List of exercises #2 (3D dynamics) - 7600018

1. **A**, **B**, and **C** are any three vectors not lying in a single plane.
   (a) Prove that $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ is the volume of the parallelepiped whose edges are **A**, **B**, and **C** with positive or negative sign according to whether a right-hand screw rotated from **A** toward **B** would advance along **C** in the positive or negative direction.
   (b) Use this result to prove geometrically that $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$.

2. A particle is acted on by a force whose components are $F_x = ax^3 + bxy^2 + ez$, $F_y = ay^3 + bx^2y$, $F_z = ex$.
   (a) Calculate the work done by this force when the particle moves along a straight line from the origin to the point (x_0, y_0, z_0).
   (b) What is the other force acting on the particle so it moves along the tractory with constant acceleration.

3. A particle in the xy-plane is attracted toward the origin by a force $F = k/y$, inversely proportional to its distance from the x-axis.
   (a) Calculate the work done by the force when the particle moves from the point $(0, a)$ to the point $(2a, 0)$ along a path which follows the sides of a rectangle consisting of a segment parallel to the x-axis from $(0, a)$ to $(2a, a)$, and a vertical segment from the latter point to the x-axis.
   (b) Calculate the work done by the same force when the particle moves along an ellipse of semi axes a and $2a$. Hint: Set $x = 2a\sin\theta$ and $y = a\cos\theta$.

4. Find the r- and θ-components of $\mathrm{d}\mathbf{a}/\mathrm{d}t$ in plane polar coordinates, where **a** is the acceleration of a particle.

5. (Optional) Plane parabolic coordinates f and h are defined in terms of cartesian coordinates x and y by the equations $x = f - h$ and $y = 2\sqrt{fh}$, where f and h are never negative.
   (a) Find f and h in terms of x and y.
   (b) Let unit vectors $\hat{f}$ and $\hat{h}$ be defined in the directions of increasing f and h respectively. That is, $\hat{f}$ is a unit vector in the direction in which a point would move if its f-coordinate increases slightly while its h-coordinate remains constant. Show that $\hat{f}$ and $\hat{h}$ are perpendicular at every point. Hint: $\hat{f} = \frac{\hat{x}\mathrm{d}x + \hat{y}\mathrm{d}y}{\sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}}$, when $\mathrm{d}f > 0$ and $\mathrm{d}h = 0$. Why?
   (c) Show that $\hat{f}$ and $\hat{h}$ are functions of f and h, and find their derivatives with respect to f and h.
   (d) Show that $\mathbf{r} = \sqrt{f+h}\left(\sqrt{f}\hat{f} + \sqrt{h}\hat{h}\right)$. Find the components of velocity and acceleration in parabolic coordinates.
   (e) A particle moves along the parabola $y^2 = 4f_0^2 - 4f_0x$, where f_0 is a constant. Its speed v is constant. Find its velocity and acceleration components in rectangular and in polar coordinates. Show that the equation of the parabola in polar coordinates is $r\cos^2(\theta/2) = f_0$. What is the equation of this parabola in parabolic coordinates?

6. A particle moves with varying speed along an arbitrary curve lying in the xy-plane. The position of the particle is to be specified by the distance s the particle has traveled along that curve from some fixed point on the curve. Let $\hat{\tau}(s)$ be a unit vector tangent to the curve at the point s in the direction of increasing s.
   (a) Show that $\frac{\mathrm{d}\hat{t}}{\mathrm{d}s} = \frac{\hat{\nu}}{R}$, where $\hat{\nu}(s)$ is a unit vector normal to the curve at the point s, and R is the radius of curvature at the point s, defined as the distance from the curve to the point of intersection of two nearby normals.
   (b) Show that the velocity and acceleration of the particle are, respectively, $\mathbf{v} = \dot{s}\hat{\tau}$ and $\mathbf{a} = \ddot{s}\hat{\tau} + \frac{\dot{s}^2}{R}\hat{\nu}$.

