# TIME REVERSAL SYMMETRY

≡ REVERSAL OF MOTION

- NOTICE IT IS A DISCRETE SYMMETRY

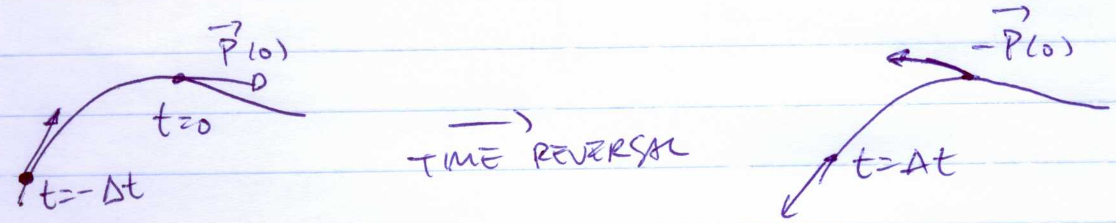
CLASSICAL MECHANICS : $\begin{cases} \vec{r}\,'(t_0) = \vec{r}(t_0) \\ \vec{p}\,'(t_0) = -\vec{p}(t_0) \end{cases}$

$t_0 \equiv$ INSTANT OF THE TIME-REVERSAL OPERATION (TAKE $= 0$)

FOR CONSERVATIVE FORCES $m \dfrac{d^2\vec{r}}{dt^2} = -\nabla V(\vec{r})$

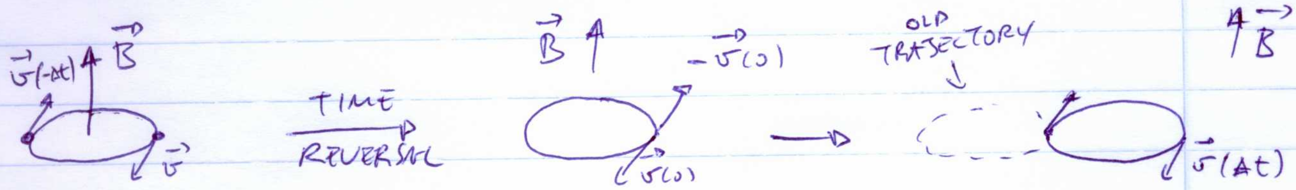
BUT $\vec{r}(-t)$ IS ALSO A SOLUTION OF NEWTON'S EQUATION

$$m \frac{d^2\vec{r}(-t)}{dt^2} = (-1)^2\, m \frac{d^2\vec{r}(t)}{dt^2} = -\nabla V(\vec{r})$$




* FRICTION FORCE BREAKS TIME-REVERSAL SYMMETRY

EXTERNAL MAGNETIC FIELD ALSO BREAKS T.R.S.




RECALL $\vec{F} = q\,\vec{v}\times\vec{B} \longrightarrow \vec{F}' = q\,\vec{v}\,'\times\vec{B}' = -q\,\vec{v}\times\vec{B} = -\vec{F}$

IN ORDER TO RECOVER INVARIANCE $\Rightarrow$ $\vec{B} \to -\vec{B}$, WHICH IS THE CASE AT MICROSCOPIC LEVEL (MAXWELL EQUATIONS)

## IN QUANTUM MECHANICS

$$i\hbar \frac{\partial \psi}{\partial t}(\vec{r},t) = H(\vec{r})\,\psi(\vec{r},t) = \frac{-\hbar^2}{2m}\nabla^2\psi + U(\vec{r})\,\psi(\vec{r},t)$$

DOES $\psi(\vec{r},-t)$ SATISFY THE SCHRÖDINGER'S EQUATION?

NO!

BECAUSE $\dfrac{\partial}{\partial t}\psi(\vec{r},t) \neq \dfrac{\partial}{\partial t}\psi(\vec{r},-t) = -\dfrac{\partial}{\partial t}\psi(\vec{r},t)$

HOWEVER, $\psi^*(\vec{r},-t)$ SATISFIES IT.

