TIME-DEPENDENT PERTURBATION THEORY

WE ARE INTERESTED IN PROBLEMS SUCH THAT

- $H = H_0 + V(t) = H(t)$

THE FORMAL SOLUTION IS VIA THE TIME DEVELOPMENT OPERATOR $U(t,t_0)$ SINCE $|\psi_s(t)\rangle = U(t,t_0)\,|\psi_s(t_0)\rangle$
(SUBINDEX S DENOTES THE SCHRÖDINGER REPRESENTATION)
AND $U(t,t_0)$ IS DETERMINED BY $H(t)$:

$$i\hbar \frac{d}{dt} U(t,t_0) = H(t)\,U(t,t_0) \Rightarrow U(t,t_0) = \mathbb{1} + \frac{1}{i\hbar}\int_{t_0}^{t} dt'\, H(t')\,U(t',t_0)$$

WHERE $U(t_0,t_0) = \mathbb{1}$

THIS EQUATION CAN BE SOLVED ITERATIVELY:

$$U(t,t_0) = \mathbb{1} + \frac{1}{i\hbar}\int_{t_0}^{t} dt_1\, H(t_1) + \left(\frac{1}{i\hbar}\right)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, H(t_1)\,H(t_2) + \cdots$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{i\hbar}\right)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n\, H(t_1)\,H(t_2)\cdots H(t_n)$$

* NOTE THAT $t > t_1 > t_2 > \cdots > t_n > t_0$ AND $[H(t_i), H(t_j)] \neq 0$

THERE IS A COMPACT WAY OF WRITING THE ABOVE RESULT

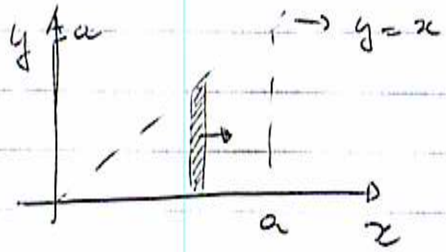
$$U(t,t_0) = T\left\{ \exp\left(\frac{1}{i\hbar}\int_{t_0}^{t} H(t')\,dt' \right) \right\}$$

WHERE $T \equiv$ TIME ORDERING OPERATOR

EXAMPLE:
$$T\{H(t_1)\,H(t_2)\} = \begin{cases} H(t_1)\,H(t_2), & \text{if } t_1 > t_2 \\ H(t_2)\,H(t_1), & \text{if } t_1 \leq t_2 \end{cases}$$
$$= \Theta(t_1-t_2)\,H(t_1)\,H(t_2) + \Theta(t_2-t_1)\,H(t_2)\,H(t_1)$$

HOW DO WE ARRIVE AT THIS RESULT ?

CONSIDER $I = \int_0^a dx \int_0^x dy \; f(x,y)$



$$\rightarrow \int_0^a dx \int_0^x dy \; f(x,y)$$

$$\rightarrow \int_0^a dy \int_y^a dx \; f(x,y)$$

$$I = \frac{1}{2} \int_0^a dx \int_0^x dy \; f(x,y) + \frac{1}{2} \int_0^a dy \int_y^a dx \; f(x,y) \quad x \leftrightarrow y$$

$$= \frac{1}{2} \int_0^a dx \int_0^x dy \; f(x,y) + \frac{1}{2} \int_0^a dx \int_x^a dy \; f(y,x)$$

$$= \frac{1}{2} \int_0^a dx \int_0^a dy \; \theta(x-y) f(x,y) + \theta(y-x) f(y,x)$$

$$= T \left\{ \frac{1}{2} \int_0^a dx \int_0^a dy \; f(x,y) \right\}$$

SAME FOR $f(x,y,z)$ AND SO ON

$$\Rightarrow \sum_{n=0}^{\infty} \int_0^x dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{n-1}} dx_n \; f(x_1, x_2, \ldots, x_n)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^x dx_1 \cdots \int_0^x dx_n \; T\{ f(x_1 \ldots x_n) \}$$

FOR $f(x_1, \ldots, x_n) = f(x_1) \ldots f(x_n)$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} T\left(\int_0^x dx' \, f(x') \right)^n = T\left\{ \sum_{n=0}^{\infty} \frac{1}{n!} \left[\int_0^x f(x') \, dx' \right]^n \right\}$$

$$= T \left\{ e^{\int_0^x f(x') \, dx'} \right\}$$

USUALLY, IT IS HARD TO INTEGRATE (OR IMPOSSIBLE) ANALITICALLY (OR EVEN NUMERICALLY) $\int_{t_0}^{t} dt'\, H(t')$ ③

AND, AT THE SAME TIME, PERFORM THE TIME ORDERING

WHAT DO WE DO? ANSWER: PERTURBATION THEORY

- EXPAND $U(t, t_0)$ UP TO SOME ORDER
- IN MANY CASES $H = H_0 + V(t)$ WITH $V(t) \ll H_0$ (GAPS IN THE SPECTRUM OF $V(t)$ $\ll$ GAPS OF SPECTRUM OF H_0)

$\Rightarrow$ TREAT $V(t)$ PERTURBATIVELY

IN OTHER WORDS, WE INVESTIGATE HOW THE DYNAMICS OF A SYSTEM GOVERNED BY H_0 IS AFFECTED BY $V(t)$

IN THIS WAY, THE INTERACTION REPRESENTATION IS VERY CONVENIENT, THE DYNAMICS OF H_0 IS EASILY INCORPORATED IN THE OPERATORS WHILE THE EFFECTS OF $V(t)$ IS ENCODED IN THE STATE VECTORS WHICH WE OBTAIN PERTURBATIVELY.

RECALL: $i\hbar \frac{d}{dt} |\psi_S\rangle = (H_0 + V)|\psi_S\rangle$, $|\psi_I(t)\rangle = e^{i\frac{H_0 t}{\hbar}} |\psi_S(t)\rangle = T_0^{+}(t)|\psi_S(t)\rangle$

$i\hbar \frac{d}{dt} U_I(t, t_0) = V_I(t)\, U_I(t, t_0)$, $V_I = T_0^{+}(t)\, V\, T_0(t)$

$|\psi_I(t)\rangle = U_I(t, t_0)\, |\psi_I(t_0)\rangle$

THUS, $U_I(t, t_0) = \mathbb{1} + \frac{1}{i\hbar}\int_{t_0}^{t} dt'\, V_I(t')\, U_I(t', t_0)$

RECURSIVELY, $$\boxed{U_I(t, t_0) = \mathbb{1} + \frac{1}{i\hbar}\int_{t_0}^{t} dt_1\, V_I(t_1) + \left(\frac{1}{i\hbar}\right)^2 \int_{0}^{t} dt_1 \int_{0}^{t_1} dt_2\, V_I(t_1)\, V_I(t_2) + \dots}$$

THIS EQUATION IS THE BASIS OF OUR FORMALISM FOR THE PERTURBATION THEORY

WE WILL BE INTERESTED IN THE TRANSITION AMPLITUDES BETWEEN STATES OF H_0:

$H_0 |m\rangle = E_m |m\rangle \longrightarrow |m(t)\rangle = U_S(t, t_0)\, |m(t_0)\rangle$
$= U_S(t, t_0)\, |m\rangle$

TRANSITION AMPLITUDE : $\langle n | m(t)\rangle = \langle n | U_S(t, t_0) | m\rangle$
$= C_{n,m}$

(4)

USING THAT $U_S(t,t_0) = T_0(t)\, U_I(t,t_0)\, T_0^{+}(t_0)$

$$= e^{-\frac{i}{\hbar}H_0 t}\, U_I(t,t_0)\, e^{\frac{i}{\hbar}H_0 t_0}$$

$$\Rightarrow C_{m,n} = \langle m | T_0(t) U_I(t,t_0) T_0^{+}(t_0)\, n \rangle$$

$$C_{m,n} = \langle m | T_0(t) T_0^{+}(t_0) | n \rangle + \frac{1}{i\hbar}\int_{t_0}^{t} dt_1 \langle m | T_0(t) V_I(t_1) T_0^{+}(t_0) | n \rangle + \cdots$$

$$= \langle m | U_0(t,t_0) | n \rangle + \frac{1}{i\hbar}\int_{t_0}^{t} dt_1 \langle m | T_0(t) T_0^{+}(t_1) V(t_1) T_0(t_1) T_0^{+}(t_0) | n \rangle + \cdots$$

$$= \langle m | U_0(t,t_0) | n \rangle + \frac{1}{i\hbar}\int_{t_0}^{t} dt_1 \langle m | U_0(t,t_1) V(t_1) U_0(t_1,t_0) | n \rangle + \left(\frac{1}{i\hbar}\right)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \langle m | U_0(t,t_1) V(t_1) U_0(t_1,t_2) V(t_2) U_0(t_2,t_0) | n \rangle + \cdots$$

$$\Rightarrow C_{m,n} = \langle m | U_0(t,t_0) + \sum_{n=1}^{\infty} \int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t_{n-1}} dt_n\, U_0(t,t_1) V(t_1) U_0(t_1,t_2) V(t_2) \cdots U_0(t_{n-1},t_n) V(t_n) U_0(t_n,t_0) | n \rangle$$

VARIOUS ORDERS OF PERTURBATION:

0th: $$C^{(0)}_{m,n} = \delta_{m,n}\, e^{-\frac{i}{\hbar}E_m (t-t_0)} = \delta_{m,n}\, \overbrace{e^{-\frac{iE_m t}{\hbar}}\, e^{\frac{iE_m t_0}{\hbar}}}^{\text{PHASE}}$$

1st: $$C^{(1)}_{m,n} = \frac{1}{i\hbar}\int dt_1 \langle m | U_0(t,t_1) V(t_1) \sum_k |k\rangle\langle k| U_0(t_1,t_0) | n \rangle$$

$$= \frac{1}{i\hbar}\sum_k \int dt_1\, e^{-\frac{iE_m}{\hbar}(t-t_1)} V_{mk}(t_1)\, e^{-\frac{iE_k}{\hbar}(t_1-t_0)}\, \delta_{k,n}$$

$$= \underbrace{e^{-\frac{iE_m t}{\hbar}}\, e^{\frac{iE_m t_0}{\hbar}}}_{\text{SAME PHASE}} * \underbrace{\frac{1}{i\hbar}\int_{t_0}^{t} dt_1\, e^{i\omega_{mn} t_1} V_{mn}(t_1)}_{\text{FOURIER TRANSFORM OF } V_{mn}}$$

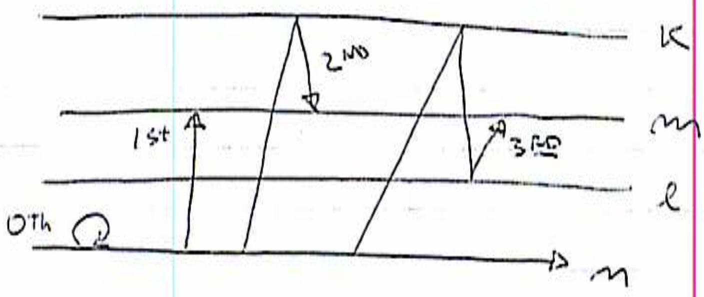
WHERE $\hbar\omega_{mn} = E_m - E_n$

2ND $= C^{(2)}_{m,n} = \left(\frac{1}{i\hbar}\right)^2 \int dt_1\, dt_2 \sum_{k\ell} e^{-\frac{iE_m}{\hbar}(t-t_1)} V_{mk}(t_1)\, e^{-\frac{iE_k}{\hbar}(t_1-t_2)} V_{k\ell}(t_2)\, e^{-\frac{iE_\ell}{\hbar}(t_2-t_0)}\, \delta_{\ell n}$

$$= \underbrace{e^{-\frac{iE_m t}{\hbar}}\, e^{\frac{iE_m t_0}{\hbar}}}_{\text{SAME PHASE}} * \left(\frac{1}{i\hbar}\right)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \sum_k e^{i\omega_{mk} t_1} V_{mk}(t_1)\, e^{i\omega_{kn} t_2} V_{kn}(t_2)$$

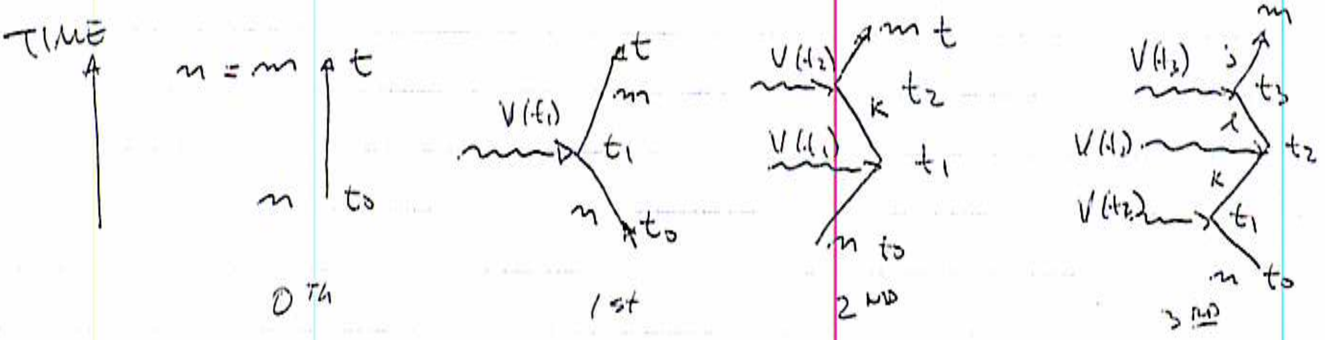
GRAPHIC INTERPRETATION (DIAGRAMS)

A SIMPLE GRAPHIC VISUALIZATION REFERS TO THE STRING OF OPERATORS THAT APPEARS IN THE VARIOUS ORDERS OF PERT. THEORY : $V_{mk} V_{kl} V_{lj} \cdots V_{in}$



VIRTUAL TRANSITIONS BEFORE REACHING THE FINAL ~~STATE~~ STATE

ANOTHER, AND MORE INTUITIVE, DESCRIPTION IS VIA DIAGRAMS



$\rightsquigarrow$ = INTERACTION (CAN CARRY ENERGY AND MOMENTUM)

$\longrightarrow$ = "FREE" PROPAGATION (ACCORDING TO H_0)
UNPERTURBED

THUS, IT IS LIKE THE SYSTEM'S DYNAMICS IS DICTATED BY H_0 AND OCCASIONALLY, V ENTERS IN THE GAME (ON INSTANTS t_1, t_2, t_3 --)

- THE GENERAL PROBLEM ONE MAY BE CONCERNED IS THE TRANSITION FROM A GENERIC STATE $|\psi_0\rangle = \sum_i \alpha_i |i\rangle$ AT $t = t_0$ TO ANOTHER STATE $|\phi\rangle = \sum_j \beta_j |j\rangle$ AT TIME $t = t$. THEN ONE HAS TO GO OVER THE SIMPLE ALGEBRA

$$\langle \phi | U_s | \psi_0 \rangle = \sum_{ij} \alpha_i \beta_j^* \langle j | U_s | i \rangle = \sum_{ij} \alpha_i \beta_j^* c_{ji}$$

WHERE THE COEFFICIENTS $c_{j,i}$ ARE CALCULATED IN ORDERS OF PERTURBATION

EXAMPLE: TIME INDEPENDENT PERTURBATION

$$V(t) = V\,\Theta(t) \rightarrow$$ PERTURBATION IS SWITCHED ON INSTANTLY AT $t=0$

TRANSITION PROBABILITY $= P_{m,n}(t) = |C_{m,n}|^2$

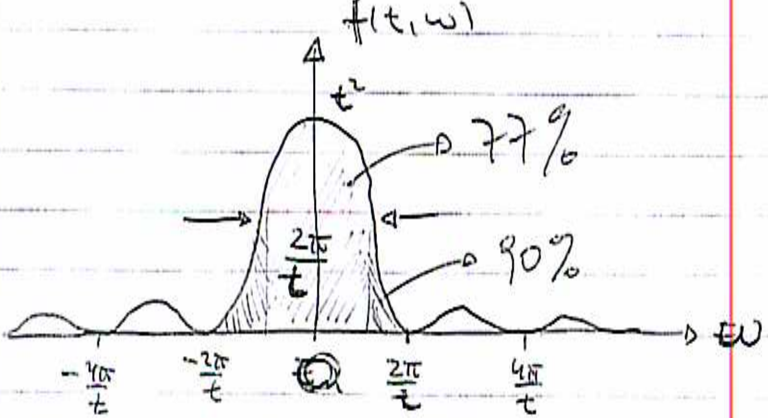
UP TO 1ST ORDER $\Rightarrow$ $P_{m,n} = \left| \delta_{m,n} + \frac{1}{i\hbar}\int_0^t dt_1\, e^{i\omega_{mn} t_1} V_{mn} \right|^2$