7. What is the geometric interpretation of the Gauss' (divergence) theorem $\iiint_V \boldsymbol{\nabla} \cdot \mathbf{A}\mathrm{d}V = \oiint_S \mathbf{A} \cdot \hat{n}\mathrm{d}S$ and of the Stoke's theorem $\iint_S (\boldsymbol{\nabla} \times \mathbf{A}) \cdot \hat{n}\mathrm{d}S = \oint_{\partial S} \mathbf{A} \cdot \mathrm{d}\mathbf{r}$

8. Calculate $\boldsymbol{\nabla} \times \mathbf{A}$ in cylindrical coordinates.

9. Using the Levi-Civita tensor, show that $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} f) = 0$ where f is a twice-differentiable function of $\mathbf{r}$, and that $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) = 0$, where $\mathbf{A}$ is a twice-differentiable vector field of $\mathbf{r}$.
Recall that the Levi-Civita tensor ε_{ijk} is

$$\varepsilon_{ijk} = \begin{cases} 1, & \text{if } (i,j,k) = (1,2,3)\,, \text{ or } (2,3,1)\,, \text{ or } (3,1,2)\,, \\ -1, & \text{if } (i,j,k) = (2,1,3)\,, \text{ or } (3,2,1)\,, \text{ or } (1,3,2)\,, \\ 0, & \text{otherwise,} \end{cases}$$

and that

$$\mathbf{A} \times \mathbf{B} = \sum_{i=1}^{3} \left(\sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{ijk} A_j B_k \right) \hat{e}_i, \text{ where } \mathbf{V} = V_x \hat{e}_1 + V_y \hat{e}_2 + V_z \hat{e}_3.$$

10. A particle of mass m moves according to the equations $x = x_0 + at^2$, $y = bt^3$, and $z = ct$. Find the angular momentum L at any time t with respect to the origin. Find the force F and from it the associated torque N acting on the particle. Verify that the angular momentum theorem $\dot{\mathbf{L}} = \mathbf{r} \times \mathbf{F} = \mathbf{N}$ is satisfied.

11. The trajectory of an harmonic oscillator is $x = A\cos(\omega t)$, $y = B\cos(\omega t + \theta)$, and $z = C\cos(2\omega t)$. Obtain the corresponding xy and xz and yz Lissajous figures for the case $\theta = 0$, $\theta = \frac{\pi}{2}$, and $\theta = \pi$.

12. (Optional) A projectile is fired from the origin with initial velocity $\mathbf{v}_0 = (v_{x0}, v_{y0}, v_{z0})$. The wind velocity is $\mathbf{v}_w = w\hat{y}$. Solve the equations of motion $m\ddot{\mathbf{r}} = -mg\hat{z} - b(\dot{\mathbf{r}} - \mathbf{v}_w)$. Find the point $(x_r, y_r, 0)$ at which the projectile will return to the horizontal plane, keeping only first-order terms in b. Show that if air resistance and wind velocity are neglected in aiming the gun, air resistance alone will cause the projectile to fall short of its target a fraction $4bv_{z0}/(3mg)$ of the target distance, and that the wind causes an additional miss in the y-coordinate of amount $2bwv_{z0}^2/(mg^2)$.

13. A projectile is to be fired from the origin in the xz-plane (z-axis vertical) with muzzle velocity v_0 to hit a target at the point $x = x_0$ and $z = 0$.
    (a) Neglecting air resistance, find the correct angle of elevation of the gun. Show that, in general, there are two such angles unless the target is at or beyond the maximum range.
    (b) Find the first-order correction to the angle of elevation due to air resistance (the force of which is viscous $-b\mathbf{v}$).

14. Determine which of the following forces are conservative, and find the potential energy for those which are:
    (a) $\mathbf{F} = (6abyz^3 - 20bx^3y^2)\,\hat{x} + (6abxz^3 - 10bx^4y)\,\hat{y} + 18abxyz^2\,\hat{z}$,
    (b) $\mathbf{F} = (18abyz^3 - 20bx^3y^2)\,\hat{x} + (18abxz^3 - 10bx^4y)\,\hat{y} + 6abxyz^2\,\hat{z}$,
    (c) $\mathbf{F} = F_x(x)\,\hat{x} + F_y(y)\,\hat{y} + F_z(z)\,\hat{z}$,
    (d) $\mathbf{F} = -2ar\sin\theta\cos\phi\hat{r} - ar\cos\theta\cos\phi\hat{\theta} + ar\sin\theta\sin\phi\hat{\phi}$.

15. Find the force associated to the following potential-energy functions:
    (a) $V = axy^2z^3$,
    (b) $V = \frac{1}{2}kr^2$,
    (c) $V = -k/r$.