SCHRODINGER EQ. FOR $\psi^*$: $i\hbar\,\partial_t \psi = H\psi = -\frac{\hbar^2}{2m}\psi + U\psi$

$$\Rightarrow -i\hbar\,\partial_t \psi^* = -\frac{\hbar^2 \psi^*}{2m} + U\psi^*$$

NOW, MAKING $t \to -t$

$$\Rightarrow i\hbar\,\frac{\partial}{\partial t}\psi^*(\vec{r},-t) = \frac{-\hbar^2}{2m}\nabla^2\psi^*(\vec{r},-t) + U(\vec{r})\,\psi^*(\vec{r},-t)$$

WHICH IS THE SCHRODINGER EQ. FOR $\psi$

$\Rightarrow$ IN CONCLUSION, IN Q.M., TIME-REVERSAL OPERATION INVOLVES
$$\begin{cases} \vec{r} \to \vec{r} \\ \vec{p} \to -\vec{p} \\ \psi \to \psi^* \\ t \to -t \end{cases}$$

• OBS: WE WILL SEE THAT $\psi \to \psi^*(\vec{r},-t)$ HAS SUBTLE BUT IMPORTANT CONSEQUENCES

## THE OPERATOR T (TIME-REVERSAL OPERATOR)

WE WANT AN OPERATOR SUCH THAT $\begin{cases} \vec{r}\,' = T\,\vec{r}\,T^{-1} = \vec{r} \\ \vec{p}\,' = T\,\vec{p}\,T^{-1} = -\vec{p} \end{cases}$

AS A CONSEQUENCE
$$\vec{L}' = T\,\vec{r}\times\vec{p}\,T^{-1} = \vec{r}\,'\times\vec{p}\,' = -\vec{L}$$
AS WE EXPECT THE ORBITAL AND SPINORIAL ANGULAR MOMENTA TO TRANSFORM IN THE SAME WAY

THEN $\begin{cases} \vec{S}' = T\vec{S}\,T^{-1} = -\vec{S} \\ \vec{J}' = T(\vec{L}+\vec{S})T^{-1} = -\vec{J} \end{cases}$ (TOTAL ANGULAR MOMENTUM)

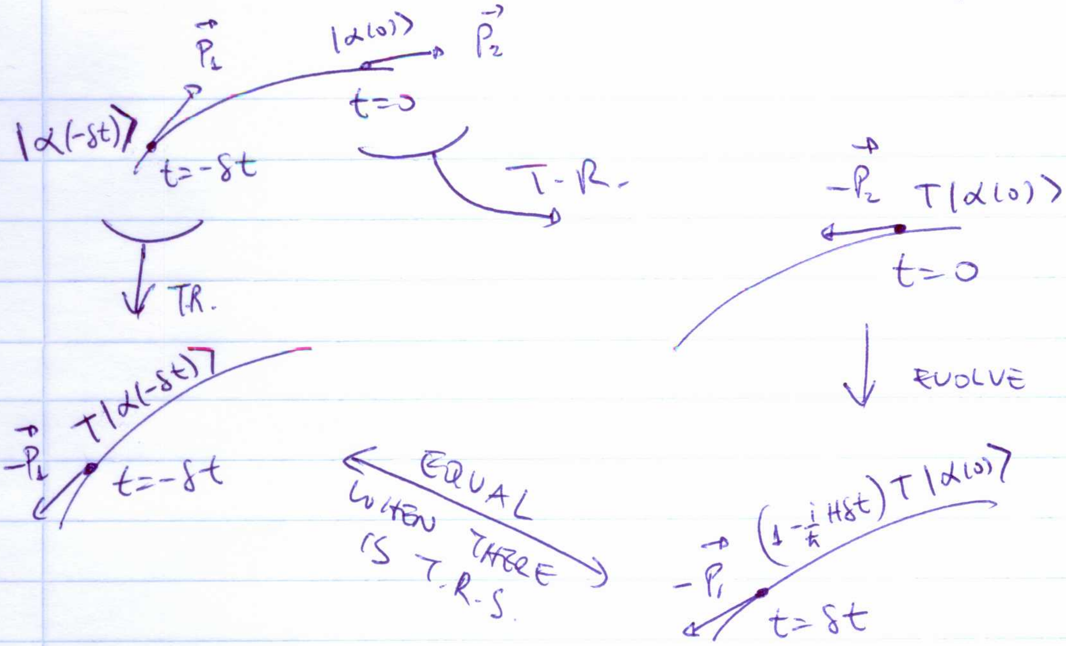
WHAT DO WE EXPECT FROM THE TIME-EVOLUTION POINT OF VIEW?

