List of exercises #6 (Small vibrations and normal modes) - 7600018

1. Closed orbits under a central force.
   (a) Find the frequency of small radial oscillation about the steady circular motion of the Kepler's problem, i.e., let $E = \frac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r)$ with $V_{\text{eff}} = \frac{L^2}{2mr^2} - \frac{K}{r}$, with $L$and K being positive constants. Show that this equals to the frequency of revolution.
   (b) Repeat the same as in the previous item but to the elastic force. What is the ratio between these frequencies? If this is a rational number, this means that the orbit is closed.

2. A mass m moving in space is subject to a force whose potential energy is $V = V_0 \exp\left[\left(5x^2 + 5y^2 + 8z^2 - 8yz - 26ya - 8za\right)/a^2\right]$, where the constants V_0 and a are positive. Show that V has one minimum point and find the normal frequencies of vibration about the minimum. What would change if $V = V_0 \exp\left[\left(-5x^2 + 5y^2 + 8z^2 - 8yz - 26ya - 8za\right)/a^2\right]$, i.e., with only one slight change from $5x^2 \to -5x^2$? Explain.

3. (Optional) The vector potential due to a magnetic dipole of magnetic moment μ is, in spherical coordinates relative to the dipole axis, $\mathbf{A} = \frac{1}{2\pi r^2}\mu \sin\theta\hat{\phi}$.
   (a) Find the corresponding magnetic field $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$.
   (b) Find the steady motions for a charged particle moving in such a field.
   Hint: Show that $\mathbf{B}$, in the dipole's xy-plane, is perpendicular to that plane. Thus, circular orbits exists in that plane. Find the corresponding radii R and velocities of those orbits.
   (c) Show that those steady motions are unstable.
   Hint: From the Lagrange's equations, allow to perturbations on the orbit and show that they diverge, i.e., let $r = R + \delta r$ and insert it on the Lagrange's equations. Expand for small δr and show that the resulting equation of motion for δr is that in which δr diverges with time.

4. Consider the following mechanical system of 3 identical and ideal springs of relaxed length ℓ and to identical point-like masses constrained to move freely in 1D: the first spring is connected to a wall and to the first mass, the second spring is connected to the first and second mass, and the third spring is connected to the second mass and to another wall 3ℓ distant apart from the first wall.
   (a) Set up the Lagrangian function for this system. Find the natural frequencies and the corresponding normal modes.
   (b) Now consider that both masses as subjected to the same viscous force $-b(\dot{x}_1 + \dot{x}_2)$. Set up the Lagrangian function in terms of the matrices $\mathbb{M}$, $\mathbb{B}$ and $\mathbb{K}$ can be diagonalized simultaneously, and find the normal modes. The damping rate of the fast and slow modes are the same?
   (c) The system is initially at rest when the second wall starts to move harmonically, i.e., its position is $3\ell + a\sin\Omega t$, with A and Ω being constants. Find the system's motion.

5. Set up the Lagrangian function for the mechanical system of 3 masses constrained to move freely in 1D and connected by two ideal springs. Choose your coordinates so that as many as possible will be ignorable, i.e., a coordinate in which the corresponding momentum is conserved. Find the natural frequencies and the corresponding normal modes.

6. An ion of mass m and charge q is held by a linear attractive force $F = -kr$ to a point A, where r is the distance from the ion to the point A. An identical ion is similarly bound to a second point B a distance ℓ from A. The two ions move (in three-dimensional space) under the action of these forces and their mutual electrostatic repulsion.
   (a) Find the normal modes of vibration, and write down the most general solution for small vibrations about the equilibrium point.
   (b) This pair of ions is now subject to a plane polarized electromagnetic wave incident perpendicular to the line AB, whose electric field is $E_0\cos\Omega t$ is directed at 45° to the line AB. Find the steady-state motion.

7. A mass m slides without friction on a smooth horizontal table. It is tied to a weightless string of total length ℓ which passes through a hole in the table and is tied at its lower end to a mass M which hangs below the table. Set up the Hamiltonian function using as coordinates the polar coordinates r, α of the mass m relative to the hole, and the spherical angles θ, ϕ of the mass M relative to the hole. Find the steady motions and the normal frequencies of small vibrations about a steady motion.

8. A 1D chain of particles of alternating masses m and M are connected by identical and ideal springs of constant k and relaxed length ℓ. Let u_n (v_n) be the deviation from the equilibrium position of the nth particles m (M).

   (a) Find the Lagrange's equations. (Disregard the first and the last particles.)

   (b) Using the Ansatz $u_n = Ue^{i(qn\ell-\omega t)}$ and $v_n = ve^{i(qn\ell-\omega t)}$, find the relation between q and ω. Show the existence of two types of modes: acoustic and optical, in which $\omega_{\text{ac}}(q) < \omega_{\text{op}}(q)$. Make sense of the corresponding normal modes.

   (c) Assuming periodic boundary conditions ($u_{N+1} = u_1$ and $v_{N+1} = v_1$) and that there are N pairs of masses in the ring, find the possible values of q.