16. The potential energy for an isotropic harmonic oscillator is $V = \frac{1}{2}kr^2$. Plot the effective potential energy for the r-motion when a particle of mass m moves with this potential energy and with angular momentum L about the origin. Discuss the types of motion that are possible, giving as complete a description as is possible without carrying out the solution. Find the frequency of revolution for circular motion and the frequency of small radial oscillations about this circular motion. Hence describe the nature of the orbits which differ slightly from circular orbits. Find $r(t)$ and $\theta(t)$.

17. (Optional) According to Yukawa's theory of nuclear forces, the attractive force between a neutron and a proton has the potential $V(r) = K\frac{e^{-\alpha r}}{r}$, where $\alpha > 0$ and $K < 0$.

(a) Find the force, and compare it with an inverse square law of force.

(b) Discuss the types of motion which can occur if a particle of mass m moves under such a force.

(c) Discuss how the motions will be expected to differ from the corresponding types of motion for an inverse square law of force.

(d) Find L and E for motion in a circle of radius R.

(e) Find the period of circular motion and the period of small radial oscillations.

(f) Show that the nearly circular orbits are almost closed when R is very small. (Small compared to what?)

18. (Optional) The effect of a uniform distribution of dust of density ρ about the sun is to add to the gravitational attraction of the sun on a planet of mass m an additional attractive central force $F' = -mkr$, where $k = \frac{4}{3}\pi\rho G$.

(a) If the mass of the sun is M, find the angular velocity of revolution of the planet in a circular orbit of radius R, and find the angular frequency of small radial oscillations. Hence show that if F' is much less than the attraction due to the sun, a nearly circular orbit will be approximately an ellipse whose major axis precesses slowly with angular velocity $2\pi\rho\sqrt{\frac{R^3 G}{M}}$.

(b) Does the axis precess in the same or in the opposite direction to the orbital angular velocity? Look up M and the radius of the orbit of Mercury, and calculate the density of dust required to cause a precession of 41 seconds of arc per century.

19. (Optional) Consider a particle moving under an attractive central force inversely proportional to the cube of the radius

$$F = -\frac{k}{r^3}, \text{ with } k > 0.$$

(a) Using the method of the effective potential, discuss the types of motion.

(b) Find the ranges of energy and angular momentum for each type of motion.

(c) Show that the orbit is one of the forms:

$$\begin{aligned} r^{-1} &= A\cos\left[\beta\left(\theta-\theta_0\right)\right], \\ r^{-1} &= A\cosh\left[\beta\left(\theta-\theta_0\right)\right], \\ r^{-1} &= A\sinh\left[\beta\left(\theta-\theta_0\right)\right], \\ r^{-1} &= A\left(\theta-\theta_0\right), \\ r^{-1} &= r_0^{-1}e^{\pm\beta\theta}. \end{aligned}$$

(d) For what values of L and E does each of the above types of motion occur? Express the constants A and β in terms of E and L for each case.

(e) Sketch a typical orbit of each type.

20. Consider a particle of mass m moving under the central force $F(r) = -\frac{k}{r^2} + \frac{q}{r^3}$, where k and q are constants.

(a) Discuss the types of motion that can occur assuming that $k > 0$, and consider both signs for q.

(b) Solve the orbital equation, and show that the bounded orbits have the form $r = \frac{a\left(1-\epsilon^2\right)}{1+\epsilon\cos\alpha\theta}$ if $L^2 > -mq$.

(c) Show that this is a precessing ellipse, determine the angular velocity of precession, and state whether the precession is in the same or in the opposite direction to the orbital angular velocity.

21. Sputnik I had a perigee (point of closest approach to the earth) 227 km above the earth's surface, at which point its speed was 28,710 km/h. Find its apogee (maximum) distance from the earth's surface and its period of revolution. (Assume the earth is a sphere, and neglect air resistance. You need only look up g and the earth's radius to solve this problem.)

22. A comet is observed a distance of 1.00×10^8 km from the sun, traveling toward the sun with a velocity of 51.6 km/s at an angle of 45° with the radius from the sun. Work out an equation for the orbit of the comet in polar coordinates with origin at the sun and x-axis through the observed position of the comet. (The mass of the sun is 2.00×10^{30} kg.)