LET $|\alpha(t=\delta t)\rangle = \left(1 - \frac{i}{\hbar} H\delta t\right)|\alpha(t=0)\rangle$

THEN, IF THERE IS T.R.S., WE EXPECT THAT

$$\underbrace{T\underbrace{\left(1 + \frac{i}{\hbar} H\,\delta t\right)|\alpha(0)\rangle}_{|\alpha(-\delta t)\rangle}}_{\text{TIME-REVERSED STATE AT } t=-\delta t} = \underbrace{\left(1 - i H\delta t/\hbar\right)\underbrace{T|\alpha(0)\rangle}_{\text{TIME-REVERSED STATE AT } t=0}}_{\text{EVOLVED STATE AT } t=\delta t \text{ STARTING FROM THE REVERSED-TIME STATE}}$$

SCHEMATICALLY, WE HAVE THAT




THEREFORE,

$$T\left(1+\frac{i}{\hbar}H\delta t\right) = \left(1-\frac{i}{\hbar}H\delta t\right)T \Rightarrow \boxed{TiH = -iHT}$$

IN ADDITION, THE SYSTEM IS T.R. SYMMETRIC $\Rightarrow [H,T]=0$

THUS, $TiH = -iHT = -iTH$

$$\Rightarrow \boxed{Ti = -iT} \Rightarrow T \text{ IS AN ANTI-LINEAR OPERATOR}$$

ANOTHER VIEW OF ALL THIS

WE EXPECT THAT $\boxed{\frac{\partial}{\partial t}\rho + \nabla\cdot\vec{j} = 0}$ BE ALWAYS VALID

CONTINUITY EQUATION

WHERE $\rho = \psi^*\psi$

$$\vec{j}(\vec{r},t) = \frac{\hbar}{2mi}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) = \frac{1}{2m}\left(\psi^*\vec{p}\psi - \psi\vec{p}\psi^*\right)$$

$$= \frac{1}{2m}\left(\langle\psi|\vec{r}\rangle\langle\vec{r}|\vec{p}|\psi\rangle + \langle\psi|\vec{p}|\vec{r}\rangle\langle\vec{r}|\psi\rangle\right)$$

$$= \frac{1}{m}\,\mathrm{Re}\left(\langle\psi|\vec{r}\rangle\langle\vec{r}|\vec{p}|\psi\rangle\right)$$

TIME-REVERSING: $\begin{cases}\rho' = \rho \\ \vec{j}' = -\vec{j}\end{cases}$ $\Rightarrow$ $\rho'$ AND $\vec{j}'$ SATISFY THE CONTINUITY EQUATION WITH $t\to -t$

$$\Rightarrow \frac{\partial\rho'}{\partial(-t)} + \nabla\cdot\vec{j}' = -\left[\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{j}\right] = 0$$

IN SHORT, $\begin{cases} \rho' = \rho \\ \vec{J}' = \vec{J} \\ t' = -t \end{cases}$

~~DEFINING $|\psi'(t)\rangle = T|\psi(-t)\rangle$ THEN FROM~~ EXPLORING THIS A LITTLE FURTHER, THEN

① $\rho' = \rho \Rightarrow |\langle \vec{r}|T|\psi\rangle|^2 = |\langle \vec{r}|\psi\rangle|^2$
(T DOES NOT ALTER THE PROB. DENSITY)

② $\vec{J}' = -\vec{J} \Rightarrow \vec{J}' = \frac{1}{m} Re\left(\langle \psi'|\vec{r}\rangle\langle \vec{r}|\vec{P}|\psi'\rangle\right)$

$$= \frac{1}{m} Re\left(\langle \psi|T^{+}|\vec{r}\rangle\langle \vec{r}|T\,T^{+}\vec{P}\,T|\psi\rangle\right)$$

$$= \frac{1}{m} Re\left(\langle \psi|T^{+}|\vec{r}\rangle\langle \vec{r}|T\,\vec{P}'|\psi\rangle\right)$$

$$\vec{J} = \frac{1}{m} Re\left(\langle \psi|\vec{r}\rangle\langle \vec{r}|\vec{P}|\psi\rangle\right)$$

