$

IN TERMS OF THE FREE SPHERICAL WAVES $\psi^{(0)}_{k\ell m}$

THE RESULT IS $\langle \vec{r}|\vec{k}\rangle = \sum_\ell \frac{2\ell+1}{4\pi} \frac{\hbar}{\sqrt{mk}}\, c_\ell\, j_\ell(kr)\, P_\ell(\hat{k}\cdot\hat{r})$

with $c_\ell = \frac{i^\ell}{\hbar}\sqrt{\frac{2mk}{\pi}}$

LEGENDRE POLYNOMIAL

$= \sqrt{\frac{4\pi}{2\ell+1}}\, \underbrace{Y_{\ell,0}(\theta,\phi)}_{Y_{\ell,0}(\theta)}$

IF WE CHOOSE $\vec{k} = k\hat{z}$

$\Rightarrow$ THERE WILL BE NO DEPENDENCE ON ϕ

$\langle \vec{r}|0,0,k\rangle = \frac{e^{ikz}}{(2\pi)^{3/2}}$ (NOTICE $L_z|k\hat{z}\rangle = 0$ BECAUSE $L_z = xp_y - yp_x$)

AND $e^{ikz} = \sum_{\ell=0}^{\infty} i^\ell (2\ell+1)\, j_\ell(kr)\, P_\ell(\cos\theta)$

$$\boxed{e^{ikz} = \sum_{\ell=0}^{\infty} i^\ell \sqrt{4\pi(2\ell+1)}\; j_\ell(kr)\, Y_{\ell,0}(\theta)}$$

WE HAVE FINALLY SOLVED THE $V=0$ CASE

$$\boxed{\psi^{(0)} = \frac{e^{ikz}}{(2\pi)^{3/2}} = \sum_{\ell=0}^{\infty} i^\ell \sqrt{\frac{2\ell+1}{2\pi^2}}\, j_\ell(kr)\, Y_{\ell,0}(\theta)}$$

PARTIAL WAVES IN THE $V(r)\neq 0$ POTENTIAL

$$\psi_{k\ell m} = R_{k\ell}(r)\, Y_{\ell m}(\theta,\phi) = \frac{1}{r}\, u_{k\ell}(r)\, Y_{\ell m}(\theta,\phi)$$

$$\Rightarrow H\psi = E\psi \longrightarrow \left(-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{\ell(\ell+1)\hbar^2}{2mr^2} + V(r)\right) u_{k\ell}(r) = \frac{\hbar^2 k^2}{2m} u_{k\ell}(r)$$

CONSTRAINED TO THE BOUNDARY CONDITION $u_{k\ell}(0) = 0$

FOR $r > a \Rightarrow V(r)$ IS NEGLIGIBLE

$\rightarrow$ SOLUTIONS ARE $j_\ell(kr)$ AND $n_\ell(kr)$

WE CAN THEN TAKE THE $r\to\infty$ LIMIT AND SIMPLIFY EVEN FURTHER

$$\boxed{-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} u_{k\ell} = \frac{\hbar^2 k^2}{2m} u_{k\ell}}$$

$$\Rightarrow u_{k\ell} = A_\ell e^{ikr} + B_\ell e^{-ikr}$$

WHICH ARE THE BESSEL AND NEUMANN FOR $r\to\infty$

A AND B <u>CANNOT</u> BE ARBITRARY

SINCE THE INCOMING WAVE IS TOTALLY REFLECTED INTO THE OUTGOING WAVE (NO TRANSMISSION FOR $r<0$) $\Rightarrow |A_\ell| = |B_\ell|$

$$\Rightarrow u_{k\ell} = C_\ell \sin\left(kr - \frac{\pi}{2}\ell + \delta_\ell\right)$$

