List of exercises #5 (Lagrange's equations) - 7600018

For Test3, consider only the problems marked with ♦.

1. Set up the Lagrangian function for the mechanical system of 3 masses constrained to move freely in 1D and connected by two ideal springs. Derive the equations of motion, and show that they are equivalent to the equations that would be written down directly from Newton's law of motion.

2. ♦ Coordinates u and w are defined in terms of plane polar coordinates ρ and θ by the equations $u = \ln \rho/a - \theta \cot \zeta$ and $w = \ln \rho/a + \theta \tan \zeta$, where a and ζ are constants. Sketch the curves of constant u and of constant w. Find the kinetic energy for a particle of mass m in terms of u, w, $\dot{u}$, and $\dot{w}$. Find expressions for the generalized forces Q_u and Q_w in terms of the polar force components F_ρ and F_θ. Find the generalized moments P_u and P_w. Find the forces Q_u and Q_w required to make the particle move with constant speed $\dot{s}$ along a spiral of constant $u = u_0$.

3. ♦ Two masses m_1 and m_2 move under their mutual gravitational attraction in a uniform external gravitational field whose acceleration is $\mathbf{g} = g\hat{z}$. Choose as generalized coordinates the Cartesian coordinates X, Y, and Z of the center of mass, the distance r between m_1 and m_2, and the polar angles θ and ϕ which specify the direction of the line from m_1 to m_2. Write expressions for the kinetic energy, the six generalized forces Q_X, ..., Q_ϕ, and the six momenta. Write out the six Lagrange's equations of motion.

4. (Optional) Set up the expression for the kinetic energy of a particle of mass m in terms of plane parabolic coordinates f and h as defined in Problem 5 of the List of Exercises #2. Find the momenta P_f and P_h. Write out the Lagrange's equations in these coordinates if the particle is not acted on by any force. Find the forces Q_f and Q_h required to make this particle to move along a parabola $f = f_0 =$ a constant, with constant generalized velocity $\dot{h} = \dot{h}_0$, starting from $h = 0$ at $t = 0$. Find the corresponding forces F_x and F_y relative to a Cartesian coordinate system.

5. ♦ Masses m and $2m$ are suspended from a string of length ℓ_1 which passes over a pulley. Masses $3m$ and $4m$ are similarly suspended by a string of length ℓ_2 over another pulley. These two pulleys hang from the ends of a string of length ℓ_3 over a third fixed pulley. Set up Lagrange's equations, and find the accelerations and the tensions in the strings.

6. A massless tube is hinged at one end. A uniform rod of mass m, length L, slides freely in it. The axis about which the tube rotates is horizontal, so that the motion is confined to a plane. Choose a suitable set of generalized coordinates, one for each degree of freedom, and set up Lagrange's equations.

7. Set up Lagrange's equations for a uniform door whose axis is slightly out of plumb. What is the period of small vibrations?

8. A double pendulum is formed by suspending a mass m_2 by a string of length ℓ_2 from a mass m_1 which in turn is suspended from a fixed support by a string of length ℓ_1.

   (a) Choose a suitable set of coordinates, and write the Lagrangian function, assuming the double pendulum swings in a single vertical plane.

   (b) Write out Lagrange's equations, and show that they reduce to the equations for a pair of coupled oscillators if the strings remain nearly vertical.

   (c) Find the normal frequencies for small vibrations of the double pendulum. Describe the nature of the corresponding vibrations. Find the limiting values of these frequencies when $m_1 \gg m_2$ and when $m_2 \gg m_1$. Show that these limiting values are to be expected on physical grounds by considering the nature of the normal modes of vibration when either mass becomes vanishingly small.

9. A particle moves in a plane under the influence of a force $f = -Ar^{\alpha-1}$ directed toward the origin; A and $\alpha > 0$ are constants. Choose appropriate generalized coordinates, and let the potential energy be zero at the origin. Find the Lagrangian equations of motion. Is the angular momentum about the origin conserved? Is the total energy conserved?

10. ♦ A hoop of mass m and radius R rolls without slipping down an inclined plane of mass M, which makes an angle α with the horizontal. Find the Lagrange equations and the integrals of the motion if the plane can slide without friction along a horizontal surface.

11. Consider a region of space divided by a plane. The potential energy of a particle in region 1 is U_1 and in region 2 it is U_2. If a particle of mass m and with speed v_1 in region 1 passes from region 1 to region 2 such that its path in region 1 makes an angle θ_1 with the normal to the plane of separation and an angle θ_2 with the normal when in region 2, show that

$$\frac{\sin\theta_1}{\sin\theta_2} = \sqrt{1 + \frac{U_1 - U_2}{T_1}},$$

where $T_1 = \frac{1}{2}mv_1^2$. What is the optical analog of this problem?

12. ♦ A particle of mass m rests on a smooth plane. The plane is raised to an inclination angle θ at a constant rate α ($\theta = 0$ at $t = 0$), causing the particle to move down the plane. Determine the motion of the particle.

13. A pendulum of length ℓ and mass m is oscillating at small angles when the length of the pendulum string is shortened at a constant rate α ($\mathrm{d}\ell/\mathrm{d}t = -\alpha$). Find Lagrange's equations of motion. Compare the Hamiltonian and the total energy, and discuss the conservation of energy for the system.

14. (Optional) A disk rolls without slipping across a horizontal plane. The plane of the disk remains vertical, but it is free to rotate about a vertical axis. What generalized coordinates may be used to describe the motion? Write a differential equation describing the rolling constraint. Is this equation integrable? Justify your answer by a physical argument. Is the constraint holonomic?

15. ♦ A ladder rests against a smooth wall and slides without friction on wall and floor. Set up the equation of motion, assuming that the ladder maintains contact with the wall. If initially the ladder is at rest at an angle α with the floor, at what angle, if any, will it leave the wall?

16. A pendulum bob of mass m is suspended by a string of length ℓ from a point of support. The point of support moves along a horizontal x-axis according to the equation $x = a\cos\omega t$. Assume that the pendulum swings only in a vertical plane containing the x-axis. Let the position of the pendulum be described by the angle θ which the string makes with a line vertically downward.

    (a) Set up the Lagrangian function and write out the Lagrange equation.

    (b) Show that for small values of θ, the equation reduces to that of a forced harmonic oscillator, and find the corresponding steady-state motion. How does the amplitude of the steady-state oscillation depend on m, ℓ, a, and ω?

17. ♦ A particle of mass m slides over the inner surface of an inverted cone of half-angle α. The apex of the cone is at the origin, and the axis of the cone extends vertically upward. The only force acting on the particle, other than the force of constraint, is the force of gravity.

    (a) Set up the equations of motion, using as coordinates the horizontal distance ρ of the particle from the axis, and the angle θ measured in a horizontal circle around the cone. Show that θ is ignorable, and discuss the motion by the method of the effective potential.

    (b) For a given radius ρ_0, find the angular velocity $\dot{\theta}_0$ of revolution in a horizontal circle, and the angular frequency ω of small oscillations about this circular motion. Show that the small oscillations are a wobbling or an up-and-down spiraling motion, depending on whether the angle α is greater than or less than the angle $\alpha_c = \arcsin\left(1/\sqrt{3}\right)$.

18. Write down the Hamiltonian function for the spherical pendulum and the Hamiltonian equations of motion.

19. ♦ An extremely limber rope of uniform mass density, mass m and total length ℓ lies on a table with a length z hanging over the edge of the table. Only gravity acts on the rope. Find Lagrange's equation of motion.