FOR $m \neq n$, IT SIMPLIFIES TO

$$P_{m,n} = \underbrace{\frac{1}{\hbar^2}\left|\langle m|V|n\rangle\right|^2}_{\text{YIELDS TO SELECTION RULES}} * \left| \frac{1}{i\omega_{mn}}\left(e^{i\omega_{mn}t} - 1\right)\right|^2 = \frac{4|V_{mn}|^2}{(\hbar\omega_{mn})^2}\left(\sin\frac{\omega_{mn}t}{2}\right)^2$$

$$= \frac{1}{\hbar^2}|V_{mn}|^2 f(t,\omega_{mn})$$

LET US ANALYZE $f(t,\omega) = \dfrac{4\sin^2(\omega t/2)}{\omega^2}$



NOTICE THAT $\lim\limits_{t\to\infty} f(t,\omega) = t^2$

WIDTH:

~~$(\Delta\omega)$ IS SUCH THAT $f(t,\frac{\Delta\omega}{2}) \approx t^2/2$~~

SINCE THE 1st MINIMUM IS OF ORDER $2\pi/t$ $\Rightarrow$ WIDTH $\approx \dfrac{2\pi}{t}$

SINCE THE PEAK GROWS $\sim t^2$ AND THE WIDTH DIMINISHES $\sim 1/t$ $\Rightarrow$ THE AREA OF THE CENTRAL GROWS $\sim t$ $\Rightarrow$ FOR LARGE t, ALL TRANSITIONS REMAIN CLOSE TO THE INITIAL STATE

PRECISELY $\displaystyle\int_{-\frac{2\pi}{t}}^{\frac{2\pi}{t}} \frac{4\sin^2(\omega t/2)}{\omega^2}\,d\omega \approx 4.86\,t$ / $\displaystyle\int_{-\frac{2\pi}{t}}^{\frac{2\pi}{t}} f(t,\omega)\,d\omega = 5.6t$

NOTICE THAT $\displaystyle\int_{-\infty}^{\infty} \frac{4\sin^2(\frac{\omega t}{2})}{\omega^2}\,d\omega = 2\pi t = 6.28\,t$ $\qquad \dfrac{5.6t}{6.28} \approx 0.9$

$\Rightarrow \dfrac{4.86\,t}{6.28\,t} \simeq 0.77 \Rightarrow$ 77% OF THE TRANSITION STATES ARE WITHIN $-\frac{2\pi}{t} < \omega < \frac{2\pi}{t}$

CONCLUSIONS:

(a) FOR $\omega t \ll 1 \Rightarrow P_{m,n} \sim t^2$ (GROWS RAPIDLY)

(b) THE MOST IMPORTANT TRANSITIONS ARE SUCH THAT

$$|\omega_{mn}| \lesssim \frac{2\pi}{t} \Rightarrow \boxed{|E_m - E_n|\, t \lesssim 2\pi\hbar} \to \Delta E \Delta t \gtrsim 2\pi\hbar = h$$

(WE ENSURE THE CENTRAL PEAK IS CAPTURE)

LOOKS LIKE AN INCERTAINTY RELATION

BUT RECALL IT IS NOT. t IS NOT AN OPERATOR

$\Delta E \equiv$ UNCERTAINTY ON THE SYSTEM ENERGY

$t \equiv \Delta t \equiv$ TIME THAT THE SYSTEM IS UNDER V

(c) SYSTEM'S ENERGY IS CONSERVED WITHIN THE ERROR $\frac{2\pi\hbar}{t}$

(THE ERROR IS DUE TO THE INTERACTION V WHICH IS CAUSED BY AN EXTERNAL AGENT)

REWRITE $P_{m,n} = \dfrac{4|V_{mn}|^2}{(\hbar\omega_{mn}t)^2}\, t^2 \sin^2\left(\dfrac{\omega_{mn}t}{2}\right)$, CALL $\theta = \dfrac{\omega_{mn}t}{2}$

$$\Rightarrow \boxed{P_{mn} = t^2 \left|\frac{V_{mn}}{\hbar}\right|^2 \left(\frac{\sin\theta}{\theta}\right)^2}$$

* LIMIT CASE $\theta \ll 1 \Rightarrow |\omega_{mn}t| \ll 1 \Rightarrow P_{mn} \simeq \left|\frac{V_{mn}}{\hbar}\right|^2 t^2$

TRANSITION RATE (IN 1st ORDER) $\Gamma_{mn} = \dfrac{P_{m,n}}{t} = \left|\dfrac{V_{mn}}{\hbar}\right|^2 t$

$\Rightarrow$ TRANSITION RATE GROWS LINEARLY IN TIME

* LIMIT CASE $\theta \gg 1 \Rightarrow |\omega_{mn}t| \gg 1$,

THE MAJORITY OF TRANSITIONS ARE SUCH THAT $|E_m - E_n| < \dfrac{2\pi\hbar}{t}$

THIS IS VALID AS LONG AS $P_{m,n} \ll 1$, OTHERWISE HIGHER ORDER CONTRIBUTIONS ARE NEEDED

WHILE THIS LIMIT IS VALID, THEN

WE USE THAT $$\frac{1}{\omega_{mn}^2 t}\sin^2\left(\frac{\omega_{mn}t}{2}\right) \xrightarrow{t\to\infty} \frac{\pi}{2}\delta(\omega_{mn})$$

THIS COMES FROM $$\int_{-\infty}^{\infty}\frac{\sin^2 x}{x^2}\,dx = \pi$$

$$\Rightarrow \lim_{\alpha\to\infty}\frac{1}{\pi}\frac{\sin^2 \alpha x}{\alpha x^2} = \delta(x)$$

$$\Rightarrow P_{m,n} = 4\left|\frac{V_{mn}}{\hbar}\right|^2\left(\frac{\sin^2 \omega_{mn}t/2}{\omega_{mn} t}\right)t \longrightarrow 2\pi\left|\frac{V_{mn}}{\hbar}\right|^2\delta(\omega_{mn})\,t$$

$$= \Gamma_{mn}\,t \quad , \quad \Gamma_{mn} = \frac{2\pi}{\hbar^2}|V_{mn}|^2\delta(\omega_m-\omega_n)$$

$$\boxed{\Gamma_{m,n} = \frac{2\pi}{\hbar}|V_{m,n}|^2\delta(E_m-E_n)}$$

APPLICATION: DECAY TO A CONTINUUM (AND AUGER EFFECT)

SUPPOSE THERE IS A CONTINUUM OF STATES IN WHICH V CONNECTS THE INITIAL STATE $|n\rangle$

QUESTION: WHAT IS THE DECAY PROBABILITY?

$$P_n(t) = \sum_m P_{mn}(t)$$

FOR A CONTINUUM $\sum_m \longrightarrow \int dE_m\,\rho(E_m)$

↳ DENSITY OF STATES

$$\Rightarrow P_n(t) \longrightarrow \int dE_m\,\rho(E_m)\,\frac{2\pi}{\hbar}|V_{mn}|^2\delta(E_m-E_n)\,t$$

$$= \frac{2\pi}{\hbar}|V_{mn}|^2\rho(E_n)\,t = \Gamma_n\,t$$ → LINEAR IN t

(RECALL THE FIRST PEAK INTEGRAL $\sim t^2\times\frac{1}{t}\sim t$)

$$\boxed{\Gamma_n \equiv \text{TRANSITION RATE} = \frac{2\pi}{\hbar}|\langle m|V|n\rangle|^2\rho(E_n)}$$

FERMI'S GOLDEN RULE

"FORMAL" DERIVATION :

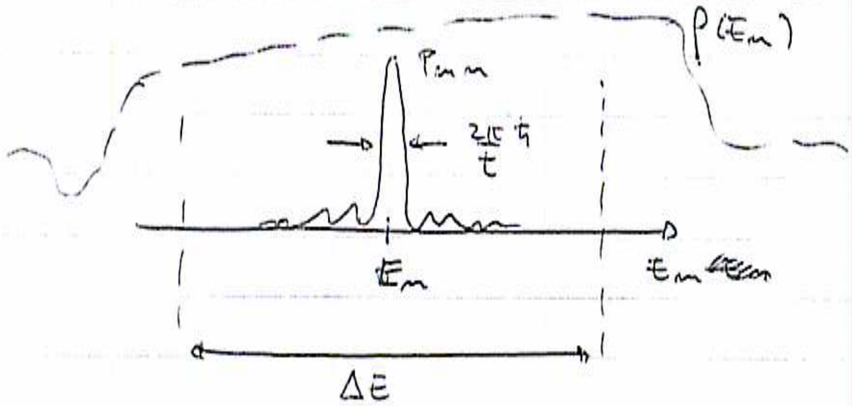
$$P_m = \int dE_m \, \rho(E_m) \frac{|V_{mn}|^2}{(E_m - E_n)^2} 4 \sin^2\left(\frac{(E_m - E_n) t}{2\hbar}\right)$$

ASSUMPTIONS :
- $\rho(E_m)$ VARIES SLOWLY AROUND THE FINAL STATES
- $|V_{mn}|^2$ VARIES WEAKLY ACCROSS THE FINAL STATES

$$\Rightarrow P_m \approx \rho \, |V_{mn}|^2 \int_{-\infty}^{\infty} dE_m \frac{4 \sin^2(\;)}{(E_m - E_n)^2}$$

$\hookrightarrow$ THE LIMITS OF INTEGRATION ARE OK AS LONG AS $\frac{\sin^2}{\Delta E^2}$ VANISHES SUFFICIENTLY RAPID

$$\Rightarrow P_m(t) = \rho \, |V_{mn}|^2 \cdot 2\pi \frac{t}{\hbar}$$



VALIDITY : ~~1ST ORDER VALID WHEN $\left|\frac{V_{mn}}{E_m - E_n}\right|^2 \ll 1$~~

① $P_m(t) = \Gamma_m t \quad \Rightarrow \quad t \ll \frac{1}{\Gamma_m}$ = "LIFE TIME"

② $\frac{2\pi\hbar}{t} \gg \delta E \quad \Rightarrow \quad t \ll \frac{h}{\delta E}$ (CONTINUUM LIMIT EXISTS)

③ $\Delta E \gg \Gamma_m \hbar$ (SHARP TRANSITION)

IT ENSURES THE SYSTEM GOES TO THE CONTINUUM AND DO NOT CAMES BACK TO $|n\rangle$

ANOTHER DERIVATION

$$\Gamma_n = \frac{d}{dt} P_n(t) = \frac{d}{dt} \int dE_m \, \rho(E_m) \frac{1}{\hbar^2} \left| \int_0^t V_{mn} e^{i\omega_{mn}t'} dt' \right|^2$$

$$= \frac{d}{dt} \int dE_m \frac{\rho(E_m)}{\hbar^2} \int_0^t dt_1 \int_0^t dt_2 \, \underset{(t_1)}{\langle n|V|m\rangle} \underset{(t_2)}{\langle m|V|n\rangle} e^{i\omega_{mn}(t_1 - t_2)}$$

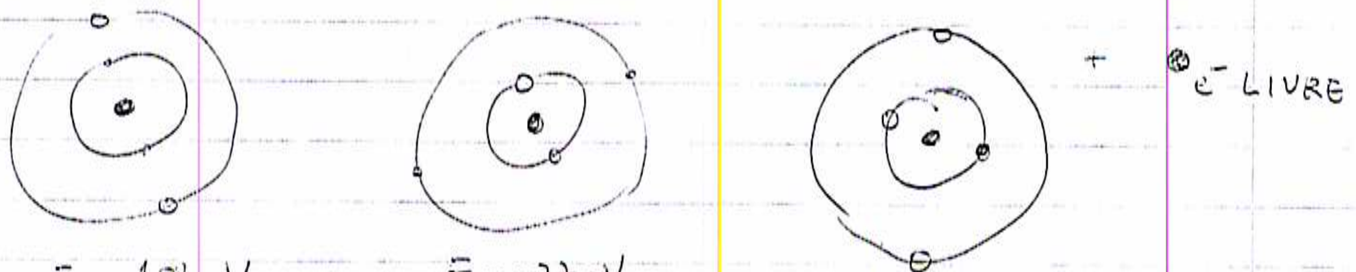
FOR $\rho(E_m)$ AND $\langle m|V|n\rangle$ SLOWLY VARYING WITH RESPECT TO $e^{i\omega_{mn}(t_1-t_2)}$ $\Rightarrow$ INTEGRATE OVER dE_m

$$\int_{-\infty}^{\infty} \frac{dE_m}{\hbar} e^{i\frac{E_m - E_n}{\hbar}(t_1 - t_2)} = \int_{-\infty}^{\infty} dx \, e^{ix(t_1-t_2)} = 2\pi\delta(t_1 - t_2)$$

$$\Rightarrow \Gamma_n = \frac{d}{dt} \int_0^t dt_1 \left| \langle m | \overset{\text{DOES NOT DEPEND ON } t}{\widehat{V(t_1)}} | n\rangle \right|^2 \frac{2\pi}{\hbar} \rho(E_m)$$

$$\Rightarrow \boxed{\Gamma_n = \frac{2\pi}{\hbar} \left| \langle m|V|n\rangle \right|^2 \rho(E_m)}$$

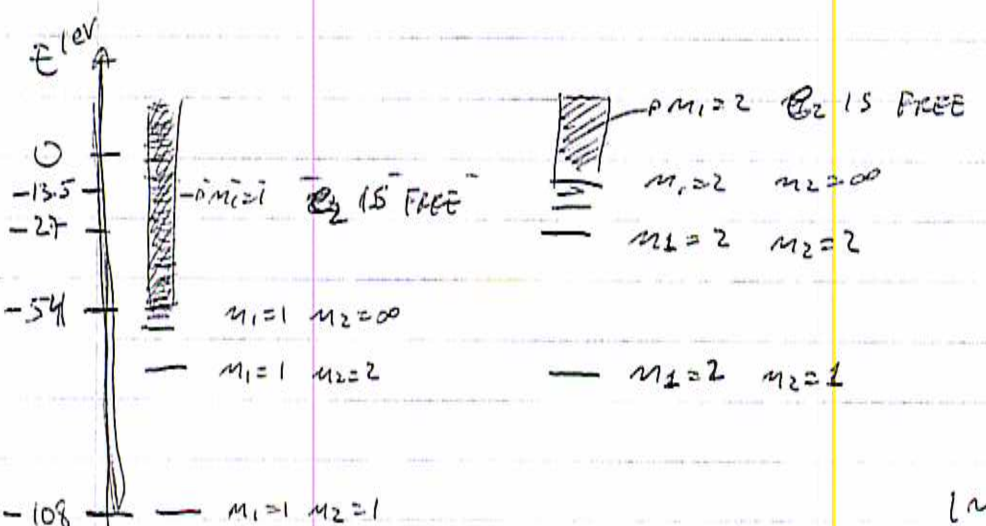
AUGER EFFECT FOR He



$E = -108\,eV$ — $n_1 = n_2 = 1$

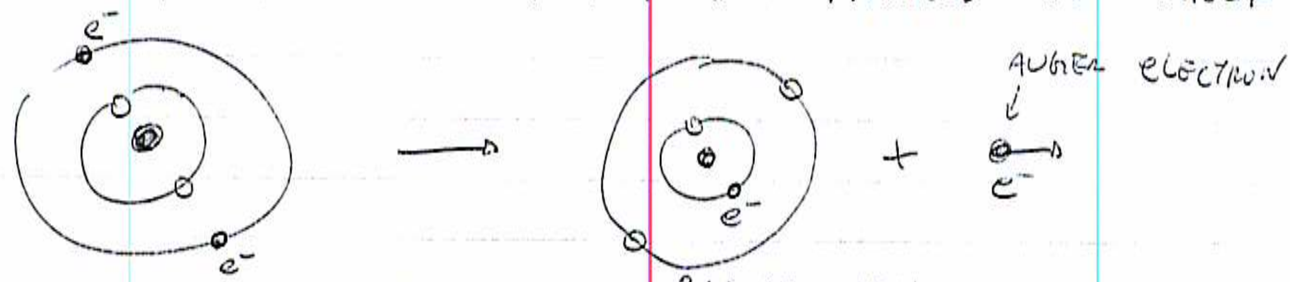
$E = -27\,eV$ — $n_1 = n_2 = 2$

$E = -54\,eV$ — $n_1 = 1$, $n_2 = \infty$



NOTICE THAT THE STATE $|n_1 = 2, n_2 = 2\rangle = |n\rangle$ CAN DECAY INTO A CONTINUUM IN WHICH $|m\rangle = |n_1 = 1\rangle \otimes \underbrace{|k_2\rangle}_{\text{PLANE WAVES}}$