$\downarrow$ $(\delta_\ell \in \mathbb{R})$

PHASE SHIFT DUE TO $V(r)$

$$\Rightarrow \psi_{k\ell m} \propto Y_{\ell m}(\theta,\phi)\, \underline{e^{i(kr-\frac{\ell\pi}{2}+\delta_\ell)} - e^{-i(kr-\frac{\ell\pi}{2}+\delta_\ell)}} \propto Y_{\ell m}(\theta,\phi)\, \underline{e^{ikr} - e^{-i(kr-\ell\pi-2\delta_\ell)}}$$

⇒ COMPARING WITH THE ASYMPTOTIC BEHAVIOR OF THE FREE SPHERICAL WAVE, THIS NEW ONE CAN BE THOUGHT AS AN INCOMING SPHERICAL PACKET THAT FOCUSED ON THE ORIGIN AND IS TOTALLY REFLECTED AS AN OUTGOING WAVE. THE NET EFFECT OF THE POTENTIAL WAS THE ACCUMULATION OF AN EXTRA PHASE $2\delta_\ell$ RELATIVE TO THE $V=0$ CASE

– CONNECTION WITH S MATRIX → SEE PAGE 18.a

RELATION WITH THE SCATTERING AMPLITUDE

$$\psi_k = \frac{1}{(2\pi)^{3/2}}\left[e^{ikz} + \underset{\substack{\downarrow \\ \text{RECALL } f_{\vec{k}}(\theta,\phi) = f_{\vec{k}}(\theta) \text{ FOR } V(\vec{x}) = V(r)}}{f_{\vec{k}}(\theta)}\,\frac{e^{ikr}}{r}\right]$$

WE HAVE FOUND THAT

ONLY $m=0$ BECAUSE THE INCOMING WAVE HAS $L_z=0$

$$\psi_k = \sum_{\substack{\ell=0\\ m}}^{\infty} \underbrace{b_\ell\sqrt{\frac{2k^2}{\pi}}}_{c_\ell}\;\frac{\sin\left(kr-\frac{\pi}{2}\ell+\delta_\ell\right)}{kr}\,Y_{\ell m}(\theta,\phi)\,\delta_{m,0}$$

$$= \frac{1}{(2\pi)^{3/2}}\left[\sum_{\ell=0}^{\infty} i^\ell\sqrt{4\pi(2\ell+1)}\;\underset{\substack{\downarrow\\ \simeq \sin(kr-\frac{\pi}{2}\ell)/kr}}{j_\ell(kr)}\,Y_{\ell 0}(\theta) + f(\theta)\,\frac{e^{ikr}}{r}\right]$$

NOW, WE DO THE MATCHING:

- IDENTIFYING THE TERM $\propto e^{-i(kr-\ell\frac{\pi}{2})}$

$$\Rightarrow \frac{-1}{(2\pi)^{3/2}}\,\frac{i^\ell\sqrt{4\pi(2\ell+1)}\;Y_{\ell,0}}{2ikr} = -c_\ell\,\frac{Y_{\ell 0}}{2ikr}\,e^{-i\delta_\ell}$$

$$\Rightarrow c_\ell = \frac{e^{i\delta_\ell}}{(2\pi)^{3/2}}\,i^\ell\sqrt{4\pi(2\ell+1)}$$

IT IS INTERESTING TO NOTICE THE CONNECTION WITH THE S (SCATTERING) MATRIX OPERATOR DEFINED BY

$$S = \lim_{\substack{t_f \to +\infty \\ t_i \to -\infty}} U(t_f, t_i)$$

where $U(t_f, t_i)$ IS THE TIME EVOLUTION OPERATOR $= e^{-i\frac{H}{\hbar}(t_f - t_i)}$

SINCE THE ONLY EFFECT OF SCATTERING IS A SIMPLE PHASE SHIFT (THIS SHOULD BE THE CASE BECAUSE S IS UNITARY AND CONSERVES ENERGY $\Rightarrow$ THE EINGENVALUES OF S ARE SIMPLE PHASES $e^{i\alpha}$), THEN

WE CONCLUDE THAT THE MATRIX ELEMENT OF S WITH RESPECT TO THE WAVE OF ANGULAR MOMENTUM ℓ IS $\boxed{S_\ell = e^{2i\delta_\ell}}$