23. (Optional) It can be shown that the correction to the potential energy of a mass m in the earth's gravitational field, due to the oblate shape of the earth, is approximately, in spherical coordinates, relative to the polar axis of the earth,

$$V' = -\eta \frac{GMm}{5r^3}\left(1 - 3\cos^2\theta\right),$$

where M is the mass of the earth and $2R$ and $2R(1-\eta)$ are, respectively, the equatorial and polar diameters of the earth.

(a) Calculate the rate of precession of the perigee (point of closest approach) of an earth satellite moving in a nearly circular orbit in the equatorial plane. Look up the equatorial and polar diameters of the earth, and estimate the rate of precession in degrees per revolution for a satellite 400 miles above the earth.

(b) Calculate the torque on another earth satellite moving in a circular orbit of radius r_0 whose plane is inclined so that its normal makes an angle α with the polar axis. Assume that the orbit is very little affected in one revolution, and calculate the average torque during a revolution. Show that the effect of such a torque is to make the normal to the orbit precess in a cone of half angle α about the polar axis, and find a formula for the rate of precession in degrees per revolution. Calculate the rate for a satellite 400 miles above the earth, using suitable values for M, η, and R.

24. Show that the scattering angle for a particle of mass m subject to a central force $F(r)$ is

$$\Theta = \left|\pi - \int_0^{u_0} \frac{2s}{\sqrt{1 - s^2u^2 - \frac{V(1/u)}{\frac{1}{2}mv_0^2}}}\mathrm{d}u\right|,$$

where $V(r) = \int_r^\infty F(r)\,\mathrm{d}r$ is the potential energy, s is the impact parameter, v_0 is the particle's speed far from the scattering center, and u_0 is such that $1 - s^2u_0^2 - \frac{V(1/u_0)}{\frac{1}{2}mv_0^2} = 0$.

Hint: The orbit equation is $\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} = -u - \frac{m}{L^2u^2}F(1/u) = \mathcal{F}_R(u)$. Show that $\mathcal{F}_R(u)\,du = d\left(\frac{1}{2}w^2\right)$, where $w = \frac{\mathrm{d}u}{\mathrm{d}\theta}$. Show that $\mathcal{E} = \frac{1}{2}w^2 + \frac{1}{2}u^2 + \frac{m}{L^2}V(1/u)$ is conserved along the orbit. Show that, for $u \to 0$, then $w \to 1/s$, and, thus, $\mathcal{E} = \frac{1}{2s^2}$. Show that $L = mv_0s$. Finally, show that $\Delta\theta = \pi - \Theta$.
Notice that, in essence, this is the energy trick if we replace $(x,t) \to (u,\theta)$.

25. (Optional) A particle is reflected from the surface of a hard sphere of radius R in such a way that the incident and reflected lines of travel lie in a common plane with the radius to the point of impact and make equal angles with the radius.

(a) Find the cross-section $\mathrm{d}\sigma$ for scattering through an angle between Θ and $\Theta + \mathrm{d}\Theta$. Integrate $\mathrm{d}\sigma$ over all angles and show that the total cross-section has the expected value πR^2.

(b) The potential of the hard sphere is simply

$$V_{\mathrm{HS}}(r) = \begin{cases} 0, & \text{if } r > R, \\ \infty, & \text{if } r \leq R. \end{cases}$$

Using the result of problem 24, show that such a potential gives the same law of reflection as specified in the statement of this problem.

26. Show that for a repulsive central force inversely proportional to the cube of the radius,

$$F = \frac{k}{r^3}, \text{ with } k > 0,$$

the orbits are of the form $r^{-1} = A\cos\left[\beta(\theta - \theta_0)\right]$ and express β in terms of k, E, L, and the mass m of the incident particle. Show that the cross-section for scattering through an angle between Θ and $\Theta + \mathrm{d}\Theta$ for a particle subject to this force is

$$\mathrm{d}\sigma = \frac{2\pi^3 k(\pi - \Theta)}{mv_0^2\Theta^2(2\pi - \Theta)^2}\mathrm{d}\Theta.$$

27. (Optional) A particle of charge q, mass m at rest in a constant, uniform magnetic field $\mathbf{B} = B_0\hat{z}$ is subject, beginning at $t = 0$, to an oscillating electric field $\mathbf{E} = E_0\sin\omega t\hat{x}$. Find its motion.