THEREFORE, THERE IS A PHYSICAL PROCESS IN WHICH



RADIATIONLESS EMISSION OF ELECTRON (AUTOIONIZING OR AUGER TRANSITION)

$$H_0 = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} - \frac{2e^2}{r_1} - \frac{2e^2}{r_2}$$

$$V = \frac{e^2}{|\vec{r}_1 - \vec{r}_2|}$$ → COULOMB REPULSION IS SUFFICIENT TO PROMOTE THE DECAY

THE TRANSITION $|2s, 2s\rangle \to |1s, 2s\rangle + \gamma$
$\to |1s, 1s\rangle + 2\gamma$
ARE POSSIBLE, BUT MUCH LESS PROBABLE

LATER, WE WILL COME BACK TO THE CALCULATION OF THE TRANSITION RATE $|2s, 2s\rangle \to |1s\rangle \otimes |\vec{k}_e\rangle$

NOW, WE WANT TO GENERALIZE THE FERMI'S GOLDEN RULE TO PERTURBATION THAT DEPEND ON TIME IN A HARMONIC FASHION

$$V(t) = V_0 \cos(\omega t + \phi)\,\theta(t) = \frac{1}{2}\left(V_0 e^{i\omega t} + V_0^+ e^{-i\omega t}\right)\theta(t)$$

1st ORDER: $$P_{mn}(t) = \frac{|\langle m|V_0|n\rangle|^2}{\hbar^2} \left| \int_0^t \frac{e^{i(\omega+\omega_{mn})t'} + e^{i(\omega_{mn}-\omega)t'}}{2} dt' \right|$$

(CROSS TERM WILL GIVE $\delta(\omega_{mn}-\omega)\,\delta(\omega_{mn}+\omega)$ → VERY SMALL) $$= \frac{2|V^0_{mn}|^2}{\hbar^2}\left[\frac{\sin^2(\omega_{mn}+\omega)\frac{t}{2}}{(\omega_{mn}+\omega)^2} + \frac{\sin^2(\omega_{mn}-\omega)\frac{t}{2}}{(\omega_{mn}-\omega)^2}\right]$$

SAME ANALYSIS AS BEFORE BUT WE HAVE TO TAKE INTO ACCOUNT TWO PEAK AT
$\omega_m = \omega_n + \omega$ AND $\omega_m = \omega_n - \omega$

$$I = \int_0^t e^{iAt} \pm e^{iBt}\, dt = \frac{1}{iA}\left(e^{iAt}-1\right) \pm \frac{1}{iB}\left(e^{iBt}-1\right)$$

$$|I|^2 = \left| 2e^{i\frac{At}{2}} \frac{\sin(At/2)}{A} \pm 2e^{i\frac{Bt}{2}} \frac{\sin Bt/2}{B} \right|^2$$

$$= \frac{4\sin^2\frac{At}{2}}{A^2} + \frac{4\sin^2\frac{Bt}{2}}{B^2} \pm \frac{8\sin\frac{At}{2}\sin\frac{Bt}{2}\cos\frac{(A-B)t}{2}}{AB}$$

$$|e^{i\theta}A \pm e^{i\gamma}B|^2 = |A\cos\theta \pm B\cos\gamma + i(A\sin\theta \pm B\sin\gamma)|^2$$

$$= \underbrace{\begin{matrix} A^2\cos^2\theta + B^2\cos^2\gamma \\ + A^2\sin^2\theta + B^2\sin^2\gamma \end{matrix}}_{A^2+B^2} \pm \underbrace{2AB(\cos\theta\cos\gamma + \sin\theta\sin\gamma)}_{}$$

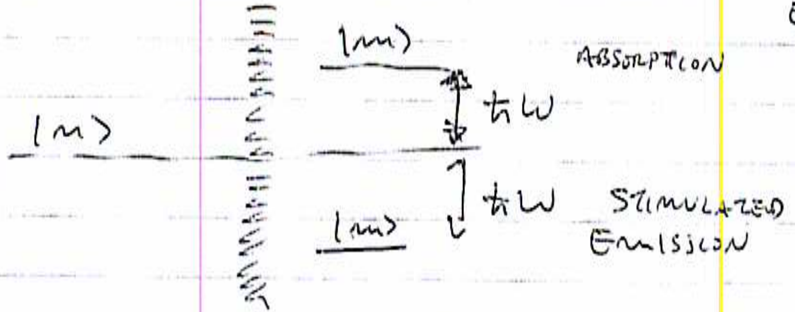
$$A^2 + B^2 \pm 2AB\cos(\theta-\gamma)$$

$$\frac{\sin^2(\omega_{mn}+\omega)\frac{t}{2}}{(\omega_{mn}+\omega)^2} + \frac{\sin^2(\omega_{mn}-\omega)\frac{t}{2}}{(\omega_{mn}-\omega)^2} \pm \frac{2\sin(\omega_{mn}+\omega)\frac{t}{2}\sin(\omega_{mn}-\omega)\frac{t}{2}\cos\omega t}{\omega_{mn}^2-\omega^2}$$

FOR TRANSITIONS INTO THE CONTINUUM LIMIT

$$\Rightarrow \quad \Gamma_n = \frac{\pi}{\hbar} |\langle m | V_0 | n \rangle|^2 \left\{ \rho(E_n - \omega) + \rho(E_n + \omega) \right\}$$

EMISSION   ABSORPTION



EXAMPLE : PHOTO ELECTRIC EFFECT (WE WILL COME BACK TO THIS LATER

EXPONENTIAL DECAY

FOR STOCHASTIC PROCESSES, WE CAN COMPUTE THE PROBABILITY THAT THE SYSTEM IS IN STATE $|n\rangle$ GIVEN THAT IT'S DECAY RATE IS Γ ~~[illegible]~~

$$P_n(t+dt) = \underbrace{P_n(t)(1-\Gamma_n dt)}_{\text{PROBABILITY OF NOT DECAYING}} + \underbrace{\sum_{m \neq n} P_m(t)\, \Gamma_{mn}\, dt}_{\text{PROBABILITY TO RETURN TO } |n\rangle}$$

$$\Rightarrow \quad P_n(t+dt) = \cancel{P_n(t)} + \frac{dP_n}{dt} dt = \cancel{P_n(t)} - P_n(t)\Gamma_n dt$$

(LET'S NEGLECT IT, IRREVERSIBLE DECAY)

$$\Rightarrow \quad \frac{dP_n}{dt} = -P_n \Gamma_n \quad \Rightarrow \quad \boxed{P_n(t) = P_n(0)\, e^{-\Gamma_n t}}$$

EXPONENTIAL DECAY LAW

OBSERVATIONS: (a) THE FACT THE STATE $|n\rangle$ IS NOT REPLENISHED IS NECESSARY
HOW DOES IT APPLY TO Q.M. PROBLEMS?
(b) IT IS ALSO NECESSARY THAT THE DETERMINATION (OR MEASUREMENT) ~~[illegible]~~ OF P_n AT TIME t DOES NOT AFFECT P_n AT $t+dt$.

OF COURSE, OBSERVATION (b) IS NOT SATISFIED IN Q.M. SINCE ~~A~~ MEASUREMENT OF THE SYSTEM IS INDEED IN $|m\rangle$ NECESSARILY IMPLIES THAT $P_S(t) = 1$ (THIS IS DUE TO THE WAVE FUNCTION COLLAPSE)

NATURALLY, IT APPLIES TO Q.M. PROBLEMS IF A MEASUREMENT BETWEEN THE INITIAL AND FINAL TIMES IS NOT PERFORMED.

IRREVERSIBLE DECAY AND Q.M

IF IRREV. DECAY IS TO TAKE PLACE, THEN

$$P_m(t+dt) = P_m(t)\,(1-\Gamma_m dt)$$

$$\Rightarrow \quad |\langle m|\, U_I(t+dt, t_0)\,|m\rangle|^2 = |\langle m|\, U_I(t,t_0)\,|m\rangle|^2\,(1-\Gamma_m\, dt)$$

THERE IS NO REASON, A PRIORI, THAT THIS SHOULD HOLD HOWEVER, THE SAME CONDITIONS THAT ENSURE THE FERMI'S GOLDEN RULE ENSURE THIS RELATION

IN ORDER TO SEE THIS, LET'S FOCUS ON

$$i\hbar \frac{d}{dt}\langle m|\, U_I(t,t_0)\,|m\rangle = i\hbar\langle m|\,\frac{d}{dt}U_I\,|m\rangle = \langle m|\, V_I(t)\, U_I(t,t_0)\,|m\rangle$$
$$= \sum_k \langle m|\, V_I(t)\,|k\rangle\langle k|\, U_I(t,t_0)\,|m\rangle$$
$$= \sum_k e^{i\omega_{km}}\langle m|\, V(t)\,|k\rangle\langle k|\, U_I(t,t_0)\,|m\rangle$$

① FOR $k \neq m$ → GREATEST CONTRIBUTION IS FROM $k=m$ BECAUSE $\langle k\neq m|\, U_I(t,t_0)\,|m\rangle \ll \langle m|\, U_I(t,t_0)\,|m\rangle$ FOR SHORT TIMES

$$\Rightarrow \quad i\hbar\frac{d}{dt}\langle m|\, U_I(t,t_0)\,|m\rangle \simeq e^{i\omega_{mm}t}\cdot\langle m|\, V(t)\,|m\rangle\langle m|\, U_I(t,t_0)\,|m\rangle$$

FOR CONSTANT IN TIME $V(t) \equiv V$

$$\Rightarrow \quad \langle m|\, U_I(t,t_0)\,|m\rangle = \frac{1}{i\hbar}\langle m|V|m\rangle\int_{t_0}^{t} dt'\, e^{i\omega_{mm}t'}\langle m|\, U_I(t',t_0)\,|m\rangle$$

② FOR $m=n$

$$\Rightarrow \quad i\hbar\frac{d}{dt}\langle m|U_I|m\rangle = \langle m|V|m\rangle\langle m|U_I|m\rangle + \sum_{k\neq m} e^{i\omega_{km}t}\langle m|V|k\rangle\langle k|\, U_I|m\rangle$$

NOW USE THE RESULT OF ①

$$\Rightarrow i\hbar\frac{d}{dt}\langle m|U_I|m\rangle = \langle m|V|m\rangle\langle m|U_I|m\rangle - \frac{1}{i\hbar}\sum_{k\neq m}\langle m|V|k\rangle\langle k|V|m\rangle\int_{t_0}^{t} dt'\, e^{i\omega_{km}(t'-t)} \times \langle m|\, U_I(t',t_0)\,|m\rangle$$

CASE $\langle m|U_I(t',t_0)|m\rangle$ VARIES SLOWLY COMPARED TO $e^{i\omega_{km}(t'-t_0)}$ IN THE TIME INTERVAL $t_0<t'<t$

$$\Rightarrow i\hbar \frac{d}{dt}\langle m|U_I|m\rangle \simeq \left[\langle m|V|m\rangle + \frac{1}{i\hbar}\sum_{k\neq m}|\langle k|V|m\rangle|^2 \int_{t_0}^{t} dt'\, e^{i\omega_{km}(t'-t)}\right]\langle m|U_I|m\rangle$$

THIS ASSUMPTION IS TRUE WHEN $|\omega_{km}(t'-t)| \gg 1$, IN WHICH WE USE

$$\cancel{I = \int_{t_0}^{t} dt'\, e^{i\omega_{km}(t'-t)} \longrightarrow \pi\delta(\omega_{km}) + i\,\mathcal{P}\left(\frac{1}{\omega}\right)}$$

$$I = \int_{t_0}^{t} dt'\, e^{i\omega_{km}(t'-t)} = \int_{t_0-t}^{0} dx\, e^{i\omega_{km}x} \longrightarrow \int_{-\infty}^{0} dx\, e^{i\omega x}$$

$$\lim_{\epsilon\to 0^+}\int_{-\infty}^{0} dx\, e^{i(\omega - i\epsilon)x} = \lim_{\epsilon\to 0^+}\frac{1}{i\omega+\epsilon} = \pi\delta(\omega) - i\,\mathcal{P}\left(\frac{1}{\omega}\right)$$

CAUCHY'S PRINCIPAL VALUE

$$\Rightarrow i\hbar\frac{d}{dt}\langle m|U_I|m\rangle = \left[\langle m|V|m\rangle + \frac{1}{i\hbar}\sum_{k\neq m}|\langle k|V|m\rangle|^2\left(\pi\delta(\omega_{km}) - i\,\mathcal{P}(\omega_{km})\right)\right]\langle m|U_I|m\rangle$$

$$\boxed{\begin{aligned}\Rightarrow \frac{d}{dt}\langle m|U_I|m\rangle &= \left[-\frac{\pi}{\hbar^2}\sum_{k\neq m}|V_{km}|^2\delta(\omega_{km}) - \frac{i}{\hbar}\left(\langle m|V|m\rangle + \sum_{k\neq m}|V_{km}|^2\frac{1}{\omega_{km}}\right)\right]\langle m|U_I|m\rangle \\ &= \left[-\frac{W}{2} - \frac{i}{\hbar}\Delta E_m\right]\langle m|U_I|m\rangle\end{aligned}}$$

THIS LOOKS LIKE $dP_m(t) = (1-\Gamma_m\, dt)\,P_m(t)$

SOLVING : $\langle m|U_I(t,t_0)|m\rangle = e^{-\left(\frac{W}{2} + i\frac{\Delta E_m}{\hbar}\right)t}$

WHERE $W = \frac{2\pi}{\hbar^2}\sum_{k\neq m}|V_{mk}|^2\delta(\omega_{km}) = \frac{2\pi}{\hbar}\sum_{k\neq m}|V_{km}|^2\delta(E_{km})$

$$\longrightarrow \frac{2\pi}{\hbar}|\langle k|V|m\rangle|^2\rho(E_m)$$ FERMI'S GOLDEN RULE

$$\Delta E_m = \sum_{k\neq m}\frac{|V_{km}|^2}{\hbar\omega_{km}} + V_{mm} = V_{mm} + \sum_{k\neq m}\frac{|\langle k|V|m\rangle|^2}{E_k - E_m}$$

$\equiv$ ENERGY SHIFT OF STATE $|m\rangle$ UP TO 2ND ORDER IN PERT. THEORY

FINALLY, $P_{mm}(t) = |\langle m|U_I|m\rangle|^2 = e^{-Wt}$

THAT'S THE EXPONENTIAL DECAY OF AN UNSTABLE STATE

EXAMPLE: RADIOACTIVE EXPONENTIAL DECAY.
IT IS TRUE THAT WE NEED Q.M. IN ORDER TO HAVE TUNNELING, AND THUS, THE DECAY. HOWEVER THE EXPONENTIAL DECAY IS A CONSEQUENCE THAT THE INITIAL STATE IS COUPLED TO MANY (A LARGE NUMBER) OF FINAL (TARGET) STATES WITH SIMILAR ENERGY.
IN THIS MANNER, THE EXPONENTIAL DECAY IS POSSIBLE DUE TO A SERIES OF DELICATE APPROXIMATIONS.