NOTICE THAT L^2 AND L_z ~~ARE EIG~~ COMMUTE WITH H, THUS THE S-MATRIX IS DIAGONAL IN THE ANGULAR MOMENTUM BASIS. THIS IS NOT THE CASE IN THE MOMENTUM BASIS SINCE $\vec{P}$ IS NOT CONSERVED

WE CAN ~~THEREFORE~~ DEFINE

$$f(\theta) = \sum_{\ell=0}^{\infty} (2\ell+1) f_\ell(k) P_\ell(\cos\theta)$$

SEE PAGE 19 $\Rightarrow f_\ell(k) = \frac{e^{i\delta_\ell}\sin\delta_\ell}{k} = \frac{e^{2i\delta_\ell} - 1}{2ik} \Rightarrow \boxed{S_\ell = 1 + 2ik f_\ell(k)}$

* IDENTIFYING THE TERM $\propto e^{ikr}$

$$\Rightarrow \sum_l c_l \frac{e^{i\delta_l} e^{-\frac{i\pi l}{2}}}{2ikr} Y_{l,0} = \frac{1}{(2\pi)^{3/2}} \left[\sum_l i^l \frac{\sqrt{4\pi(2l+1)}\, e^{-\frac{i\pi l}{2}}}{2ikr} Y_{l,0} + \frac{f(\theta)}{r} \right]$$

$$c_l \to \frac{e^{i\delta_l}}{(2\pi)^{3/2}} i^l \sqrt{4\pi(2l+1)}$$

$$\Rightarrow \sum_{l=0}^{\infty} \frac{\cancel{i^l} \sqrt{4\pi(2l+1)}\, \cancel{e^{-\frac{i\pi l}{2}}}}{2ik} \left(1 - e^{2i\delta_l} \right) Y_{l,0} + f(\theta) = 0$$

$$\Rightarrow \boxed{f(\theta) = \sum_{l=0}^{\infty} \frac{\sqrt{4\pi(2l+1)}}{k} \sin\delta_l\, e^{i\delta_l}\, Y_{l,0}(\theta)}$$

RECALLING THAT $P_l(\cos\theta) = \sqrt{\frac{4\pi}{2l+1}}\, Y_{l,0}(\theta)$

$$\Rightarrow \boxed{f(\theta) = \frac{1}{k} \sum_{l=0}^{\infty} (2l+1) e^{i\delta_l} \sin\delta_l\, P_l(\cos\theta)}$$

~~DIFF.~~ CROSS-SECTION : $\sigma(\theta) = \frac{1}{k^2} \left| \sum_{l=0}^{\infty} (2l+1) e^{i\delta_l} \sin\delta_l\, P_l(\cos\theta) \right|^2$

$$= k^{-2} \sum_{l,l'} (2l+1)(2l'+1) e^{i(\delta_l - \delta_{l'})} \sin\delta_l \sin\delta_{l'}\, P_l(\cos\theta)^* P_{l'}(\cos\theta)$$

TOTAL CROSS-SECTION

$$\sigma_{tot} = \int \sigma(\theta)\, d\Omega = \frac{2\pi}{k^2} \sum_{l,l'} (2l+1)(2l'+1) e^{i(\delta_l - \delta_{l'})} \sin\delta_l \sin\delta_{l'} \underbrace{\int P_l(\cos\theta) P_{l'}(\cos\theta)\, d(\cos\theta)}_{\frac{2}{2l+1}\delta_{l,l'}}$$

$$\Rightarrow \boxed{\sigma_{tot} = \frac{4\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) \sin^2\delta_l}$$

NOTICE THIS RESULT SATISFIES THE <u>OPTICAL THEOREM</u>

$$\operatorname{Im} f(\theta=0) = \frac{k}{4\pi}\sigma_{tot}$$

PROOF: $\operatorname{Im} f(0) = \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2\delta_\ell \, P_\ell(1)$