FROM ①, A POSSIBLE SOLUTION IS THAT

$$\langle \vec{r}|T|\psi\rangle = \langle \vec{r}|\psi\rangle * \underbrace{e^{i\theta(\vec{r})}}_{\text{PHASE}} \Rightarrow T|\vec{r}\rangle = e^{-i\theta(\vec{r})}|\vec{r}\rangle$$

$\Rightarrow$ T IS A UNITARY OPERATOR

NOTICE THAT THIS YIELDS TO ~~[illegible]~~

$$\langle \psi|T^{+}|\vec{r}\rangle\langle r|T\,\vec{P}'|\psi\rangle = e^{-i\theta(\vec{r})}\langle \psi|\vec{r}\rangle\, e^{i\theta(\vec{r})}\langle \vec{r}|\vec{P}'|\psi\rangle$$

$$= \langle \psi|\vec{r}\rangle\langle \vec{r}|\vec{P}'|\psi\rangle$$

NOW USING ② $(\vec{J}' = -\vec{J})$

$$\Rightarrow \langle \psi|\vec{r}\rangle\langle \vec{r}|\vec{P}'|\psi\rangle = -\langle \psi|\vec{r}\rangle\langle \vec{r}|\vec{P}|\psi\rangle \Rightarrow \boxed{\vec{P}' = -\vec{P}}$$

HOWEVER, THE SOLUTION OF T BEING UNITARY YIELDS TO A CONTRADICTION:

$$[x_j, P_k] = i\hbar\delta_{jk} \Rightarrow T^{+}[x_j, P_k]T = T^{+} i\hbar\delta_{jk} T$$

$$= i\hbar\delta_{jk}$$

BUT $T^{+}[x_j, P_k]T =$

$$= T^{+}x_j T\, T^{+}P_k T - T^{+}P_k T\, T^{+}x_j T$$

$$= x_j(-P_k) - (-P_k)x_j = -[x_j, P_k] = -i\hbar\delta_{jk}$$

IN ORDER TO CONCILIATE THE CONTRADICTION, WE CAN ASSUME THAT $T^{+} i\hbar\delta_{jk} T = T^{+} i T \hbar\delta_{jk}$

$= - i \hbar\delta_{jk}$

$$\Rightarrow \quad T^{+} i T = - i \quad \Rightarrow \boxed{iT = -Ti}$$

IN SUM, $\begin{cases} T T^{+} = T T^{-1} = \mathbb{1} & \text{(UNITARY-LIKE)} \\ i T = - T i & \text{(ANTI-LINEAR)} \end{cases}$

CONSEQUENCES:

* $T(a_1|\alpha\rangle + a_2|\beta\rangle) = a_1^* T|\alpha\rangle + a_2^* T|\beta\rangle$

* THERE IS NO CONSERVATION LAW (LIKE A TIME-REVERSAL QUANTUM NUMBER)

LET $[H,T] = 0$

$$\Rightarrow H'\psi' = T H T^{-1} T \psi = T H \psi = T\, i\hbar \frac{\partial}{\partial t} \psi = - i\hbar \frac{\partial}{\partial t} \psi'$$

$$\Rightarrow H'\psi' = H\psi' = - i\hbar\frac{\partial}{\partial t}\psi' \neq i\hbar\frac{\partial}{\partial t}\psi$$

THUS, THE DYNAMICS OF $\psi'$ IS NOT THE SAME OF $\psi$

$\Rightarrow$ NO CONSERVATION LAW $\Rightarrow$ NO DEGENERACY

(OBS: SEE KRAMERS THEOREM)