COMMING BACK TO THE $n \neq m$ CASE

$$\langle n|U_I|m\rangle \simeq \frac{1}{i\hbar}\langle n|V|m\rangle \int_{t_0}^{t} dt'\, e^{i\omega_{nm}t'} \langle m|U_I|m\rangle$$

$$= \frac{1}{i\hbar} V_{nm} \int dt'\, e^{i\omega_{nm}t'}\, e^{-\left(\frac{W}{2}+i\frac{\Delta E_m}{\hbar}\right)t'}$$

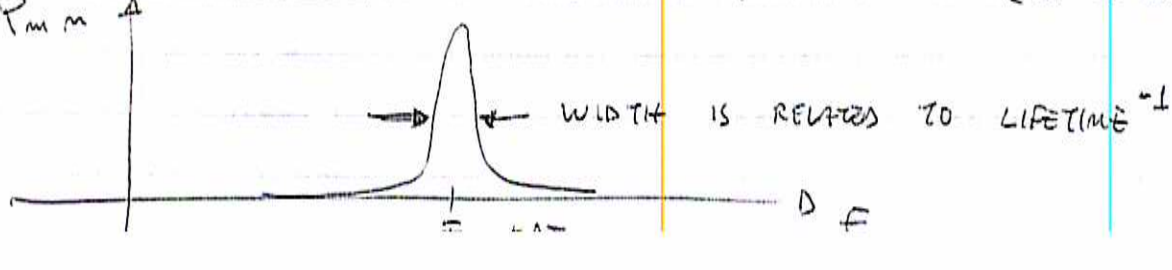
$$= \frac{\frac{1}{i\hbar} V_{nm}\left[e^{-\frac{Wt}{2}}\, e^{\frac{-i}{\hbar}(E_m+\Delta E_m-E_n)t} - 1\right]}{-\frac{W}{2} - \frac{i}{\hbar}(E_m+\Delta E_m - E_n)}$$

$$\Rightarrow P_{nm}(t) = \frac{|\langle n|V|m\rangle|^2}{\hbar^2} * \left(\frac{1 - 2\cos\Omega_{nm}t\; e^{-\frac{Wt}{2}} + e^{-Wt}}{\Omega_{nm}^2 + W^2/4} \right)$$

Where $\hbar\Omega_{nm} = E_m + \Delta E_m - E_n$ = ENERGY DIFFERENCE

FOR LONG TIMES $Wt \gg 1$

$$\Rightarrow P_{nm} \longrightarrow \left|\frac{\langle n|V|m\rangle}{\hbar}\right|^2 * \frac{1}{\frac{W^2}{4} + \Omega_{nm}^2} = \frac{|\langle n|V|m\rangle|^2}{(E_m+\Delta E_m - E_n)^2 + \frac{(\hbar W)^2}{4}}$$



~~PARAD~~ ZENO'S ~~PARADOX~~

- HOW DOES A STATE DECAY IF IT IS CONTINUOUSLY MEASURED ?

$$P_N(t) \equiv \left(|\langle \psi(0) | \psi(t/N) \rangle|^2 \right)^N \equiv$$ PROBABILITY OF BEING AT $|\psi\rangle$ AFTER N MEASUREMENTS EQUALLY SPACED IN THE TIME INTERVAL $[0,t]$

↳ N MEASUREMENT IN THE INTERVAL $[0,t]$

$$= \left(\langle \psi_0 | U(t/N) | \psi(0) \rangle \langle \psi(0) | U^\dagger(t/N) | \psi(0) \rangle \right)^N$$

AS $U(t/N) = \mathbb{1} + \frac{1}{i\hbar} \frac{t}{N} H + \frac{1}{2}\left(\frac{1}{i\hbar}\right)^2 \left(\frac{t}{N}\right)^2 H^2 + \dots$

↓ RECALL THE TIME ORDERING

FOR $\frac{t}{N} << 1$

$$\Rightarrow P_N \approx \left(\left| 1 + \frac{1}{i\hbar}\left(\frac{t}{N}\right) \langle \psi_0 | H | \psi_0 \rangle + \frac{1}{2}\left(\frac{1}{i\hbar}\right)^2 \left(\frac{t}{N}\right)^2 \langle \psi_0 | H^2 | \psi_0 \rangle + \dots \right|^2 \right)^N$$

$$= \left(1 + \frac{1}{\hbar^2} \langle H \rangle^2 \left(\frac{t}{N}\right)^2 - 2 * \frac{1}{2} \frac{1}{\hbar^2} \langle H^2 \rangle \left(\frac{t}{N}\right)^2 + \dots \right)^N \equiv$$ CUMULANT EXPANTION

(NOTICE THE LINEAR TERM VANISHES $\frac{1}{i\hbar} \langle H \rangle \left(\frac{t}{N}\right) + \left(\frac{1}{i\hbar} \langle H \rangle \frac{t}{N}\right)^* = 0$)

$$\Rightarrow P_N \approx 1 - N \frac{\langle \Delta H \rangle^2}{\hbar^2} \left(\frac{t}{N}\right)^2 + O\left(\frac{t}{N^3}\right) , \quad \langle \Delta H \rangle^2 = \langle H^2 \rangle - \langle H \rangle^2$$ = 2ND CUMULANT

$$\Rightarrow \lim_{N \to \infty} P_N = 1 - \frac{(\Delta H)^2}{\hbar^2} \frac{t^2}{N} \to \underline{1}$$

$\Rightarrow$ THE STATE DOES <u>NOT</u> DECAY (NO PERTURBATION THEORY NEEDED)

* IN CLASSICAL MECHANICS

$$P_N(t) = \left(\langle \psi_0 | \psi\left(\tfrac{t}{N}\right) \rangle \right)^N$$, NO SQUARE NEEDED SINCE $\langle \psi | \psi \rangle$ IS ALREADY THE PROBABILITY

$$\Rightarrow P_N = \left(1 + \frac{d\psi}{d\epsilon} \frac{t}{N} + \dots \right)^N \underset{\epsilon = \frac{t}{N}}{\approx} 1 + \left(\frac{d\psi}{d\epsilon}\right)^* t < 1$$

↓ $\psi'(0) < 0$

$\Rightarrow$ STATE DECAYS

EXAMPLE (APLICATION):

~~ELECTRON~~ PARTICLE IN A FIELD (NEGLECT SPIN)

$$H = \frac{1}{2m}\left(\vec{P} - q\vec{A}\right) + q\phi$$

COULOMB GAUGE : $\phi = 0$, $\nabla \cdot \vec{A} = 0 \Rightarrow \nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = 0$

$$\Rightarrow H = \frac{P^2}{2m} - \frac{q}{2m}\left(\vec{P}\cdot\vec{A} + \vec{A}\cdot\vec{P}\right) + \frac{q^2}{2m}A^2$$

IN THIS GAUGE $[\vec{P}, \vec{A}] = 0$

$$\vec{P}\cdot\vec{A}|\psi\rangle = -i\hbar\nabla\cdot(\vec{A}|\psi\rangle) = -i\hbar\left[(\nabla\cdot\vec{A})|\psi\rangle + \vec{A}\cdot\nabla|\psi\rangle\right] = \vec{A}\cdot(-i\hbar\nabla|\psi\rangle)$$
$$= \vec{A}\cdot\vec{P}|\psi\rangle$$

$$\Rightarrow H = \frac{P^2}{2m} - \frac{q}{m}\vec{A}\cdot\vec{P} + \underbrace{O(A^2)}_{\to 0 \text{ (SMALL)}}$$

PARTICLE IN A E.M. FIELD + ~~CENTRAL~~ EXTERNAL POTENTIAL

$$\Rightarrow H = \underbrace{\frac{P^2}{2m} + V_0}_{\text{HYDROGEN ATOM, FOR INSTANCE}} - \frac{q}{m}\vec{A}\cdot\vec{P}$$

DEFINING THE VECTOR POTENTIAL (RADIATION FIELD)

$$A(\vec{r},t) = \int_{-\infty}^{\infty} d\omega' \, \vec{A}(\omega') \, e^{-i\omega'\left(t - \frac{\hat{n}\cdot\vec{r}}{c}\right)}$$

where $\begin{cases} \hat{n} \equiv \text{DIRECTION OF PROPAGATION} \\ \vec{A}\cdot\hat{n} = 0 \end{cases}$

$$A(\vec{r},t) \in \mathbb{R} \Rightarrow \vec{A}^*(\omega') = \vec{A}(-\omega')$$

EXAMPLE : $\hat{n} = \hat{z}$, $\vec{A}(\omega') = A_0\left[\delta(\omega' - \omega) + \delta(\omega' + \omega)\right]\hat{x}$

$$\Rightarrow \begin{cases} \vec{E}(\vec{r},t) = -\frac{\partial}{\partial t}\vec{A} = \omega A_0 \sin\left(\omega t - \frac{\omega}{c}z\right)\hat{x} \\ \vec{B}(\vec{r},t) = \nabla\times\vec{A} = \frac{\omega}{c}A_0 \sin\left(\omega t - \frac{\omega}{c}z\right)\hat{y} \end{cases}$$

POYNTING VECTOR: $\vec{N} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{-1}{\mu_0}\frac{\partial}{\partial t}\vec{A}\times(\nabla\times\vec{A}) = \frac{\omega^2 A_0^2}{\mu_0 c}\sin^2(\ldots)\hat{z}$

TRANSITION PROBABILITY

$$C_{fi} = \frac{\langle f | V_I(-\infty, \infty) | i \rangle}{i\hbar}$$

1st ORDER IN PERTURBATION THEORY

$$= \frac{1}{i\hbar} \int_{-\infty}^{\infty} dt \; V_{fi}(t) \, e^{i\omega_{fi} t}$$

$\equiv$ FOURIER TRANSFORM OF THE PERTURBATION

WITH $V(t) = -\frac{q}{m} \int_{-\infty}^{\infty} d\omega' \, e^{-i\omega'(t - \frac{\hat{n}\cdot\vec{r}}{c})} \vec{A}(\omega') \cdot \vec{p}$

INTEGRATING OVER TIME:

$$\int_{-\infty}^{\infty} dt \; e^{i(\omega_{fi} - \omega')t} = 2\pi \, \delta(\omega_{fi} - \omega')$$

$$\Rightarrow C_{fi} = \frac{-2\pi q}{i m \hbar} \langle f | e^{i\omega_{fi} \frac{\hat{n}\cdot\vec{r}}{c}} \vec{A}(\omega_{fi}) \cdot \vec{p} \, | i \rangle$$

LET $\vec{A}(\omega_{fi}) = A(\omega_{fi}) \, \hat{e}$, WITH $\hat{e} \cdot \hat{n} = 0$

$$\Rightarrow \boxed{P_{f \leftarrow i} = \frac{4\pi^2 q^2}{m^2 \hbar^2} |A(\omega_{fi})|^2 \; |\langle f | e^{i\omega_{fi} \frac{\hat{n}\cdot\vec{r}}{c}} \, \hat{p} \cdot \hat{e} \, | i \rangle|^2}$$

THIS CAN BE RELATED TO THE POYNTING VECTOR

$$\vec{N} = -\frac{1}{\mu_0} \left(\frac{\partial}{\partial t} \vec{A} \right) \times \left(\nabla \times \vec{A} \right)$$

$$= -\frac{1}{\mu_0} \left[\int_{-\infty}^{\infty} d\omega' (-i\omega') A(\omega') \, \hat{e} \, e^{-i\omega'(t - \frac{\hat{n}\cdot\vec{r}}{c})} \right] \times \left[\int_{-\infty}^{\infty} d\omega'' \left(\frac{i\omega''}{c} \right) A(\omega'') e^{-i\omega''(t - \frac{\hat{n}\cdot\vec{r}}{c})} \hat{n} \times \hat{e} \right]$$

where WE USED THAT $\nabla \times (f \vec{C}) = f \, \nabla \times \vec{C} + \nabla f \times \vec{C}$

NOW WE USE $\hat{e} \times (\hat{n} \times \hat{e}) = (\hat{e} \cdot \hat{e}) \hat{n} - (\hat{e} \cdot \hat{n}) \hat{e} = \hat{n}$

$$\Rightarrow \vec{N} = -\frac{\hat{n}}{\mu_0 c} \int_{-\infty}^{\infty} d\omega' d\omega'' \; A(\omega') A(\omega'') \, e^{-i(\omega' + \omega'')(t - \frac{\hat{n}\cdot\vec{r}}{c})} \omega' \omega''$$

RADIATION PRESSURE $\equiv \frac{\vec{N} \cdot \hat{n}}{c} = \frac{\text{ENERGY}}{\text{VOLUME}}$

DEFINITION: $c \int_{-\infty}^{\infty} \frac{\vec{N} \cdot \hat{n}}{c} dt \equiv \int_{0}^{\infty} \tilde{N}(\omega) \, d\omega$

WHAT IS $\tilde{N}(\omega)$?

$$[\tilde{N}] = \frac{\text{ENERGY}}{\text{VOLUME}} \times \frac{\text{LENGTH}}{\text{TIME}} \times \frac{\text{TIME}}{\text{FREQUENCY}} = \frac{\text{ENERGY}}{\text{AREA} \times \text{FREQUENCY}}$$

$$\int_{-\infty}^{\infty} N(\omega)\, d\omega = \frac{-1}{\mu_0 c} \int_{-\infty}^{\infty} d\omega' d\omega''\, A(\omega') A(\omega'')\, \omega'\omega'' \underbrace{\int_{-\infty}^{\infty} dt\, e^{-i(\omega'+\omega'')(t - \frac{\hat{n}\cdot\vec{x}}{c})}}_{2\pi\delta(\omega'+\omega'')}$$

$$= \frac{2\pi}{\mu_0 c} \int_{-\infty}^{\infty} d\omega\, A(\omega) A(-\omega)\, \omega^2$$

AS $A(\omega) = A^*(-\omega)$

$$\Rightarrow \boxed{N(\omega) = \frac{4\pi\omega^2 |A(\omega)|^2}{\mu_0 c}}$$

THUS, $P_{f\leftarrow i} = \frac{\pi q^2}{m^2\hbar^2} \mu_0 c \frac{N(\omega)}{\omega^2} |\langle f| e^{i\omega_{fi} \frac{\hat{n}\cdot\vec{x}}{c}} \vec{p}\cdot\hat{e}|i\rangle|^2$

ABSORBING CROSS SECTION: $\int_0^{\infty} \sigma(\omega) N(\omega)\, d\omega \equiv$ ABSORBED ENERGY

$$[\sigma(\omega)] \equiv \frac{\text{ABSORBED ENERGY}}{\text{INCIDENT ENERGY / AREA}}, \quad N(\omega) \equiv \frac{\text{INCIDENT ENERGY}}{\text{AREA} \times \text{FREQ}}$$

$$\Rightarrow \int_0^{\infty} \sigma(\omega) N(\omega)\, d\omega = \int_0^{\infty} \underbrace{\hbar\omega_{fi}}_{\text{INCIDENT ENERGY}}\ \underbrace{P_{f\leftarrow i}}_{\text{ABSOPTION PROBABILITY}}\, d\omega_{fi}$$

$$\Rightarrow \sigma(\omega) = \frac{\pi q^2 \mu_0 c}{m^2 \hbar^2} \frac{\hbar\omega}{\omega^2} |\langle f| e^{i\omega \frac{\hat{n}\cdot\vec{x}}{c}} \vec{p}\cdot\hat{e}|i\rangle|^2$$

FINE STRUCTURE CONSTANT $\alpha \equiv \frac{q^2}{4\pi\epsilon_0} \times \frac{1}{\hbar c} \approx \frac{1}{137} = \frac{1}{137.035999074(44)}$

$$\Rightarrow \boxed{\sigma(\omega)_{fi} = \frac{4\pi^2 \alpha}{m^2 \omega_{fi}} \left| \langle f| e^{i\omega_{fi} \frac{\hat{n}\cdot\vec{x}}{c}} \vec{p}\cdot\hat{e}|i\rangle \right|^2}$$

COMPUTING THE MATRIX ELEMENT

WE NEED TO EXPAND $e^{i\frac{\omega_{fi}}{c}\hat{m}\cdot\vec{r}} = 1 + i\omega_{fi}\frac{\vec{m}\cdot\vec{r}}{c} + \cdots$

TYPICALLY, $\frac{\omega_{fi}}{c}\hat{m}\cdot\vec{r} = \frac{\text{IONIZATION FREQ} \times \text{ATOM RADIUS}}{c}$

$$= \frac{1}{\hbar}\left(\frac{Z^2 q^2}{4\pi\epsilon_0(2a_0)}\right)\times\left(\frac{a_0}{Zc}\right), \quad a_0 = \frac{4\pi\epsilon_0\hbar^2}{mq^2} = \frac{\hbar}{mc\alpha}$$

$$= \frac{Z}{2}\frac{q^2}{4\pi\epsilon_0\hbar c} = \frac{1}{2}Z\alpha \equiv \frac{Z}{2\times 137}$$