SINCE $P_\ell(x) = 2^\ell \sum_{k=0}^{\ell} x^k \binom{\ell}{k}\binom{\frac{\ell+k-1}{2}}{\ell}$

$$\Rightarrow P_\ell(1) = 2^\ell \sum_{k=0}^{\ell}\binom{\ell}{k}\binom{\frac{\ell+k-1}{2}}{\ell} = 2^\ell \times 2^{-\ell} = 1$$

ALTERNATIVELY, WE COULD USE THAT

$$(\ell+1)P_{\ell+1}(x) = (2\ell+1)\,x\,P_\ell(x) - \ell P_{\ell-1}(x)$$

SINCE $P_0(x) = 1$ AND $P_1(x) = x$

$\Rightarrow$ FOR $\ell=1 \rightarrow 2P_2(1) = 3-1 = 2 \Rightarrow P_2(1) = 1$

$\boxed{\text{INDUTIVE HYPOTHESIS } P_\ell(1) = 1 = P_{\ell-1}(1)} \Rightarrow (\ell+1)P_{\ell+1}(1) = 2\ell+1 - \ell = \ell+1$

$\Rightarrow \boxed{P_{\ell+1}(1) = 1}$

$$\Rightarrow \operatorname{Im} f(0) = \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2\delta_\ell = \frac{k}{4\pi}\left(\frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2\delta_\ell\right)$$

$$= \frac{k}{4\pi}\sigma_{tot}$$

<u>PARTIAL CROSS-SECTION</u> $\sigma_{tot} = \sum_\ell \sigma_{\ell,tot} \Rightarrow \sigma_{\ell,tot} = \left(\frac{4\pi}{k^2}\right)(2\ell+1)\sin^2\delta_\ell$

$$= (2\ell+1)\sin^2\delta_\ell \times 4\pi\lambda\!\!\!^{-2}$$

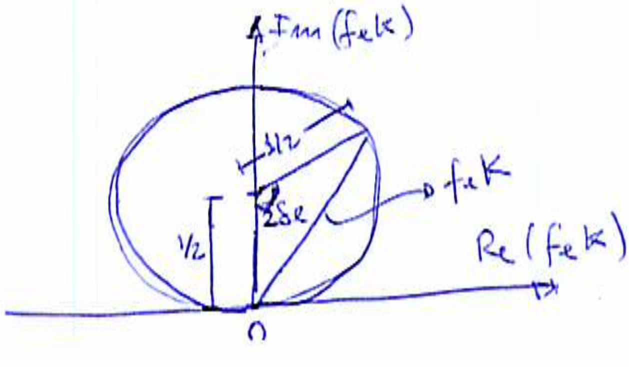
where $\lambda\!\!\!^{-} = \frac{1}{k}$. NOTICE THE MAXIMUM PARTIAL CROSS SECTION IS ACHIEVED WHEN $\delta_\ell = \pm\frac{\pi}{2}$, IMPLYING A MAXIMAL $|f_\ell k|$.

RECALL $f_\ell k = e^{i\delta_\ell}\sin\delta_\ell$

$$= \frac{1}{2i}\left(e^{2i\delta_\ell} - 1\right)$$

$$= \frac{\cos(2\delta_\ell) - 1 + i\sin(2\delta_\ell)}{2i}$$

$$= \frac{1}{2}\left(\sin(2\delta_\ell) + i\left(1-\cos(2\delta_\ell)\right)\right)$$



NOW, WE NEED TO COMPUTE THE PHASE SHIFTS (OTHERWISE, WE HAVE DONE NOTHING)

THIS IS NOT A SIMPLE TASK IN GENERAL. THUS, WE WILL FOCUS IN ONE EXAMPLE: HARD-SPHERE POTENTIAL

$$V(r) = \begin{cases} \infty, & r < a \\ 0, & r > a \end{cases}$$

AS DISCUSSED ON PAGE 17, $R_{k\ell}(r) = A_\ell\, j_\ell(kr) + B_\ell\, n_\ell(kr)$