ALTERNATIVELY, THIS RESULT CAN BE OBTAINED FROM

$[H,T] = 0$ BUT $[U(t,t_0), T] \neq 0$

T COMMUTES WITH THE GENERATOR OF THE TIME TRANSLATIONS

T DOES NOT COMMUTE WITH THE OPERATORS OF THE (REPRESENTATION OF THE) TIME-TRANSLATION GROUP

$$\boxed{U' = T^{+} e^{-i\frac{H}{\hbar}(t-t_0)} T = e^{+\frac{i}{\hbar}H(t-t_0)} \neq U}$$

DEFINITION: AN ANTI-UNITARY TRANSFORMATION IS THAT IN WHICH

$$\begin{cases} (\beta|\alpha') = \langle\beta|\alpha\rangle^* \\ (a_1|\alpha\rangle + a_2|\beta\rangle)' = a_1^*|\alpha'\rangle + a_2^*|\beta'\rangle \end{cases}$$

IN GENERAL, THIS TRANSFORMATION CAN BE WRITTEN AS

$$T = UK,$$

WHERE

$$\begin{cases} U \equiv \text{UNITARY OPERATOR} \\ K \equiv \text{COMPLEX CONJUGATE OPERATOR} \end{cases}$$

<u>PROPERTIES OF K</u> $\quad K a|\alpha\rangle = a^* K|\alpha\rangle$ (FOLLOW FROM $Ki = -iK$)

WHAT DOES HAPPEN TO $|\alpha\rangle$ WHEN $K$ ACTS ON IT?

SUPPOSE WE CAN EXPAND $|\alpha\rangle$ IN A REAL KET BASIS

$$\Rightarrow |\alpha\rangle = \sum_\ell a_\ell |\ell\rangle$$

$$\Rightarrow K|\ell\rangle = |\ell\rangle \quad \Rightarrow \quad K|\alpha\rangle = \sum_\ell a_\ell^* |\ell\rangle$$

EXAMPLE: $|+\rangle$ AND $|-\rangle$ OF THE $\sigma_z$ PAULI MATRIX

$$\Rightarrow K|\pm\rangle = |\pm\rangle$$

EIGENKETS OF $\sigma_x$: $|\pm\rangle_x = \frac{1}{\sqrt{2}}\left(|+\rangle \pm |-\rangle\right)$

$$\Rightarrow K|\pm\rangle_x = |\pm\rangle_x$$

EIGENKETS OF $\sigma_y$: $|\pm\rangle_y = \frac{1}{\sqrt{2}}\left(|+\rangle \pm i|-\rangle\right)$

$$\Rightarrow K|\pm\rangle_y = |\mp\rangle_y$$

THUS, $K|\ell\rangle$ DEPENDS ON THE CHOICE OF THE BASIS. IF WE HAD CHOSEN $|\pm\rangle_y$ TO BE THE REAL KET BASIS, THEN WE WOULD HAD

$$K|\pm\rangle_y = |\pm\rangle_y$$

IN ORDER TO HAVE A UNIQUE DEFINITION FOR $T = UK \Rightarrow U$ ALSO HAS TO DEPEND ON THE BASIS SUCH THAT IT CANCELS THE NON PHYSICAL EFFECT OF K.

① $UK\left(a_1|\alpha\rangle + a_2|\beta\rangle\right) = U\left(a_1^* K|\alpha\rangle + a_2^* K|\beta\rangle\right)$
$= a_1^*\left(UK|\alpha\rangle\right) + a_2^*\left(UK|\beta\rangle\right)$

② $\langle\beta'|\alpha'\rangle = \langle\beta|\alpha\rangle^*$

NOTICE THAT $|\alpha'\rangle = UK|\alpha\rangle = UK\sum_\ell c_\ell|\ell\rangle = \sum_\ell c_\ell^*\left(U|\ell\rangle\right)$

LIKEWISE, $|\beta'\rangle = \sum_m d_m^*\left(U|m\rangle\right) \Rightarrow \langle\beta'| = \sum_m d_m\left(\langle m|U^+\right)$

$\Rightarrow \langle\beta'|\alpha'\rangle = \sum_{\ell m} d_m\left(\langle m|U^+\right) c_\ell^*\left(U|\ell\rangle\right)$