$\Rightarrow$ THE SERIES EXPANSION CONVERGES AS LONG AS $Z \ll 137$

1st TERM: ELECTRIC DIPOLE

$\langle f|\vec{p}|i\rangle\cdot\hat{e}$. RECALL THAT $H_0 = \frac{p^2}{2m} + V(\vec{r})$

$\Rightarrow [\vec{r}, H_0] = \frac{1}{2m}[\vec{r}, p^2] = i\hbar/m\,\vec{p}$

$\Rightarrow \langle f|\vec{p}|i\rangle = \frac{m}{i\hbar}\langle f|[\vec{r}, H_0]|i\rangle = \frac{m}{i}\omega_{if}\langle f|\vec{r}|i\rangle = im\omega_{fi}\langle f|\vec{r}|i\rangle$

THEREFORE, ONLY TRANSITIONS IN WHICH $\langle f|\vec{r}|i\rangle \neq 0$ ARE ALLOWED (SELECTION RULES)

IN GENERAL, THE SELECTION RULES COMES FROM INTEGRALS OF TYPE $\int_{-\infty}^{\infty}\psi_f^* O \psi_i \, dr$, WITH O BEING THE "TRANSITION" OPERATOR

IT IS THEN USEFUL TO USE THE SYMMETRIES OF $\psi_{i,f}$ IN ORDER TO KNOW THE ALLOWED TRANSITIONS

(21)

EXAMPLE: INITIAL STATE = $|i\rangle = |1s\rangle$

FINAL STATE = $|f\rangle = |n\, l\, m\rangle$ = GENERIC STATE OF HYDROGEN ATOM

$$\langle \vec{r}\,|1s\rangle = \frac{1}{\sqrt{\pi}}\left(\frac{1}{a_0}\right)^{\frac{3}{2}} e^{-r/a_0} = \langle \vec{r}\,|100\rangle$$

$$\langle \vec{r}\,|n l m\rangle = \left[\left(\frac{2}{n a_0}\right)^3 \frac{(n-l-1)!}{2n\,(n+l)!}\right]^{1/2} e^{-\frac{1}{2}\rho}\, L^{2l+1}_{n-l-1}(\rho)\, Y_l^m(\theta,\varphi)$$

where $\rho = \dfrac{2r}{n a_0}$

SELECTION RULES:

LET $\underset{\substack{\downarrow \\ \text{ELECTRIC FIELD POLARIZATION}}}{\vec{r}\cdot\hat{e}} = z = r\cos\theta = r\sqrt{\frac{4\pi}{3}}\,Y_{10}(\theta,\varphi)$

$$\Rightarrow \langle n\, l, m\,|\,z\,|\,100\rangle = \underset{\substack{\downarrow \\ \text{RADIAL INTEGRAL}}}{\#} \int d\Omega\; Y_l^{m*}(\theta,\varphi)\, Y_1^0(\theta,\varphi)\, \overbrace{Y_0^0(\theta,\varphi)}^{1}$$

USING THE ORTHOGONALITY PROPERTIES OF Y_l^m, WE CONCLUDE THAT ONLY WHEN $l=1$ AND $m=0$, THE TRANSITION MATRIX IS DIFFERENT FROM ZERO

IF WE HAD CHOSEN $\hat{e} = \hat{x} \Rightarrow \vec{r}\cdot\hat{e} = x = r\sin\theta\cos\varphi$
$= \#\, r\,(Y_1^{+1} + Y_1^{-1})$

$\Rightarrow$ ALLOWED TRANSITIONS WOULD BE $l=1$ AND $m=\pm 1$

(SAME FOR $\hat{y} = \hat{e}$, $y = r\sin\theta\sin\varphi \sim Y_1^{+1} - Y_1^{-1}$

IN GENERAL THE DIPOLE ELECTRIC TRANSITIONS ARE SUCH THAT $\Delta l = \pm 1$ AND $\Delta m = -1, 0, +1$

ANOTHER EXAMPLE: AUGER EFFECT

$$H = \underbrace{\frac{1}{2m}P_1^2 + \frac{1}{2m}P_2^2 + V(1-\text{NUCLEOUS}) + V(2-\text{NUCLEOUS}) + \frac{P^2_{\text{NUCLEOUS}}}{2M}}_{H_0} + V(1-2)$$

$$V(1-2) = \frac{e^2}{4\pi\epsilon_0}\frac{1}{|\vec{r}_1-\vec{r}_2|} = \frac{e^2}{4\pi\epsilon_0} * 4\pi\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}\frac{1}{2l+1}\sum_{m=-l}^{l} Y_l^{m*}(\hat{r}_1)\,Y_l^m(\hat{r}_2)$$

INITIAL STATE = $|i\rangle = |2S, 2S\rangle$

FINAL STATE = $|f\rangle = |1S, \vec{k}\rangle = |1S\rangle \otimes |\text{PLANE WAVE}\rangle$

<u>1st ORDER</u> : $\langle 100, \vec{k} \,|\, V(1\leftrightarrow 2) \,|\, 200, 200\rangle$

$$\langle \vec{r} \,|\, n\ell m\rangle = R_{n\ell}(r)\, Y_{\ell m}(\hat{r})$$

$$\langle \vec{k} \,|\, \vec{r}\rangle = \frac{1}{\sqrt{V}} e^{i\vec{k}\cdot\vec{r}} = \frac{4\pi}{\sqrt{V}} \sum_{\ell=0}^{\infty} i^{\ell} j_{\ell}(kr) \sum_{m=-\ell}^{\ell} Y^{*}_{\ell m}(\hat{k})\, Y_{\ell m}(\hat{r})$$

FERMI'S GOLDEN RULE : $E_i = E_f \Rightarrow 2E_{2S} = E_{1S} + \frac{\hbar^2 k^2}{2m}$

RECALL THAT $E_n = -\frac{Z^2 m}{2\hbar^2}\left(\frac{q^2}{4\pi\varepsilon_0}\right)^2 \times \frac{1}{n^2} = -\frac{Z^2}{n^2} \times 13.6\ \text{eV}$

$$\Rightarrow \frac{\hbar^2 k^2}{2m} = 2E_{2S} - E_{1S} = -2^2\left(2 \times \frac{1}{4} - 1\right) * 13.6\ \text{eV}$$

$$= 27.2\ \text{eV} = \frac{1}{2} E_{\text{IONIZATION}} = \frac{1}{2}\, \frac{Z^2 m c^2 \alpha^2}{2}$$

$$\Rightarrow k^2 = \frac{Z^2 m^2 c^2 \alpha^2}{2\hbar^2} \Rightarrow \boxed{k = \frac{Z m c \alpha}{\sqrt{2}\,\hbar}} = 2.7\ 10^{-10}\ \text{m}^{-1}$$

$$\Rightarrow \lambda = 2.3\ \text{Å}$$

SELECTION RULES:

$\langle 100 \,|\, V \,|\, 200\rangle \to$ SELECTS $\ell = 0$ IN $V(1-2)$

$$\Rightarrow V(1\text{-}2) \longrightarrow \frac{q^2}{4\pi\varepsilon_0} * 4\pi \times \frac{1}{r_>} \underbrace{Y^{*}_{00}(\hat{r}_1)\, Y_{00}(\hat{r}_2)}_{\frac{1}{4\pi}}$$

$\langle \vec{k} \,|\, V \,|\, 100\rangle \to$ SELECTS THE $\ell = 0$ COMPONENT OF THE $\vec{k}$-WAVE

$$\Rightarrow \langle \vec{k} \,|\, \vec{r}_2\rangle \longrightarrow \frac{4\pi}{\sqrt{V}}\, j_0(kr_2)\, \underbrace{Y^{*}_{00}(\hat{k})\, Y_{00}(\hat{r}_2)}_{1/4\pi}$$

<u>DECAY RATE</u> :

$$W = \frac{2\pi}{\hbar} \left|\langle 1S, \vec{k} \,|\, V(1\text{-}2) \,|\, 2S, 2S\rangle\right|^2 \rho(E_k)$$

$$\hookrightarrow \frac{q^2}{\cdot} \int dr_1 dr_2\, r_1^2 r_2^2\, \frac{1}{\cdot}\, R_{10}(r_1) R_{20}(r_1)\, j_0(kr_2) R_{20}(r_2) * \underbrace{\int d\Omega_1 d\Omega_2\, |Y_{00}(\hat{r}_1)|^2\, Y_{00}(\hat{r}_2)}_{\cdots}$$

NOW WE USE THAT

$$\begin{cases} R_{10}(r) = 2\left(\frac{Z}{a_0}\right)^{3/2} e^{-\frac{Zr}{a_0}} \\ R_{20}(r) = \left(\frac{Z}{2a_0}\right)^{3/2}\left(2 - \frac{Zr}{a_0}\right) e^{-\frac{Zr}{2a_0}} \end{cases}$$

RECALL THAT $\dfrac{\frac{d^3k}{(2\pi)^3}}{V} = \rho(E)\, d\Omega\, dE$

AND $\rho(E_k) = \dfrac{V m \sqrt{2mE_k}}{(2\pi\hbar)^3}$

$$\Rightarrow W = \frac{2\pi}{\hbar}\left|\frac{e^2}{4\pi\varepsilon_0}\int\cdots\right|^2 \frac{Vm\sqrt{2m \times E_k}}{(2\pi\hbar)^3} \cong \underbrace{10^{-3}\frac{m}{\hbar^3}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2}_{4\cdot 10^{16}\ \frac{1}{\text{SEC}}}$$

SELECTION RULES (RELOAD)

CONSIDER THE ATOMIC TRANSITIONS VIA ELECTRIC DIPOLE WHEN $\langle n' l' m' | \vec{r} | n\, l\, m\rangle \neq 0$

- LETS FOCUS ON L_z AND z : $L_z |n\, l\, m\rangle = m\hbar |n\, l\, m\rangle$

RECALL THAT $[L_z, z] = 0$

$\Rightarrow \langle f | [L_z, z] | i\rangle = 0 = \hbar(m' - m)\langle n' l' m' | z | n\, l\, m\rangle$

$\Rightarrow \langle n' l' m' | z | n\, l\, m\rangle \neq 0$ ONLY WHEN $m = m'$

$\Rightarrow \Delta m = 0$

NOW FOCUS ON $r_\pm = x \pm iy$

$\Rightarrow [L_z, r_\pm] = [x p_y - y p_x, x \pm iy] = \pm\hbar r_\pm$

$\Rightarrow \langle f | [L_z, r_\pm] \mp \hbar r_\pm | i\rangle = 0 = \hbar[m' - (m \pm 1)]\langle n' l' m' | r_\pm | n\, l\, m\rangle$

$\Rightarrow$ ALLOWED TRANSITIONS ARE SUCH THAT $\Delta m = \pm 1$

WHAT CAN BE SAID ABOUT l AND l'?

THE STRATEGY IS TO USE L^2 : $L^2 |n\, l\, m\rangle = l(l+1)\hbar^2 |n\, l\, m\rangle$

$[L^2, z] = 2i\hbar\,(x L_y - L_x y)$

ON THE RHS, L^2 OR z DID NOT APPEARED $\Rightarrow$ COMMUTE ONCE MORE :

$$[L^2, [L^2, z]] = 2\hbar^2\{L^2, z\}$$

$\Rightarrow$ ALLOWED TRANSITIONS ARE SUCH THAT $\Delta l = \pm 1$ AND $\Delta m = 0, \pm 1$

23.5

→ CLASSICAL ELECTRON RADIUS

$$r_0 = \frac{q^2}{4\pi\epsilon_0} \times \frac{1}{mc^2} = \frac{q^2}{4\pi\epsilon_0 \hbar} \times \frac{\hbar}{mc} = \frac{\alpha \hbar}{mc}$$

$$\Rightarrow \sum_f \sigma(\omega_{fi}) = \text{TOTAL CROSS SECTION} = 2\pi^2 r_0 c$$

ELECTRIC QUADRUPOLE

$\langle f | xy, xz, yz, x^2, y^2, z^2 | i \rangle$

- $[L_z, z^2] = 0 \Rightarrow \Delta m = 0$
- $[L_z, r_\pm^2] = \pm 2\hbar\, r_\pm^2 \Rightarrow \Delta m = \pm 2$
- $[L_z, r_\pm z] = \pm \hbar\, r_\pm z \Rightarrow \Delta m = \pm 1$

FOR THE QUANTUM NUMBER $l \Rightarrow$ COMPUTE $[L^2, [L^2, z^2]]$ AND THE OTHERS

SUM RULES :

SOMETIMES IT IS VERY USEFUL

LET $H_0 = \frac{p^2}{2m} + V(\vec{r})$

$$\Rightarrow [H_0, x] = \frac{1}{2m}\left(p_x [p_x, x] + [p_x, x] p_x \right) = -\frac{i\hbar}{m} p_x$$

$$\Rightarrow [x, [H_0, x]] = \frac{\hbar^2}{m}$$

$$\Rightarrow \langle i | [x, [H_0, x]] | i \rangle = \frac{\hbar^2}{m} = \langle i | x H_0 x - x x H_0 - H_0 x x + x H_0 x | i \rangle$$

$$= 2 \left(\langle i | x \underset{\sum_f |f\rangle\langle f|}{\uparrow} H_0 x i \rangle - E_i \langle i | x \underset{\sum_f |f\rangle\langle f|}{\uparrow} x | i \rangle \right)$$

$$= 2 \sum_f (E_f - E_i) |\langle f | x | i \rangle|^2 = 2\hbar \sum_f \omega_{fi} |\langle f | x | i \rangle|^2$$

$$\Rightarrow \sum_f \sigma(\omega_{fi}) \simeq \sum_f \frac{4\pi^2}{m^2 \omega_{fi}} \alpha \underbrace{|\langle f | \underbrace{e^{i\omega_{fi} \frac{\hat{n}\cdot\vec{r}}{c}}}_{\simeq 1} \vec{p}\cdot\hat{e} | i \rangle}_{i m \omega_{fi} \langle f | \vec{r}\cdot\hat{e} | i \rangle} |^2$$

$$\Rightarrow \sum_f \sigma(\omega_{fi}) \simeq \sum_f 4\pi^2 \alpha\, \omega_{fi} |\langle f | x | i \rangle|^2$$

TOTAL CROSS SECTION / FREQUENCY

$$= 4\pi^2 \alpha * \frac{\hbar^2}{m} \times \frac{1}{2\hbar} = \boxed{\frac{2\pi^2 \hbar}{m}\alpha = \sum_f \sigma(\omega_{fi})}$$

GEOMETRIC PHASES

SUDDEN AND ADIABATIC APROXIMATIONS

MESSIAH (QUANTUM MECHANICS) - CHAP XXVII (VOL 2)
GALINDO & PASCUAL (QUANTUM MECHANICS) CHAP 11 (VOL 2)
LEVIN (INTROD. TO Q.M.) CHAP 16

SUDDEN APROXIMATION

IN THIS CASE, THE SITUATION IS SIMPLIFIED TO

$$H(t) = \begin{cases} H_1, & t < t_0 \\ H_2, & t > t_0 \end{cases}$$

where H_1 AND H_2 ARE TIME INDEPENDENT

THUS, THE TIME EVOLUTION IS SIMPLE

FOR $t < t_0 \rightarrow |\psi(t)\rangle = e^{-i H_1 t/\hbar} |\psi_0\rangle$

FOR $t > t_0 \rightarrow |\psi(t)\rangle = e^{-\frac{i}{\hbar} H_2 (t-t_0)} e^{-\frac{i}{\hbar} H_1 t_0} |\psi_0\rangle$

EXAMPLE: β-DECAY OF TRITIUM

$$^3H \longrightarrow {}^3He^+ + e^- + \bar{\nu}_e$$

INITIAL STATE: NEUTRAL $1e^-$ AND $1p$ $(Z=1)$ + 2 NEUTRONS

FINAL STATE: CHARGED He ATOM: $2p + 1e^-$ $(Z=2)$ + 1 NEUTRON + $\underbrace{1e^- + 1\bar{\nu}_e}_{\text{FREE}}$