FOR $r > a$, AND $R(r) = 0$, $r \leq a$

FROM THE CONTINUITY OF $R_{k\ell} \Rightarrow$ $\boxed{A_\ell\, j_\ell(ka) + B_\ell\, n_\ell(ka) = 0}$

THE ASYMPTOTIC BEHAVIOR OF $R_{k\ell} \xrightarrow{r\to\infty} \dfrac{A_\ell}{kr}\sin\left(kr - \dfrac{\ell\pi}{2}\right) - \dfrac{B_\ell}{kr}\cos\left(kr - \dfrac{\ell\pi}{2}\right)$

CALLING $A_\ell = C_\ell \cos\delta_\ell$

$B_\ell = -C_\ell \sin\delta_\ell$ $\Rightarrow R_{k\ell} \to \dfrac{C_\ell}{kr}\sin\left(kr - \dfrac{\ell\pi}{2} + \delta_\ell\right)$

THEREFORE, $\tan\delta_\ell = -\dfrac{B_\ell}{A_\ell} = \boxed{\dfrac{j_\ell(ka)}{n_\ell(ka)} = \tan\delta_\ell}$

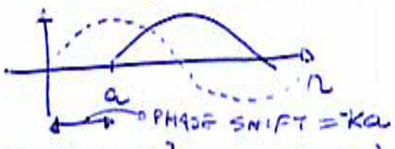
FOR SMALL ka $\Rightarrow$

$$j_\ell = \frac{(ka)^\ell}{(2\ell+1)!!}\left(1 - \frac{(ka)^2}{2(2\ell+3)} + O(ka)^4\right)$$

$$n_\ell = -\frac{(2\ell-1)!!}{(ka)^{\ell+1}}\left(1 + \frac{(ka)^2}{2(2\ell-1)} + O(ka)^4\right)$$

$$\Rightarrow \tan\delta_\ell \cong -\frac{(ka)^{2\ell+1}}{2\ell+1}\left(\frac{1}{(2\ell-1)!!}\right)^2$$

$\Rightarrow \delta_0 \approx -ka$



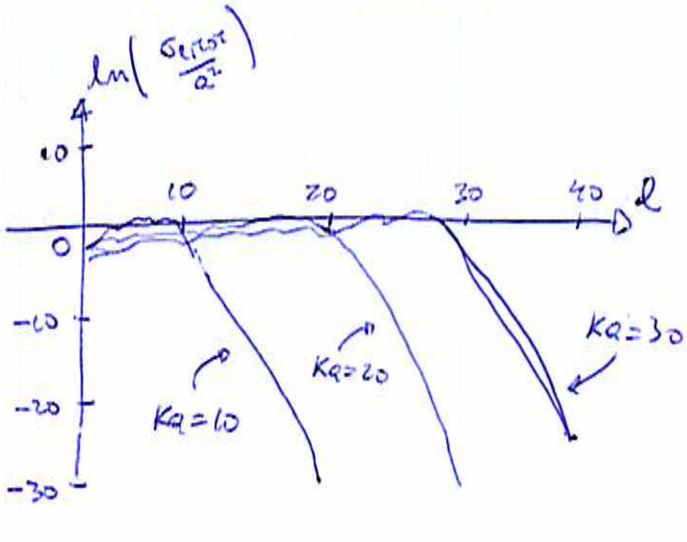
$$\Rightarrow \sigma_{TOT} \approx \sigma_{\ell=0,TOT} \cong \frac{4\pi}{k^2}(ka)^2 = 4(\pi a^2)$$