$= \sum_{\ell m} d_m c_\ell^* \langle m|\ell\rangle = \sum_\ell d_\ell c_\ell^* = \langle\alpha|\beta\rangle$

$= \langle\beta|\alpha\rangle^*$ (SINCE $U^+ c^* U = c^*$)

$\Rightarrow$ ② IS PRESERVED

THE COMMUTATION RELATION IS ALSO PRESERVED

$[r_j', p_k'] = T[r_j, p_k]T^+ = UK(i\hbar\delta_{jk})K^+U^+$
$\quad\| \qquad = -i\hbar\delta_{jk}$
$[r_j, -p_k] = -i\hbar\delta_{jk}$

WAVE FUNCTION (RELOAD)

$|\psi\rangle = \int d\vec{r}\,|\vec{r}\rangle\langle\vec{r}|\psi\rangle = \int d\vec{r}\,\psi(\vec{r})|\vec{r}\rangle$

$\Rightarrow T|\psi\rangle = \int d\vec{r}\,T\left(\psi(\vec{r})|\vec{r}\rangle\right) = \int d\vec{r}\,\psi^*(\vec{r})\,T|\vec{r}\rangle$

CHOOSING $T|\vec{r}\rangle = |\vec{r}\rangle \Rightarrow T|\psi\rangle = \int d\vec{r}\,\psi^*(\vec{r})|\vec{r}\rangle$

$\Rightarrow$ UNDER TIME REVERSAL, $\psi \to \psi^*$ AS ARGUED WHEN WE ANALYZED THE SCHRÖDINGER EQUATION

EXAMPLE: SPHERICAL HARMONIC

$$Y_{\ell m}(\theta,\phi) \Rightarrow Y^*_{\ell m}(\theta,\phi) = (-1)^m Y_{\ell -m}(\theta,\phi)$$

$$\Rightarrow \boxed{T|\ell, m\rangle = (-1)^m |\ell, -m\rangle}$$

## TIME REVERSAL OPERATOR FOR A SPIN-1/2 PARTICLE

WHAT WE HAVE: $\begin{cases} \vec{r} \to \vec{r} \\ \vec{p} \to -\vec{p} \\ \vec{\sigma} \to -\vec{\sigma} \end{cases}$

THUS, WE WANT AN OPERATOR THAT

$$T^\dagger \vec{\sigma} T = -\vec{\sigma} = K^\dagger U^\dagger \vec{\sigma} U K \quad \text{WHERE}$$

K IS THE COMPLEX-CONJUGATION OPERATOR

$$\Rightarrow \begin{cases} K^\dagger \sigma_x K = K^\dagger \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} K = \sigma_x \\ K^\dagger \sigma_y K = K^\dagger \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} K = -\sigma_y \\ K^\dagger \sigma_z K = K^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} K = \sigma_z \end{cases}$$

(* NOTICE WE USED THE EIGENBASIS OF $\sigma_z$)

WITH THAT, WE NEED

$$\begin{cases} U^\dagger \sigma_x U = -\sigma_x \\ U^\dagger \sigma_y U = \sigma_y \\ U^\dagger \sigma_z U = -\sigma_z \end{cases}$$

$\Rightarrow$ U IS JUST LIKE A ROTATION OF

$\pi$ AROUND THE y-AXIS

(* A ROTATION... THIS APPEARS A LOT)

THEREFORE $U = e^{\frac{i}{\hbar}\pi \hat{y}\cdot\vec{S}}$

$$= e^{i\frac{\pi}{2}\sigma_y} = i\,\sigma_y$$

NOW, NOTICE THAT $T^2 = i\sigma_y\, i\sigma_y = -\sigma_y^2 = -\mathbb{1}$

THIS IS A REMARKABLE RESULT FOR SPIN-1/2 SYSTEM

HOW ABOUT FOR A SYSTEM OF 2 SPIN-1/2 PARTICLES?

