List of exercises #4 - 7600037

1. **Perturbed 1D Harmonic Oscillator.** (Cohen-Tannoudji, Chap XIII, ex. 1)
   Consider a one-dimensional harmonic oscillator of mass m, angular frequency ω_0 and charge q. Let $|n\rangle$ and $E_n = \left(n + \frac{1}{2}\right)\hbar\omega_0$ be the eigenstates and eigenvalues of its Hamiltonian H_0. For $t < 0$, the oscillator is in the ground state $|0\rangle$. At $t = 0$, it is subjected to an electric field "pulse" of duration τ. The corresponding perturbation is $W(t) = -qE_0x\Theta(t)\,\Theta(\tau - t)$, where $\Theta(x)$ is the Heaviside step function, E_0 is the electric field magnitude, and x is the position observable. Let $P_{0\to n}$ be the probability of finding the oscillator in the state $|n\rangle$ after the pulse.

   (a) Calculate $P_{0\to 1}$ by using first-order time-dependent perturbation theory. How does it vary with τ, for fixed ω_0?

   (b) Show that, to obtain $P_{0\to 2}$, the time-dependent perturbation theory calculation must be pursued at least to second order. Calculate $P_{0\to 2}$ to this perturbation order.

   (c) Give the exact expressions for $P_{0\to 1}$ and $P_{0\to 2}$ in which the translation operator used in complement $\mathrm{F_V}$ appears explicitly. By making a limited power series expansion in E_0 of these expressions, find the results of the preceding questions.

2. **Time-dependent interacting spins.** (Cohen-Tannoudji, Chap XIII, ex. 2)
   Consider two spin 1/2's, $\mathbf{S}_1$ and $\mathbf{S}_2$, coupled by an interaction of the form $\varepsilon(t)\,\mathbf{S}_1\cdot\mathbf{S}_2/\hbar^2$ where $\varepsilon(t)$ is a function of time which approaches zero when $|t|$ approaches infinity, and takes on non-negligible values (on the order of ε_0) only inside an interval of width τ about $t = 0$.

   (a) At $t = -\infty$, the system is in the state $|+-\rangle$ (in the S^z basis). Calculate, without approximations, the state of the system at $t = +\infty$. Show that the probability $P_{|+-\rangle\to|-+\rangle}$ of finding the system in the state $|-+\rangle$ at $t = +\infty$ depends only on the integral $\int_{-\infty}^{+\infty}\varepsilon(t)\,dt$.

   (b) Calculate $P_{|+-\rangle\to|-+\rangle}$ by using first order time-dependent perturbation theory. Discuss the validity conditions for such an approximation by comparing the results obtained with those of the preceding question.

   (c) Now assume that the two spins are also interacting with a static magnetic field $B\hat{z}$. The corresponding Zeeman Hamiltonian is $H_0 = -\frac{\mu_B}{\hbar}B\left(g_1S_1^z + g_2S_2^z\right)$ where $g_{1,2}$ are the gyromagnetic ratios of the two spins, assumed to be different. Assume that $\varepsilon(t) = \varepsilon_0 e^{-(t/\tau)^2}$ and calculate $P_{|+-\rangle\to|-+\rangle}$ by first-order time-dependent perturbation theory. With fixed ε_0 and τ, discuss the variation of $P_{|+-\rangle\to|-+\rangle}$ with respect to B.

3. **Two-photon transitions between non-equidistant levels.** (Cohen-Tannoudji, Chap XIII, ex. 3)
   Consider an atomic level of angular momentum $J = 1$, subject to static electric and magnetic fields, both parallel to the z axis. It can be shown that three non-equidistant energy levels are then obtained. The eigenstates $|m\rangle$ of J^z ($m = -1, 0, +1$) of energies E_m correspond to them and are such that $E_1 - E_0 = \hbar\omega_0$ and $E_0 - E_{-1} = \hbar\omega_0'$ with $\omega_0 \neq \omega_0'$. The atom is also subjected to a radiofrequency field rotating at the angular frequency ω in the xy plane. The corresponding perturbation is $W(t) = \frac{1}{2}\omega_1\left(J^+e^{-i\omega t} + J^-e^{i\omega t}\right)$ where ω_1 is a constant proportional to the amplitude of the rotating field.

   (a) Let $|\psi(t)\rangle = \sum_m b_m(t)\,e^{-iE_mt/\hbar}\,|m\rangle$. Write the system of differential equations satisfied by the $b(t)$.