$$= 4 * (\text{CLASSICAL } \sigma_{TOT})$$

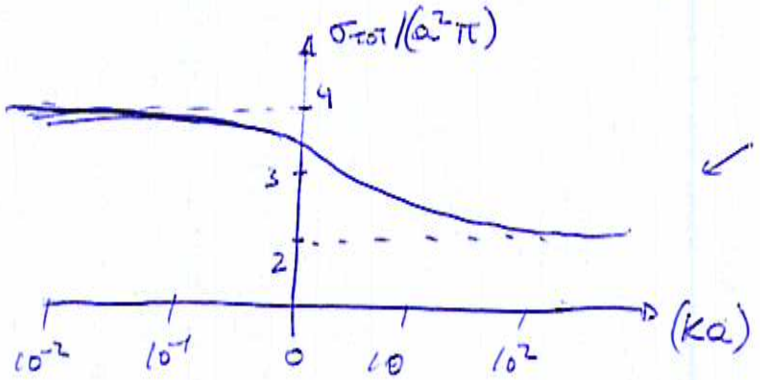
HOW MANY ℓ's WE HAVE TO KEEP?

APPROXIMATION IS GOOD UP TO $\ell \sim ka$, i.e., WE HAVE TO CONSIDER $\sigma_{\ell,TOT}$ UP TO $\ell \sim ka$

$$\sigma_{TOT} \approx \sum_{\ell=0}^{ka} \sigma_{\ell,TOT}$$



THIS RESULT, AGAIN, IS OKAY WITH THE INTUITION THAT FOR $b_\ell = \frac{L}{p} = \frac{\hbar\sqrt{\ell(\ell+1)}}{\hbar k} \sim \frac{\ell}{k}$ GREATER THAN THE RADIUS $a \Rightarrow \delta_\ell \simeq 0$ AND NOTHING HAPPENS FOR THOSE PARTIAL WAVES ($\ell > \ell_{max} \sim ka$)



FOR LARGE ka, IT IS NOT CONVERGING TO πa^2, BUT INSTEAD TO $2\pi a^2$

FOR LARGE ka $\rightarrow$ $j_\ell(ka) \approx \frac{1}{ka}\sin\left(ka - \frac{\ell\pi}{2}\right)$

$n_\ell(ka) = \frac{-1}{ka}\cos\left(ka - \frac{\ell\pi}{2}\right)$

$$\sin^2\delta_\ell = \frac{\tan^2\delta_\ell}{1+\tan^2\delta_\ell} = \frac{j_\ell^2}{j_\ell^2 + n_\ell^2} \approx \sin^2\left(ka - \ell\frac{\pi}{2}\right)$$

$$\Rightarrow \sin^2\delta_\ell + \sin^2\delta_{\ell+1} \simeq \sin^2\delta_\ell + \sin^2\left(\delta_\ell - \frac{\pi}{2}\right) = \sin^2\delta_\ell + \cos^2\delta_\ell = 1$$

$$\Rightarrow \sigma_{tot} = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2\delta_\ell = \frac{4\pi}{k^2}\Big(\sin^2\delta_0 + \sin^2\delta_1 + 2\sin^2\delta_1 + 2\sin^2\delta_2 + 3\sin^2\delta_2 + 3\sin^2\delta_3 + 4\sin^2\delta_3 + \cdots + (ka-1)\sin^2\delta_{ka-1} + ka\sin^2\delta_{ka-1} + ka\sin^2\delta_{ka} + (ka+1)\sin^2\delta_{ka}\Big)$$

$$\approx \frac{4\pi}{k^2}\sum_{\ell=0}^{ka}(2\ell+1)\sin^2\delta_\ell$$

$$\approx \frac{4\pi}{k^2}\left(1+2+3+4+\cdots+(ka-1)+ka+(ka+1)\sin^2\delta_{ka}\right)$$

$$\approx \frac{4\pi}{k^2}\times\frac{1}{2}(ka+2)ka$$

$$\boxed{\sigma_{tot} \approx 2\pi a^2}$$

NOT THE GEOMETRIC CROSS SECTION. WHY NOT?

$$f(\theta) = \frac{1}{k}\sum_{\ell=0}^{ka}(2\ell+1)e^{i\delta_\ell}\sin\delta_\ell P_\ell(\cos\theta) = \frac{1}{2ik}\sum_\ell(2\ell+1)e^{2i\delta_\ell}P_\ell + \frac{i}{2k}\sum_\ell(2\ell+1)P_\ell(\cos\theta)$$