LETS FOCUS ONLY IN THE BOUNDED ELECTRON

$$\Rightarrow \begin{cases} |i\rangle = |100\rangle_{Z=1}, \\ H_1 = \frac{p^2}{2m} + V_{Z=1}(r) \end{cases}$$

JUST AFTER THE DECAY, $H_2 = \frac{p^2}{2m} + V_{Z=2}(r)$

$$\Rightarrow |f(t)\rangle = \sum_{c_{nlm}} c_{nlm}\, e^{-\frac{i}{\hbar} E_n t} |nlm\rangle_{Z=2}$$

JUST AFTER THE DECAY, THE STATE OF $^3He^+$ IS MEASURED

WHAT IS THE PROBABILITY IT IS FOUND IN $|f\rangle = |100\rangle_{Z=2}$?

$$\Rightarrow P = |\langle 100_{Z=2} | 100_{Z=1}\rangle|^2 = \left| \int_0^\infty dr\, r^2 \underbrace{\left(\frac{2}{a_0}\right)^{3/2} 2 e^{-\frac{2r}{a_0}}}_{R_{10}(r)|_{Z=2}} \times \underbrace{\left(\frac{1}{a_0}\right)^{3/2} 2 e^{-\frac{r}{a_0}}}_{R_{10}(r)|_{Z=1}} \right|^2$$

$$= \left| \frac{16\sqrt{2}}{27} \right|^2 = 70{,}23\,\%$$

ADIABATIC APPROXIMATION

THIS IS THE OPPOSITE OF THE SUDDEN CHANGE

HERE, $H_1(t_0) \to H_2(t_1)$ EVOLVES VERY SMOOTHLY, $\tau = t_1 - t_0 \to \infty$

FOR $\tau \to \infty$ WE WILL HAVE THAT $|\psi(\vec{r}, t_0)|^2 \neq |\psi(\vec{r}, t_1)|^2$

AS WE HAVE SEEN IN THE SUDDEN CHANGE CASE, $|\psi(\vec{r}, t_0)|^2 = |\psi(\vec{r}, t_1)|^2$

THE "ADIABATIC" TERM IS USED IN THERMODYNAMICS IN ORDER TO DESIGNATE PROCESSESES THAT DO NOT EXCHANGE HEAT. HERE, IN QM., THE CHANGE $H_1 \to H_2$ IN TIME $\tau \to \infty$ IS MORE RELATED TO THE QUASI STATIC PROCESS, AND ITS RELATION TO HEAT EXCHANGE IS NOT SO CLEAR.

- INSTANTENEOUS EIGEN BASIS : ~~[illegible]~~

$$H(t)\,|\varphi_m(t)\rangle = E_m(t)\,|\varphi_m(t)\rangle$$

* $|\varphi_m\rangle$ IS NOT SOLUTION OF THE TIME-DEPENDENT SCHRÖDINGER EQUATION : $i\hbar \frac{\partial}{\partial t}|\psi_m\rangle = H(t)\,|\psi_m\rangle$

LETS USE THE INSTANTENEOUS EIGEN BASIS TO EXPRESS THE TRUE STATE

$$|\psi(t)\rangle = \sum_m c_m(t)\,|\varphi_m(t)\rangle$$

$\Rightarrow$ (SCHROD. EQ) $\sum_m i\hbar\left(\dot{c}_m|\varphi_m\rangle + c_m\left(\frac{\partial}{\partial t}|\varphi_m\rangle\right)\right) = \sum_m c_m(t)E_m(t)\,|\varphi_m(t)\rangle$

$\overset{\langle\varphi_k|}{\Rightarrow}$ $\boxed{i\hbar\,\dot{c}_k - E_k c_k = -i\hbar\sum_m \langle\varphi_k|\frac{\partial}{\partial t}|\varphi_m\rangle c_m}$

LETS INVESTIGATE $\langle\varphi_k|\frac{\partial}{\partial t}|\varphi_m\rangle$

- $H|\varphi_m\rangle = E_m|\varphi_m\rangle \Rightarrow \left(\frac{\partial H}{\partial t}\right)|\varphi_m\rangle + H\left(\frac{\partial}{\partial t}|\varphi_m\rangle\right) = \dot{E}_m|\varphi_m\rangle + E_m\left(\frac{\partial}{\partial t}|\varphi_m\rangle\right)$

$\langle\varphi_k| \Rightarrow \langle\varphi_k|\frac{\partial}{\partial t}|\varphi_m\rangle = \dfrac{\langle\varphi_k|\left(\frac{\partial H}{\partial t}\right)|\varphi_m\rangle}{E_m - E_k} - \dfrac{\dot{E}_m}{E_m - E_k}\,\delta_{mk}$

$\Rightarrow$ FOR $m \neq k$ $\boxed{\langle\varphi_k|\frac{\partial}{\partial t}|\varphi_m\rangle = \dfrac{\langle\varphi_k|\frac{\partial H}{\partial t}|\varphi_m\rangle}{E_m - E_k}}$

FOR $m=k \Rightarrow$ ~~...~~ THERE IS NO USE OF THE PREVIOUS CALCULATION

HOWEVER, THINGS ARE MUCH SIMPLER

NOTICE THAT $\frac{\partial}{\partial t}\left(\langle \varphi_m | \varphi_m \rangle\right) = 0 = \left(\frac{\partial}{\partial t}\langle \varphi_m |\right)|\varphi_m\rangle + \langle \varphi_m |\left(\frac{\partial}{\partial t}|\varphi_m\rangle\right)$

$= 2\,\mathrm{Re}\left\{\langle \varphi_m |\left(\frac{\partial}{\partial t}|\varphi_m\rangle\right)\right\}$

THUS, WE CONCLUDE THAT

$\langle \varphi_m | \frac{\partial}{\partial t} | \varphi_m \rangle$ IS PURE IMAGINARY $\equiv i\,\alpha_m(t)$

WITH $\alpha_m \in \mathbb{R}$

COMING BACK TO THE EQUATION FOR c_k:

$$i\hbar\,\dot{c}_k - E_k c_k = -i\hbar\left\{ i\alpha_k c_k + \sum_{m\neq k} \langle \varphi_k | \frac{\partial}{\partial t} | \varphi_m \rangle c_m \right\}$$

$$= -i\hbar\left\{ i\alpha_k c_k + \sum_{m\neq k} \frac{\langle \varphi_k | \frac{\partial H}{\partial t} | \varphi_m \rangle}{E_m - E_k} c_m \right\}$$

NEW DEFINITION: $\tilde{c}_k(t) = c_k(t) * e^{\int_{t_0}^{t} -\frac{1}{i\hbar}(E_k + \hbar\alpha_k)\,dt'}$

$$\Rightarrow c_k = \tilde{c}_k(t)\, e^{\frac{1}{i\hbar}\int_{t_0}^{t} E_k(t') + \hbar\alpha_k(t')\,dt'}$$

$$\Rightarrow i\hbar\,\dot{\tilde{c}}_k e^{\frac{1}{i\hbar}()} + \cancel{\tilde{c}_k(E_k + \hbar\alpha_k)e^{\frac{1}{i\hbar}()}} - \cancel{E_k \tilde{c}_k e^{\frac{1}{i\hbar}()}} = \cancel{\hbar\alpha_k \tilde{c}_k e^{\frac{1}{i\hbar}()}} - i\hbar \sum_{m\neq k} \frac{\langle\rangle}{E_m - E_k} \tilde{c}_m e^{\frac{1}{i\hbar}()}$$

$$\Rightarrow \dot{\tilde{c}}_k = \sum_{m\neq k} \frac{\langle \varphi_k | \dot{H} | \varphi_m \rangle}{E_k - E_m} \tilde{c}_m\, e^{\frac{1}{i\hbar}\int_{t_0}^{t} dt' \left[E_m + \hbar\alpha_m - (E_k + \hbar\alpha_k)\right]}$$

$\boxed{\text{NEW DEFINITION: ADIABATIC EIGENBASIS}}$ $|\tilde{\varphi}_m\rangle \equiv |\varphi_m\rangle\, e^{\frac{1}{i\hbar}\int_{t_0}^{t} dt'\, E_m + \hbar\alpha_m}$

$\Rightarrow |\varphi_m\rangle = |\tilde{\varphi}_m\rangle\, e^{-\frac{1}{i\hbar}\int_{t_0}^{t} dt'\, E_m + \hbar\alpha_m}$

$$\Rightarrow \boxed{\dot{\tilde{c}}_k = \sum_{m\neq k} \frac{\langle \tilde{\varphi}_k | \dot{H} | \tilde{\varphi}_m \rangle}{E_k - E_m} \tilde{c}_m}$$

SOLUTION: $$\boxed{\tilde{c}_k(t) = \tilde{c}_k(t_0) + \sum_{m\neq k} \int_{t_0}^{t} dt' \frac{\langle \tilde{\varphi}_k(t') | \dot{H}(t') | \tilde{\varphi}_m(t') \rangle}{E_k(t') - E_m(t')} \tilde{c}_m}$$

WE NOW CAN COMPUTE $\tilde{c}_k$ IN A DYSON'S SERIES

1st ORDER : $$\boxed{\tilde{c}_k(t) = \tilde{c}_k(t_0) + \sum_{m\neq k} \tilde{c}_m(t_0) \int_{t_0}^{t} \frac{\langle \tilde{\varphi}_k | \dot{H} | \tilde{\varphi}_m \rangle}{E_k - E_m}}$$

MOREOVER $\tilde{c}_j(t_0) = c_j(t_0)$ -> (NO NEED)

$$\boxed{\tilde{c}_k(t) = c_k(t_0) + \sum_{m\neq k} c_m(t_0) \int_{t_0}^{t} dt' \frac{\langle \tilde{\varphi}_k | \dot{H} | \tilde{\varphi}_m \rangle}{E_k - E_m}}$$

THE SERIES CONVERGES AS LONG AS $\tau \frac{\dot{H}_{km}}{E_k - E_m} \ll 1$

WITH $\tau = t - t_0$ $\quad \to \frac{\dot{H}_{km}}{\hbar\,\omega_{km}} \ll 1$

* NOTICE THAT $|\psi\rangle$ CAN BE EXPANDED IN THE "ADIABATIC" EIGEN BASIS

$$|\psi(t)\rangle = \sum_m c_m(t)\, |\varphi_m(t)\rangle = \sum_m \tilde{c}_m(t)\, |\tilde{\varphi}_m(t)\rangle$$

INTERPRETATION :

SUPPOSE WE HAVE THAT $\tilde{c}_m \cong$ CONST IN TIME

$$\Rightarrow |\psi(t)\rangle = |\tilde{\varphi}_m(t)\rangle = \underbrace{e^{\frac{1}{i\hbar}\int_{t_0}^{t} dt'\, E_m(t')}}_{\text{USUAL DYNAMICAL PHASE}} \ast \underbrace{e^{i\gamma_m(t)}}_{\text{BERRY'S PHASE}} |\varphi_m(t)\rangle$$

$$-\gamma_m = \int_{t_0}^{t} \alpha_m(t')\, dt' \in \mathbb{R}$$

BERRY'S PHASE IS AN ADDITIONAL PHASE PREVIOUSLY UNNOTICED UNTIL 1984

WHAT IS ITS SIGNIFICANCE? IS IT OBSERVABLE?

– THE DYNAMICAL PHASE $e^{i\frac{Et}{\hbar}}$ IS UNOBSERVABLE, BUT THE BERRY'S PHASE IS
IT YIELDS OBSERVABLE EFFECTS

MORE ON THIS LATER

BEFORE GOING INTO EXAMPLES, LETS DISCUSS A BIT MORE ON WHAT MEANS ADIABATICITY

WE HAVE THAT $|\psi(t)\rangle = U(t,t_0)\,|\psi(t_0)\rangle = U(t,t_0)\,|\varphi_m(t_0)\rangle$
WHERE WE USED THE INITIAL CONDITION $|\psi(t_0)\rangle = |\varphi_m(t_0)\rangle$

$$\Rightarrow |\psi(t)\rangle = \sum_k |\varphi_k(t)\rangle\langle\varphi_k(t)|\,U(t,t_0)\,|\varphi_m(t_0)\rangle$$

THE ADIABATIC APPROXIMATION MEANS THAT

$$\langle\varphi_k(t)\,|\,U(t,t_0)\,|\varphi_m(t_0)\rangle \cong e^{i\phi_m(t,t_0)}\,\delta_{km}$$

WHICH IS: AFTER THE EVOLUTION IN A STATE $|\varphi_m\rangle$ THE SYSTEM STAY IN STATE $|\varphi_k\rangle = |\varphi_m\rangle$

ADIABATIC THEOREM:

GIVEN THE INSTANTENEOUS EIGENBASIS: $H(t)\,|\varphi_m(t)\rangle = E_m(t)\,|\varphi_m(t)\rangle$
LET THE PROJECTORS $P_m(t) = |\varphi_m(t)\rangle\langle\varphi_m(t)|$

- HYPOTHESIS: $E_i(t) \neq E_j(t) \quad \forall\, i,j \;,\; t \in [t_0, t_0+\tau]$
  $\frac{d}{dt}P_j(t)$ AND $\frac{d^2P_j(t)}{dt^2}$ ARE WELL DEFINED FUNCTIONS AND BOUNDED

$$\Rightarrow U(\tau+t_0,t_0)\,P_j(0) - P_j(t_0+\tau)\,U(\tau+t_0,t_0) = O\left(\frac{1}{\tau}\right)$$

FOR $\tau\to\infty$ THIS MEANS THAT A STATE DURING ITS EVOLUTION DOES NOT LEAVE THE SUBSPACE EXPANDED BY THE INITIAL STATE

$$\underbrace{U(\tau+t_0,t_0)\,\underbrace{P_j(0)\,|\varphi_j(0)\rangle}_{|\varphi_j(0)\rangle}}_{|\psi(\tau+t_0)\rangle} = \underbrace{P_j(t_0+\tau)\,\underbrace{U(\tau+t_0,t_0)\,|\varphi_j(0)\rangle}_{|\psi(\tau+t_0)\rangle}}_{\langle\varphi_j(\tau+t_0)|\psi(\tau+t_0)\rangle\,|\varphi_j(\tau+t_0)\rangle} \quad \text{FOR } \tau\to\infty$$

$$\Rightarrow |\langle\varphi_j(\tau+t_0)|\psi(\tau+t_0)\rangle| = 1$$

IF AT t_0, $|\psi\rangle = |\varphi_1\rangle \Rightarrow$ AT $t_0+\tau$ $|\psi\rangle = |\varphi_1\rangle$
" " " $|\psi\rangle = A|\varphi_1\rangle + B|\varphi_2\rangle \Rightarrow$ " $|\psi\rangle = A'|\varphi_1\rangle + B'|\varphi_2\rangle$

MORE ON THE
VALIDITY OF THIS THEOREM AND THAT ADIABATIC APPROXIMATION

LET $\mathcal{S} = \langle\varphi(t_0)|\, U^+(t_0+\tau, t_0)\, U(t_0+\tau, t_0)\,|\varphi(t_0)\rangle -$
$\langle\varphi(t_0)|\, U^+(t_0+\tau, t_0)\,|\varphi(t_0)\rangle\langle\varphi(t_0)|\,U(t_0+\tau, t_0)\,|\varphi(t_0)\rangle$

DYSON'S SERIES: $U = \mathbb{1} + \frac{1}{i\hbar}\int_{t_0}^{t_0+\tau} dt\, H(t) + \cdots$

KEEP ONLY 1st ORDER

$$\Rightarrow \mathcal{S} \cong \langle 0|\left(\mathbb{1} - \frac{1}{i\hbar}\int H\right)\left(\mathbb{1} + \frac{1}{i\hbar}\int H\right)|0\rangle - \langle 0|\,\mathbb{1} - \frac{1}{i\hbar}\int H\,|0\rangle\langle 0|\,\mathbb{1} + \frac{1}{i\hbar}\int H\,|0\rangle$$

$$= 1 - \left(\frac{1}{i\hbar}\right)^2 \tau^2 \langle H^2\rangle - \left(1 - \left(\frac{1}{i\hbar}\right)\tau^2 \langle H\rangle^2\right)$$

$$= \frac{\tau^2}{\hbar^2}\left(\langle H^2\rangle - \langle H\rangle^2\right) = \frac{\tau^2 \Delta H^2}{\hbar^2}$$