$\vec{S}_{tot} = \frac{\hbar}{2}(\vec{\sigma}_1 + \vec{\sigma}_2) \Rightarrow T\vec{S}_{tot}T^{+} = -\vec{S}_{tot}$

AGAIN, $\begin{cases} K^{+}S_x K = K^{+}\frac{\hbar}{2}(\sigma_{1x}+\sigma_{2x})K = S_x \\ K^{+}S_y K = -S_y \\ K^{+}S_z K = S_z \end{cases}$

THUS, AGAIN $U$ IS LIKE A ROTATION OF $\pi$ AROUND $\hat{y}$

$$U = e^{i\frac{\pi}{2}(\sigma_{1y}+\sigma_{2y})} = e^{i\frac{\pi}{2}\sigma_{1y}}\, e^{i\frac{\pi}{2}\sigma_{2y}} = (i\sigma_{1y})(i\sigma_{2y})$$
$$= -\sigma_{1y}\sigma_{2y}$$

$$\Rightarrow T^2 = (-\sigma_{1y}\sigma_{2y}K)(-\sigma_{1y}\sigma_{2y}K) = +K^2\sigma_{1y}^2\sigma_{2y}^2 = +\mathbb{1}$$

IT IS EASY TO GENERALIZE FOR A SYSTEM OF N SPIN-1/2 PARTICLES

$$T = \left(i^N \prod_{j=1}^{N} \sigma_{jy}\right) K$$

THUS, $\boxed{T^2 = (-1)^N}$

WE CAN GO EVEN FURTHER AND ASK HOW IS T FOR A SYSTEM OF PARTICLES WITH A TOTAL ANGULAR MOMENTUM EQUAL TO $\vec{J}$

FOR THAT, JUST RECALL THAT WE CAN OBTAIN A TOTAL ANGULAR MOMENTUM EQUAL TO J BY SUMMING $N = 2J$ SPIN-1/2 ANGULAR MOMENTA AND RESTRICTING OURSELVES TO THE CORRESPONDING SUBSPACE OF ALL POSSIBLE RESULTS OF $\vec{\sigma}_1 + \vec{\sigma}_2 + \cdots + \vec{\sigma}_N$

$$\Rightarrow T^2 = (-1)^{2J} = \begin{cases} +\mathbb{1} & \text{FOR INTEGER } J \\ -\mathbb{1} & \text{FOR HALF-INTEGER } J \end{cases}$$

$$\Rightarrow T^2 |J\,M\rangle = (-1)^{2J} |J\,M\rangle$$

KRAMER'S DEGENERACY: SYSTEMS OF N SPIN-1/2 PARTICLES WHICH ARE TIME-REVERSAL INVARIANT (NO EXTERNAL MAGNETIC FIELD AND $[H, T] = 0$) HAVE STATES THAT ARE, AT LEAST, DOUBLY DEGENERATE IF N IS ODD.

PROOF: $|\psi'\rangle = T|\psi\rangle$, BECAUSE $[H, T] = 0 \Rightarrow |\psi'\rangle$ HAS THE SAME ENERGY AS $|\psi\rangle$.
THIS DOES NOT ~~GUARANTE~~ WARRANTS DEGENERACY. WE NEED TO COMPUTE $\langle\psi|\psi'\rangle$ IN ORDER TO VERIFY IF $|\psi'\rangle$ IS ORTHOGONAL TO $|\psi\rangle$

$$\langle\beta|\alpha\rangle = \langle\beta'|\alpha'\rangle^* = \langle\alpha'|\beta'\rangle$$

$$\Rightarrow \langle\beta|\alpha\rangle - \langle\alpha'|\beta'\rangle = 0$$

CHOOSING $|\beta\rangle = |\psi'\rangle$, $|\alpha\rangle = |\psi\rangle$

$$\Rightarrow |\beta'\rangle = T|\psi'\rangle = T^2|\psi\rangle = (-1)^N|\psi\rangle, \quad |\alpha'\rangle = |\psi'\rangle$$

$$\Rightarrow \langle\psi'|\psi\rangle - \langle\psi'|\psi\rangle(-1)^N = 0$$

$$\Rightarrow \left(1 - (-1)^N\right)\langle\psi'|\psi\rangle = 0$$

$\Rightarrow |\psi\rangle$ AND $|\psi'\rangle$ ARE ORTHOGONAL AND DEGENERATE IF N IS ODD.