$= f_{REFLECTION} + f_{SH\ldots}$

$$\sigma_{REF,TOT} \equiv \int |f_{REF}|^2 d\Omega$$

$$= \frac{1}{4k^2} \sum_{l=0}^{ka} (2l+1)^2 \underbrace{\int P_l^2(\cos\theta)\, d\Omega}_{\frac{4\pi}{2l+1}} = \frac{\pi}{k^2} \underbrace{\sum_{l=0}^{ka} (2l+1)}_{(ka+1)ka \,\simeq\, k^2a^2}$$

$$\Rightarrow \boxed{\sigma_{REF,TOT} = \pi a^2}$$

ANALOGOUSLY, $\sigma_{SHADOW,TOT} \equiv \int |f_{SHADOW}|^2 d\Omega$

with $f_{SHADOW} \approx \frac{i}{2k} \sum_l (2l+1) P_l(\cos\theta)$

NOTICE THAT f_{SHAD} ~~FED~~ IS PURELY IMAGINARY AND ADDS UP COHERENTLY FOR $\theta = 0$ $\boxed{\Rightarrow \text{BRIGHT SPOT BEHIND THE POTENTIAL}}$

FOR LARGE l AND SMALL θ

$$P_l(\cos\theta) \approx J_0(l\theta) \longrightarrow \text{BESSEL FUNCTION}$$

$$\Rightarrow f_{SHADOW} \approx \frac{i}{2k} \sum_{l=0}^{ka} (2l+1) J_0(l\theta) \xrightarrow{\text{LARGE } ka} \frac{i}{2k} \frac{1}{\Delta l} \int_0^{ka} dl\, 2l\, J_0(l\theta), \quad \Delta l = 1$$

USING THAT $b = \frac{l}{k}$ $\longrightarrow$

$$= \frac{i}{k} \cdot k^2 \int_0^a db\, b\, J_0(k\theta b) = ik \times \frac{1}{(k\theta)^2} \left(1 - \cos(k\theta a)\right)$$

$$= \frac{ia}{\theta} \frac{1 - \cos(k\theta a)}{k\theta a} \cong \frac{ia}{\theta} J_1(k\theta a)$$ $\boxed{\rightarrow \text{FRAUNHOFER DIFFRACTION PATTERN}}$

JUST AS $\sigma_{REF} \rightarrow \sigma_{SHADOW} = \frac{1}{4k^2} \sum_l (2l+1)^2 \int P_l^2(\cos\theta)\, d\Omega$

$$= \pi a^2$$

FINALLY, THE INTERFERENCE ~~RE~~ BETWEEN σ_{REF} AND σ_{SHAD} VANISHES $\mathrm{Re}(f^*_{SHAD} f_{REF}) \propto \mathrm{Re}\left(\sum_l e^{2i\delta_l}\right) \rightarrow 0$

BECAUSE $2\delta_{l+1} = 2\delta_l - \pi \Rightarrow$ STRONG OSCILLATIONS AVERAGING TO ZERO

FINALLY, NOTICE THAT

$$\sigma_{TOT} = \frac{4\pi}{k} \operatorname{Im}\left(f(0)\right) = \frac{4\pi}{k} \operatorname{Im}\left(f_{SHADOW}(0)\right) \quad \text{BECAUSE}$$

$\operatorname{Im} f_{REF}(0) \propto \operatorname{Im}\left(\sum_{\ell} e^{2i\delta_\ell}\right)$ which STRONGLY OSCILLATES AVERAGING TO ZERO.

AND $\operatorname{Im}\left(f_{SHADOW}(0)\right) = \frac{1}{2k} \sum_{\ell=0}^{ka} 2\ell+1 \cong \frac{1}{2k} \cdot 2 \frac{(ka)^2}{2}$

$$\Rightarrow \sigma_{TOT} = \frac{4\pi}{k} \cdot \frac{1}{2} k a^2 = \underline{2\pi a^2 = \sigma_{TOT}}$$