$\mathcal{S} \ll 1 \rightarrow$ SUDDEN APPROXIMATION
$\mathcal{S} \gg 1 \rightarrow$ ADIABATIC "

EXAMPLE: 1D HARMONIC OSCILLATOR

$$H(t) = \frac{1}{2m} P^2 + \frac{1}{2} m \omega^2 (x-\bar{x})^2 \quad \text{WITH} \quad \bar{x} = v_0 t$$

RECALL THAT $a = \sqrt{\frac{m\omega}{2\hbar}} \left(x + \frac{i}{m\omega} P \right)$, $a^+ = \sqrt{\frac{m\omega}{2\hbar}} \left(x - \frac{i}{m\omega} P \right)$

INSTANTANEOUS BASIS: $|\varphi_m\rangle \to \langle x | \varphi_m(t) \rangle = \varphi_m(x-\bar{x}) \equiv$ HARMONIC OSCILLATOR WAVE FUNCTION CENTERED AT $\bar{x} = v_0 t$

AND $\varphi_m(x) = \sqrt{\frac{1}{2^m m!}} \left(\frac{m\omega}{\pi\hbar} \right)^{\frac{1}{4}} \exp\left\{ -\frac{m\omega}{2\hbar} x^2 \right\} H_m\left(-\sqrt{\frac{m\omega}{\hbar}} x \right)$

$$= \frac{1}{\sqrt{2^m m!}} \frac{1}{\pi^{1/4}} \frac{1}{x_0^{m+\frac{1}{2}}} \left(x - x_0^2 \frac{d}{dx} \right)^m \exp\left\{ -\frac{1}{2} \left(\frac{x}{x_0} \right)^2 \right\}$$

$E_n = \hbar\omega \left(n + \frac{1}{2} \right) \equiv$ CONSTANT IN TIME $\quad$ WITH $x_0 = \sqrt{\frac{\hbar}{m\omega}}$

ADIABATIC BASIS $|\tilde{\varphi}_m\rangle = e^{\frac{1}{i\hbar} \int_0^t (E_m + \hbar\alpha_m) dt'}$

NOTICE THAT $i\alpha_m = \langle \varphi_m | \frac{\partial}{\partial t} | \varphi_m \rangle$ WHICH IS PURE IMAGINARY AS $\langle x | \varphi_m \rangle \in \mathbb{R} \Rightarrow \alpha_m = 0 \to$ TYPICAL OF 1D PROBLEMS

$\Rightarrow |\tilde{\varphi}_m\rangle = e^{\frac{1}{i\hbar} E_m t} |\varphi_m\rangle$ (ONLY THE DYNAMICAL PHASE)

1st ORDER:

$$\tilde{C}_k(t) = \tilde{C}_k(0) + \sum_{m \neq k} \tilde{C}_m(0) \int_0^t dt' \frac{\langle \tilde{\varphi}_k | \dot{H} | \tilde{\varphi}_m \rangle}{E_k(t') - E_m(t')}$$

$$\dot{H} = m\omega^2 (x - \bar{x}) \dot{\bar{x}} = m\omega^2 v_0 (x - v_0 t)$$

$$\Rightarrow \langle \tilde{\varphi}_k | \dot{H} | \tilde{\varphi}_m \rangle = m\omega^2 v_0 \langle \tilde{\varphi}_k | x | \tilde{\varphi}_m \rangle = m\omega^2 \frac{v_0 x_0}{\sqrt{2}} \langle \tilde{\varphi}_k | a + a^+ | \tilde{\varphi}_m \rangle$$

$$= m\omega^2 \frac{v_0 x_0}{\sqrt{2}} \left(\sqrt{m}\, \delta_{k,m-1} + \sqrt{m+1}\, \delta_{k,m+1} \right)$$

$$\Rightarrow \tilde{C}_k(t) = \tilde{C}_k(0) + \sum_{m \neq k} \tilde{C}_m(0) \frac{m\omega^2 v_0 x_0}{\sqrt{2}} \times \frac{1}{(k-m)\hbar\omega} \left(\sqrt{m}\, \delta_{k,m-1} + \sqrt{m+1}\, \delta_{k,m+1} \right) \underbrace{\int_0^t e^{\frac{1}{i\hbar}(m-k)\hbar\omega t}}_{\frac{-2 e^{\frac{(m-k)\omega t}{2i}}}{(m-k)\omega} \sin (m-k) \frac{\omega t}{2}}$$

INITIAL CONDITIONS: $\tilde{C}_k(0) = \begin{cases} 1, & k=j \\ 0, & k\neq j \end{cases}$ (32)

$$\Rightarrow \begin{cases} \tilde{C}_{j+1}(t) = -x_0 \dfrac{\sqrt{j+1}}{\hbar\omega^2} \sin\left(\dfrac{\omega t}{2}\right) e^{i\frac{\omega t}{2}} \times m\omega^2 v_0 \sqrt{2} \\ \tilde{C}_{j-1}(t) = x_0 \dfrac{\sqrt{j}}{\hbar\omega^2} \sin\left(\dfrac{\omega t}{2}\right) e^{-i\frac{\omega t}{2}} \times m\omega^2 v_0 \sqrt{2} \end{cases}$$

TRANSITION PROBABILITY:

$$P_{j+1}(t) = |\tilde{C}_{j+1}(t)|^2 = \frac{x_0^2 m\omega^4}{\hbar^2\omega^3} \times 4 \times \frac{\frac{1}{2}mv_0^2}{\hbar\omega} \times \sin^2\left(\frac{\omega t}{2}\right) \times (j+1)$$

$$= 4(j+1)\left(\frac{\frac{1}{2}mv_0^2}{\hbar\omega}\right) \sin^2\left(\frac{\omega t}{2}\right)$$

$$P_{j-1}(t) = 4j\left(\frac{\frac{1}{2}mv_0^2}{\hbar\omega}\right) \sin^2\left(\frac{\omega t}{2}\right)$$

THE APPROXIMATION IS GOOD WHEN

$\dfrac{\frac{1}{2}mv_0^2}{\hbar\omega} \ll 1$ WHICH IS $\dfrac{\text{KINETIC ENERGY OF THE PARTICLE}}{\text{TYPICAL ENERGY OF THE H.O.}}$

$\Rightarrow v_0 \ll \sqrt{\dfrac{\hbar\omega}{m}}$

EXAMPLE: SPIN IN A MAGNETIC FIELD

BERRY'S PHASE (M.V. BERRY, PROC. ROY. SOC. LOND. A, 392, 45 (1984))

RECALL $|\psi(t)\rangle = \sum_m \tilde{c}_m(t)\, |\tilde{\varphi}_m(t)\rangle$

$$|\tilde{\varphi}_m(t)\rangle = e^{\frac{i}{\hbar}\int_0^t E_m(t')\,dt'}\; e^{i\gamma_m(t)}\; |\varphi_j(t)\rangle$$

$|\tilde{\varphi}_m(t)\rangle$ → ADIABATIC BASIS; $e^{\frac{i}{\hbar}\int_0^t E_m(t')\,dt'}$ → DYNAMICAL PHASE; $e^{i\gamma_m(t)}$ → BERRY'S PHASE; $|\varphi_j(t)\rangle$ → INSTANTANEOUS BASIS

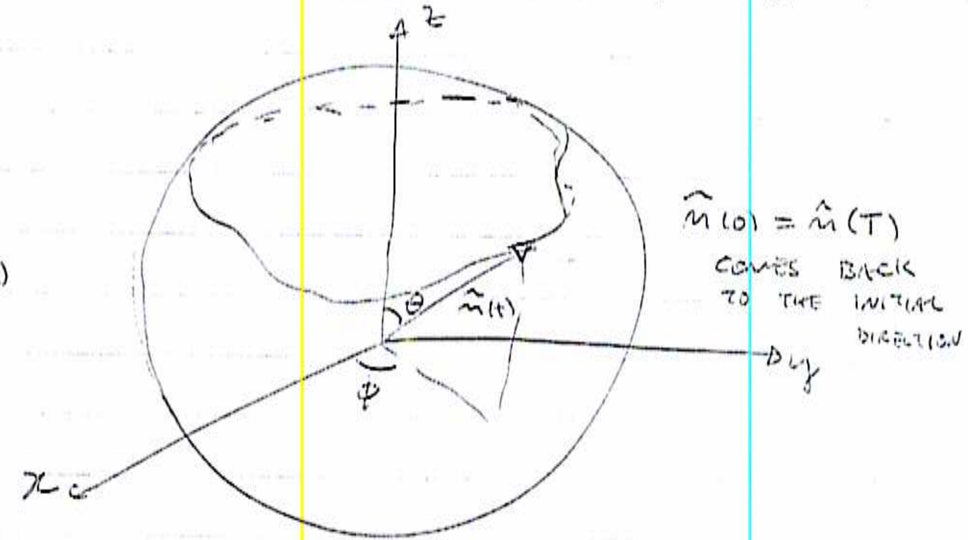
$$\gamma_m(t) = i\int_0^t \langle \varphi_m(t')|\tfrac{\partial}{\partial t'}|\varphi_m(t')\rangle\, dt' = -\mathrm{Im}\int_0^t \langle\varphi_m|\tfrac{\partial}{\partial t'}|\varphi_m\rangle\, dt'$$

$$H = -\frac{g\mu_B}{\hbar}\vec{S}\cdot\vec{B}(t)$$

SUPPOSE $\vec{B}(t) = B_0\,\hat{m}(t)$

$\Rightarrow H = -\omega\,\vec{S}\cdot\hat{m}$

with $\omega = \frac{g\mu_B B_0}{\hbar}$



$$\hat{m}(t) = \sin\theta(t)\cos\phi(t)\,\hat{x} + \sin\theta(t)\sin\phi(t)\,\hat{y} + \cos\theta(t)\,\hat{z}$$

- COMPUTE BERRY'S PHASE: 1st METHOD: PEDESTRIAN ONE

INSTANTANEOUS BASIS: $|m,\hat{m}\rangle$: $H(t)|m,\hat{m}(t)\rangle = m\hbar\omega|m,\hat{m}(t)\rangle$

HOW DO WE RELATE $|m,\hat{m}\rangle$ TO $|m,\hat{z}\rangle$?

ANSWER: GET $|m,\hat{z}\rangle$ → ROTATE θ OVER $\hat{y}$ → ROTATE ϕ OVER $\hat{z}$

$$\Rightarrow |m,\hat{m}\rangle = \exp\left\{-\frac{i}{\hbar}S_z\phi(t)\right\}\exp\left\{-\frac{i}{\hbar}S_y\theta(t)\right\}|m,\hat{z}\rangle$$

THUS, WE HAVE TO COMPUTE

$$\langle m,\hat{m}|\tfrac{\partial}{\partial t}|m,\hat{m}\rangle = \langle m,\hat{z}|e^{\frac{iS_y\theta}{\hbar}}e^{\frac{iS_z\phi}{\hbar}}\frac{\partial}{\partial t}\left(e^{-\frac{iS_z\phi}{\hbar}}e^{-\frac{iS_y\theta}{\hbar}}|m,\hat{z}\rangle\right)$$

$$= -\frac{i}{\hbar}\langle m,\hat{z}|e^{\frac{iS_y\theta}{\hbar}}e^{\frac{iS_z\phi}{\hbar}}\left(S_z\dot{\phi}\,e^{-\frac{iS_z\phi}{\hbar}}e^{-\frac{iS_y\theta}{\hbar}} + e^{-\frac{iS_z\phi}{\hbar}}S_y\dot{\theta}\,e^{-\frac{iS_y\theta}{\hbar}}\right)|m,\hat{z}\rangle$$

$$= -\frac{i}{\hbar}\left(\langle m,\hat{z}|\tilde{S}_z\dot{\phi}|m,\hat{z}\rangle + \langle m,\hat{z}|S_y\dot{\theta}|m,\hat{z}\rangle\right)$$

$\Rightarrow \gamma_m = -m\,\Omega + (m$

EXT.

PHAS

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IS THERE A PHYSICAL DIFFERENCE? $\rightarrow$ γ_m IS NOT

WITH ANOTHER STATE $\rightarrow$ WHAT IS OBSERVABLE

RECALL $S^y \sim (S^+ - S^-)$ $\Rightarrow$ $\langle m | S^y | m \rangle \sim \langle m | m+1 \rangle + \langle m | m-1 \rangle$

NOW, WE HAVE TO COMPUTE

$$\tilde{S}_z = e^{\frac{i S_y \theta}{\hbar}} S_z e^{\frac{-i S_y \theta}{\hbar}}$$

WE USE $e^{iG\lambda} A e^{-iG\lambda} = A + i\lambda [G, A] + \frac{(i\lambda)^2}{2!} [G, [G, A]] + \frac{(i\lambda)^3}{3!} [G, [G, [G, A]]] + \dots$

$$\Rightarrow \tilde{S}_z = S_z + \frac{i\theta}{\hbar} \overbrace{[S_y, S_z]}^{i\hbar S_x} + \frac{(i\theta)^2}{2!\hbar^2} \overbrace{[S_y, i\hbar S_x]}^{-(i\hbar)^2 S_z} + \frac{(i\theta)^3}{3!\hbar^3} \overbrace{[S_y, -(i\hbar)^2 S_z]}^{-(i\hbar)^3 S_x} + \frac{(i\theta)^4}{4!\hbar^4} \overbrace{[S_y, -(i\hbar)^3 S_x]}^{(i\hbar)^4 S_z} + \dots$$

$$= S_z \left(1 - \frac{i^4}{2!} \theta^2 + \frac{i^8}{4!} \theta^4 - \dots \right) + i^2 S_x \left(\theta - \frac{i^4}{3!} \theta^3 + \frac{i^8}{5!} \theta^5 + \dots \right)$$

$$= S_z \cos\theta - S_x \sin\theta$$

$$\Rightarrow \langle m, \hat{n} | \tfrac{\partial}{\partial t} | m, \hat{n} \rangle = -\frac{i}{\hbar} \dot{\phi} \langle m, \hat{z} | S_z \cos\theta - S_x \sin\theta | m, \hat{z} \rangle$$

$$= -\frac{i}{\hbar} \dot{\phi} \cos\theta \times m\hbar = -i \dot{\phi} \cos\theta \times m$$

$\Rightarrow$ BERRY'S PHASE $= \boxed{\gamma_m = m \times \int_0^T dt \; \dot{\phi}(t) \cos\theta(t)}$

NOTICE THAT IT DEPENDS ONLY ON THE TRAJECTORY

$$\gamma_m = m \times \oint_{0,\ \text{PATH}}^{2\pi} \cos\theta(\phi) \, d\phi$$

$\rightarrow$ TIME IS ELIMINATED $\Rightarrow$ AS LONG AS IT IS SUFFICIENTLY SLOW TO GUARANTEE ADIABATICITY $\Rightarrow$ IT DOES NOT MATTER HOW FAST $\hat{n}$ CHANGES

$m = -m(-2\pi\cos\theta)$

ut

$= 2\pi(1-\cos\theta)$

$\cancel{\gamma_m = m \times \frac{\Omega}{2}}$, Ω = SOLID ANGLE

T)

2ND METHOD : BERRY'S METHOD

$$\gamma_m = -\mathrm{Im} \int_0^T \langle \varphi_m | \tfrac{\partial}{\partial t'} | \varphi_m \rangle dt' = -\mathrm{Im} \oint \langle \varphi_m | \tfrac{\partial}{\partial \vec{B}} | \varphi_m \rangle \cdot \frac{d\vec{B}}{dt} dt$$

$$= -\mathrm{Im} \oint_{\text{PATH}} \langle \varphi_m | \nabla_{\vec{B}} | \varphi_m \rangle \cdot d\vec{B}$$