   (b) Assume that, at time $t = 0$, the system is in the state $|-1\rangle$. Show that if we want to calculate $b_1(t)$ by time-dependent perturbation theory, the calculation must be pursued to second order. Calculate $b_1(t)$ to this perturbation order.

   (c) For fixed t, how does the probability $P_{|-1\rangle\to|1\rangle} = |b_1(t)|^2$ of finding the system in the state $|1\rangle$ at time t vary with respect to ω? Show that a resonance appears, not only for $\omega = \omega_0$ and $\omega = \omega_0'$, but also for $2\omega = \omega_0 + \omega_0'$. Give a particle interpretation of this resonance.

4. **Hydrogen atom in an electric field**
   Consider a hydrogen atom in its ground state subject to an electric field $\mathbf{E} = \mathbf{E}_0\cos\omega t$. Assume that first order of perturbation theory is sufficient to answer the following questions.

   (a) What is the minimum frequency of the field to have ionization?

   (b) What is the transition rate (probability per unit of time) to an ionized state (assuming the electron can be represented by plane waves)?

(c) What is the angular distribution of the ejected electron in this process?

(d) Now consider that the atom is in a certain eigenstate $|n, l, m\rangle$ and that ω is lower than the corresponding ionization frequency. What can be said about the final eigenstate $|n', \ell', m'\rangle$, i.e., what are the selection rules?

5. (Optional) **The Autler-Townes effect.** (Cohen-Tannoudji, Chap XIII, ex. 5)
Consider a three-level system: $|1\rangle$, $|2\rangle$, and $|3\rangle$, of energies $E_{1,2,3}$. Assume $E_3 > E_2 > E_1$ and $E_3 - E_2 < E_2 - E_1$. This system interacts with a magnetic field oscillating at the angular frequency ω. The states $|2\rangle$ and $|3\rangle$ are assumed to have the same parity, which is the opposite of that of $|1\rangle$, so that the interaction Hamiltonian $W(t)$ with the oscillating magnetic field can connect $|2\rangle$ and $|3\rangle$ to $|1\rangle$. Assume that, in the basis of these three states $\{|1\rangle, |2\rangle, |3\rangle\}$, arranged in that order, $W(t)$ is represented by the matrix

$$W(t) = \hbar\omega_1 \sin\omega t \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

where ω_1 is a constant proportional to the amplitude of the oscillating field.

(a) Let $|\psi(t)\rangle = \sum_m b_m(t)\, e^{-iE_m t/\hbar} |m\rangle$. Write the system of differential equations satisfied by the $b(t)$.

(b) Assume that ω is very close to $\omega_{32}(E_3 - E_2)/\hbar$. Making approximations analogous to those used in complement C_{XIII}, integrate the preceding system, with the initial conditions $b_1(0) = b_2(0) = 1/\sqrt{2}$and $b_3(0) = 0$ (neglect, on the right-hand side of the differential equations, the terms whose coefficients, $e^{\pm i(\omega+\omega_{32})t}$, vary very rapidly, and keep only those whose coefficients are constant or vary very slowly, as $e^{\pm i(\omega-\omega_{32})t}$).

(c) The component D^z of the electric dipole moment of the system is represented, in the basis of the three states $\{|1\rangle, |2\rangle, |3\rangle\}$, arranged in that order, by the matrix

$$D^z = \begin{pmatrix} 0 & d & 0 \\ d & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where d is a real constant (D^z is an odd operator and can connect only states of different parities). Calculate $\langle D^z(t)\rangle = \langle\psi(t)|D^z|\psi(t)\rangle$, using the vector $|\psi(t)\rangle$ calculated in (b). Show that the time evolution of $\langle D^z(t)\rangle$ is given by a superposition of sinusoidal terms. Determine the frequencies v_k and relative intensities π_k of these terms. These are the frequencies that can be absorbed by the atom when it is placed in an oscillating electric field parallel to $\hat{z}$. Describe the modifications of this absorption spectrum when $\omega = \omega_{32}$ (and fixed), and ω_1 is increased from zero. Show that the presence of the magnetic field oscillating at the frequency $\omega_{32}/2\pi$ splits the electric dipole absorption line at the frequency $\omega_{21}/2\pi$ and that the separation of the two components of the doublet is proportional to the oscillating magnetic field amplitude (the Autler-Townes doublet). What happens when, for ω_1 fixed, $\omega - \omega_{32}$ is varied?