$\Rightarrow$ IT DEPENDS ON THE "GEOMETRY" OF THE PATH OF $\vec{B}$

7

THE TRICKS

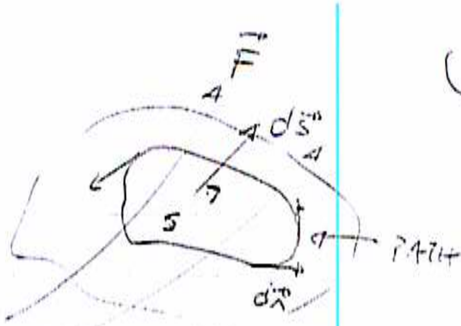
E THEOREM

SERVABLE, IN LAB ONE NEEDS TO DO INTERFERENCE

AT $\gamma - \gamma$ ... SINCE $m = m \in \mathbb{Z}$

STOKES THEOREM:

$$\iint_S \nabla\times\vec{F}\cdot d\vec{S} = \oint_{PATH} \vec{F}\cdot d\vec{r}$$



$$\Rightarrow \gamma_m = -\mathrm{Im} \iint_S \vec{V}_m(\vec{B})\cdot d\vec{S}$$

where $\vec{V}_m(\vec{B}) = \nabla_B \times \left(\langle\varphi_m|\nabla_B|\varphi_m\rangle\right) = \nabla_B\times\langle\varphi_m|\nabla_B\varphi_m\rangle$

$$= \langle\varphi_m|\underbrace{\left(\nabla_B\times|\nabla_B\varphi_m\rangle\right)}_{=0} + \langle\nabla_B\varphi_m|\times|\nabla_B\varphi_m\rangle$$

$$\Rightarrow \gamma_m = -\mathrm{Im}\iint_S \sum_K \langle\nabla_B\varphi_m|\times|\varphi_K\rangle\langle\varphi_K|\nabla_B|\varphi_m\rangle$$

$$= -\mathrm{Im}\iint_S \sum_K \langle\nabla_B\varphi_m|\varphi_K\rangle\times\langle\varphi_K|\nabla_B\varphi_m\rangle$$

FOR $K=m \Rightarrow \langle\varphi_m|\nabla_B|\varphi_m\rangle$ WHICH IS PURE IMAGINARY

$$\Rightarrow \gamma_m = -\mathrm{Im}\sum_{K\neq m}\iint_S \langle\nabla_B\varphi_m|\varphi_K\rangle\times\langle\varphi_K|\nabla_B\varphi_m\rangle$$

NOW, WE WANT TO INTRODUCE THE HAMILTONIAN INTO PLAY

– AS $H(t)|\varphi_m\rangle = E_m(t)|\varphi_m\rangle$

$$\Rightarrow (\nabla_B H)|\varphi_m\rangle + H|\nabla_B\varphi_m\rangle = (\nabla_B E_m)|\varphi_m\rangle + E_m|\nabla_B\varphi_m\rangle$$

$\Rightarrow \langle\varphi_K|$ WITH $K\neq m$

$$\Rightarrow \langle\varphi_K|\nabla_B H|\varphi_m\rangle + E_K\langle\varphi_K|\nabla_B\varphi_m\rangle = \nabla_B E_m\,\delta_{K,m} + E_m\langle\varphi_K|\nabla_B\varphi_m\rangle$$

$$(K\neq m) \Rightarrow \langle\varphi_K|\nabla_B\varphi_m\rangle = \frac{\langle\varphi_K|\nabla_B H|\varphi_m\rangle}{E_m - E_K}$$

$$\Rightarrow \gamma_m = -\mathrm{Im}\sum_{K\neq m}\iint_S d\vec{S}\cdot\frac{\langle\varphi_m|\nabla_B H|\varphi_K\rangle\times\langle\varphi_K|\nabla_B H|\varphi_m\rangle}{(E_m-E_K)^2}$$

$$= -\iint_S d\vec{S}\cdot\vec{V}_m(\vec{B})$$

WITH $\vec{V}_m = \mathrm{Im} \sum_{k \neq m} \frac{\langle \varphi_m | \nabla_B H | \varphi_k \rangle \times \langle \varphi_k | \nabla_B H | \varphi_m \rangle}{(E_m - E_k)^2}$

AS $H = -\frac{g\mu_B}{\hbar} \vec{B} \cdot \vec{S} \Rightarrow \nabla_B H = -\frac{g\mu_B}{\hbar} \vec{S}$

RECALL $S_\pm = S_x \pm i S_y \Rightarrow S_x = \frac{1}{2}(S_+ + S_-)$, $S_y = \frac{1}{2i}(S_+ - S_-)$

$$\begin{cases} S_+ |m\rangle = \hbar \sqrt{s(s+1) - m(m+1)}\, |m+1\rangle \\ S_- |m\rangle = \hbar \sqrt{s(s+1) - m(m-1)}\, |m-1\rangle \end{cases}$$

$$\Rightarrow \langle \varphi_k | \nabla_B H | \varphi_m \rangle = -\frac{g\mu_B}{2}\left(\sqrt{+}\,\delta_{k,m+1} + \sqrt{-}\,\delta_{k,m-1},\ \frac{\sqrt{+}\,\delta_{k,m+1} - \sqrt{-}\,\delta_{k,m-1}}{i},\ 2\,\delta_{k,m} \right)$$

$$\Rightarrow \vec{V}_m = \mathrm{Im}\left\{ \left(\frac{-g\mu_B}{2}\right)^2 \left[\frac{\sqrt{s(s+1)-(m+1)m}\,(1,i,0) \times \sqrt{s(s+1)-m(m+1)}\,(1,-i,0)}{\left(g\frac{\mu_B B_0}{\hbar} \times \hbar\right)^2} \right.\right.$$

$k = m+1$

$k = m-1$

$$\left.\left. + \frac{\sqrt{s(s+1)-(m-1)m}\,(1,-i,0) \times \sqrt{s(s+1)-m(m-1)}\,(1,i,0)}{(g\,\mu_B B_0)^2} \right] \right\}$$

$$\vec{V}_m = \mathrm{Im}\left\{ \frac{1}{4B_0^2} \left[(s(s+1) - m(m+1))\,(-2i\hat{z}) + (s(s+1) - m(m-1))\,(2i\hat{z}) \right] \right\}$$

$$= \mathrm{Im}\left\{ \frac{1}{4B_0^2}\, 2i\hat{z}\, (m(m+1) - m(m-1)) \right\} = \mathrm{Im}\left\{ \frac{im\hat{z}}{B_0^2} \right\}$$

$$\Rightarrow \boxed{\vec{V}_m = \frac{m}{B_0^2}\hat{z}}$$ → THIS IS COMPUTED IN THE INSTANTENEOUS BASIS WHERE $\hat{z} \cdot \hat{n}$ IS ALWAYS POINTING PARALLEL TO $\vec{B}$. IN THE LAB

$$\boxed{\vec{V}_m = m \frac{\vec{B}}{B_0^3}}$$ WHICH IS SINGULAR AT $B=0$

(NOTICE THIS IS A MONOPOLE FIELD $\sim \frac{\vec{R}}{R^3}$)

$\gamma_m = -\iint \vec{V}_m \cdot d\vec{S}$, WITH $d\vec{S} = d\vec{B}_\square = \hat{m} B_0^2\, d\Omega_B$

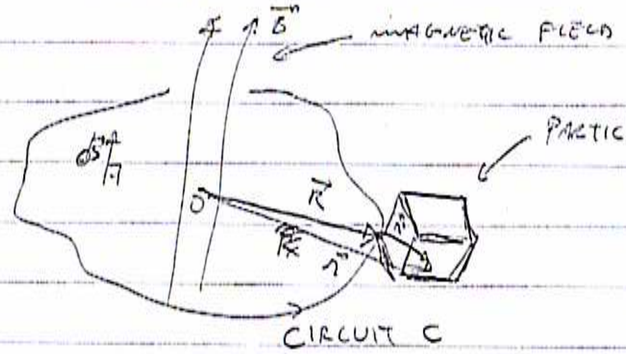
$$\Rightarrow \gamma_m = -\iint m \frac{\vec{B}}{B_0^3} \cdot \hat{m}\, B_0^2\, d\Omega_B = -m \underbrace{\Omega_B}_{\text{SOLID ANGLE}}$$

DOES NOT DEPEND ON THE DYNAMICS OF THE FIELD, ONLY ON THE "GEOMETRY"

*CLASSICAL ANALOGUE: FOCAULT PENDULUM (PARALLEL TRANSPORT)

AHARONOV – BOHM EFFECT

THE PHASE ACQUIRE BY A CHARGED PARTICLE WHEN WINDING A MAGNETIC FLUX CAN BE VIEWED AS A GEOMETRICAL PHASE



$$\text{MAGNETIC FLUX} = \Phi = \iint_S \vec{B}\cdot d\vec{s} = \iint_S \nabla\times\vec{A}\cdot d\vec{s} \overset{\text{STOKES}}{=} \oint_C \vec{A}\cdot d\vec{R}$$

EVEN THOUGH THE MAGNETIC FIELD IS ZERO AT THE CIRCUIT C. THE PARTICLE "DOES NOT KNOW" ABOUT THE MAGNETIC FIELD IN THE INNER REGION. HOWEVER A PHASE IS ACQUIRED !!! IT MEANS THAT THE VECTOR POTENTIAL HAS A PHYSICAL SIGNIFICANCE.

FOR $\vec{A}=0$ → PARTICLE IN A BOX (OR LOCALIZED PARTICLE AROUND $\vec{R}$) WAVE FUNCTIONS $\psi_m(\vec{r}) = \psi(\vec{r}-\vec{R})$

AND $$H = \frac{1}{2m}P^2 + V(\vec{r}) = H(\vec{P}, \vec{r}-\vec{R})$$

$$H|\psi_m\rangle = E_m|\psi_m\rangle$$

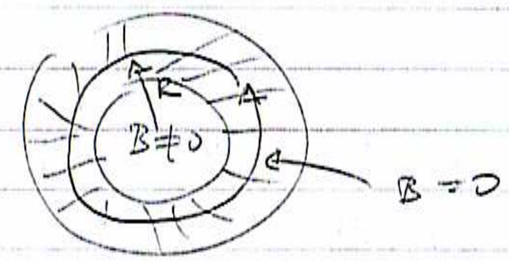
FOR $\vec{A}\neq 0$: INSTANTANEOUS BASIS:

$$H(\vec{P}-q\vec{A}, \vec{r}-\vec{R})\,|m(\vec{R})\rangle = E_m|m(\vec{R})\rangle$$

WHERE $$\langle\vec{r}\,|m(\vec{R})\rangle = \underbrace{e^{\frac{i}{\hbar}\int_{\vec{R}}^{\vec{r}} d\vec{x}\cdot(-q\vec{A}(x))}}_{\text{LIKE AN ADDITIONAL PHASE}}\psi_m(\vec{r}-\vec{R})$$

$$e^{\frac{i}{\hbar}\vec{P}\cdot(\vec{r}-\vec{R})} \to e^{\frac{i}{\hbar}\int_R^r d\vec{x}\cdot\left(\vec{P}(x) - q\vec{A}(x)\right)}$$

FLUX QUANTIZATION
IN SUPERCONDUCTORS



TRAPPED FLUX IN A CIRCULAR SUPERCONDUCTOR WHERE A DISSIPATIONLESS CURRENT FLOWS

$$\Rightarrow \frac{2q}{\hbar} \oint \vec{A} \cdot d\vec{R} = \frac{2q}{\hbar}\phi = 2\pi n \qquad n = 0, 1, 2, \ldots$$

$$\hookrightarrow \phi_0 = \frac{2\pi\hbar}{2q} = \frac{h}{2q} = 2.07\ 10^{-7}\ \text{GAUSS}\cdot$$

NOTICE THE ENERGIES ARE UNAFFECTED BY $A(\vec{r})$

FOR INSTANCE, IF $\psi_m(\vec{r})$ ARE PLANE WAVES

$\Rightarrow \psi_m = e^{i\vec{k}_m \cdot \vec{r}}$

$$H = \frac{1}{2m}(\vec{p} - q\vec{A})^2 \rightarrow H\, e^{\frac{i}{\hbar}\int_R^r d\vec{x}\cdot\left[\vec{k}_m - q\vec{A}(\vec{x})\right]} = E_m \psi_m(\vec{R}))$$

NOW LET THE BOX BE TRANSPORTED ALONG THE CIRCUIT C (NO NEED TO BE ADIABATICALLY TRANSPORTED)

$$\Rightarrow \gamma_m = i\oint_C \langle m(\vec{R})|\nabla_R|m(\vec{R})\rangle \cdot d\vec{R}$$

WHERE $\langle m(\vec{R})|\nabla_R m(\vec{R})\rangle = \int d^3r\, \psi_m^*(\vec{r}-\vec{R}) * e^{\frac{-i}{\hbar}\int_R^{\vec{r}} d\vec{x}\cdot(-q\vec{A}(x))}\, \nabla_R\left(\psi_m(\vec{r}-\vec{R})\, e^{\frac{+i}{\hbar}\int_R^{\vec{r}} d\vec{x}\cdot(-q\vec{A})}\right)$

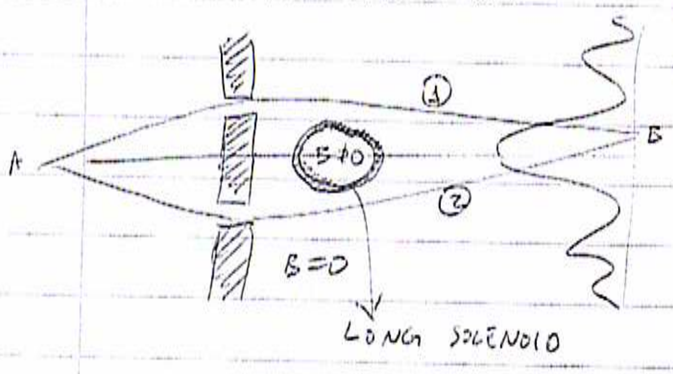
$$= \int d^3r\, \psi_m^*(\vec{r}-\vec{R})\, e^{-(\vec{A}_{PHASE})}\left[\left(\nabla_R \psi_m(\vec{r}-\vec{R})\right) e^{+(\vec{A}_{PHASE})} + \left(\frac{-1}{i\hbar}(-q\vec{A}(\vec{R}))\right)\psi_m(\vec{r}-\vec{R})\, e^{+A_{PH}}\right]$$

$$= \nabla_R \underbrace{\int d^3r\, \psi_m^*(\vec{r}-\vec{R})\psi_m(\vec{r}-\vec{R})}_{1} + \int d^3r\, \psi_m^*(\vec{r}-\vec{R})\psi_m(\vec{r}-\vec{R})\left(-\frac{i}{\hbar} q A(\vec{R})\right)$$

$$= 0 - i q A(\vec{R})/\hbar$$

$$\Rightarrow \gamma_m = i\oint_C \frac{-iq}{\hbar}\vec{A}(\vec{R})\cdot d\vec{R} = \frac{q}{\hbar}\Phi, \quad \boxed{\text{INDEPENDS ON } m \text{ AND } C}$$

DOUBLE-SLIT EXPERIMENT



INTERFERENCE PATTERN DEPENDS ON THE ADDITIONAL PHASE γ_m

$\Rightarrow$ AS THE FIELD IS INCREASED THE INTERFERENCE PATTERN CHANGES PERIODICALLY WITH A FUNDAMENTAL UNITY OF A

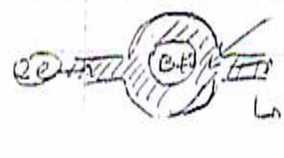
QUANTUM OF MAGNETIC FLUX : $\frac{q}{\hbar}\Phi_0 = 2\pi \rightarrow \Phi_0 = \frac{2\pi\hbar}{q} \approx 4\, 10^{-15}\, Wb$
$= 4\, 10^{-7}$ GAUSS$\cdot$cm²

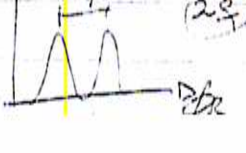
* PHASE DIFFERENCE $\Delta\Phi = \frac{q}{\hbar}\int_{A\,①}^{B}\vec{A}\cdot d\vec{x} - \frac{q}{\hbar}\int_{A\,②}^{B}\vec{A}\cdot d\vec{x} = \frac{q}{\hbar}\oint \vec{A}\cdot d\vec{x} = \frac{q}{\hbar}\Phi$

ALSO IN SUPERCONDUCTORS:

$\Phi_0 = \frac{2\pi\hbar}{(2q)} \rightarrow 2\cdot 10^{-15}\, Wb$

$(2q) \rightarrow$ COOPER PAIR





* SUGGESTION : READ