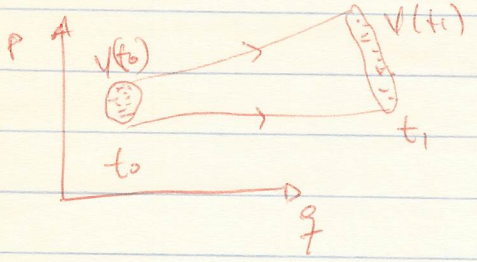
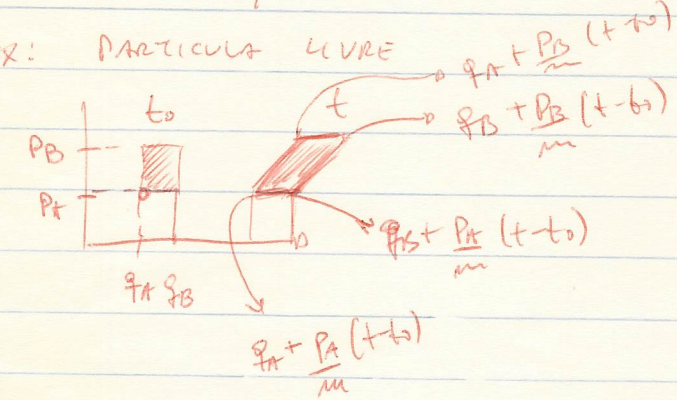
TEOREMA DE LIOUVILLE

UM VOLUME NO ESPAÇO DE FASES É PRESERVADO PELA DINÂMICA HAMILTONIANA DO SISTEMA



$$V(t_0) = V(t_1)$$

EX: PARTÍCULA LIVRE



$$V(t_0) = V(t)$$

CONSEQUÊNCIA: ① O NR. DE PONTOS DENTRO DO VOLUME (MICRO ESTADOS) NÃO SAI DA SUPERFÍCIE DELIMITADORA

⇒ DENSIDADE DE MICROESTADOS EM TORNO DE UM DADO MICRO-ESTADO É CONSTANTE NO TEMPO

* DENSIDADE DE MICROESTADOS ≡ FUNÇÃO DISTRIBUIÇÃO DO ESPAÇO DE FASES

② OS PTS SE PROPAGAM NO ESPAÇO DE FASES COMO UM FLUIDO INCOMPRESSÍVEL

PROVA: $V(t) = \int d\vec{\eta}$ ONDE $\dot{\eta}_i = f_i(\eta_1 \cdots \eta_N)$

EVOLUÇÃO INFINITESIMAL ⇒ $\eta_i(t+dt) = \eta_i(t) + f_i\, dt$

$x_i \equiv y_i + dt\, f_i(y_1 \cdots y_N)$

$$\Rightarrow V(t) = \int d\vec{x} = \int \text{DET}\, J\, d\vec{y}$$

ONDE

$$J(\vec{y}, t) = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_N} \\ \frac{\partial x_2}{\partial y_1} & & \vdots \\ \vdots & & \\ \frac{\partial x_N}{\partial y_1} & & \frac{\partial x_N}{\partial y_N} \end{pmatrix} \equiv \frac{\partial(x_1 \cdots x_N)}{\partial(y_1 \cdots y_N)}$$

MAS $\dfrac{\partial x_i}{\partial y_j} = \delta_{ij} + dt\,\dfrac{\partial f_i}{\partial y_j}$

$$\Rightarrow \text{DET} \begin{vmatrix} 1+d\theta_1 f_1 & d\theta_2 f_1 & \cdots & d\theta_N f_1 \\ d\theta_1 f_2 & 1+d\theta_2 f_2 & & \\ \vdots & & \ddots & \\ d\theta_1 f_N & & & 1+\frac{df}{\partial}_N f_N \end{vmatrix}$$

$$= 1 + dt \sum_i \frac{\partial f_i}{\partial y_i} + O(dt^2)$$

(VIDE FÓRMULA DE JACOBI)

$$\Rightarrow V(t) = \int \left(1 + dt\ \vec{\nabla}\cdot\vec{f}\right) d\vec{y}$$

$$= V(t_0) + dt \int \vec{\nabla}\cdot\vec{f}\, d\vec{y}$$

$$\Rightarrow \frac{V(t) - V(t_0)}{dt} \equiv \frac{dV}{dt} = \int \vec{\nabla}\cdot\vec{f}\, dy$$

CONDIÇÃO SUFICIENTE P/ PRESERVAÇÃO DO VOLUME DO ESPAÇO-DE FASE É

$$\vec{\nabla}\cdot\vec{f} = 0$$

(DADA POR UMA)

P/ SISTEMAS CUJA DINÂMICA É HAMILTONIANA

TEMOS QUE $\eta_i = q_i, \quad i = 1, \ldots, N$

$$\eta_{i+N} = p_i, \quad i = 1 \ldots N$$

E $f_i = \dfrac{\partial H}{\partial \eta_{i+N}} = \dfrac{\partial H}{\partial p_i}$

$f_{i+N} = \dfrac{-\partial H}{\partial \eta_i} = -\dfrac{\partial H}{\partial q_i}$

$$\Rightarrow \vec{\nabla}\cdot\vec{f} = \sum_{i=1}^{N} \frac{\partial}{\partial q_i}\left(\frac{\partial H}{\partial p_i}\right) + \frac{\partial}{\partial p_i}\left(\frac{-\partial H}{\partial q_i}\right)$$

$$= 0$$

PROVA ALTERNATIVA

$\rho \equiv \rho(q_1 \cdots q_N, p_1 \cdots p_N)$ = DENSIDADE DE PTS NO ESPAÇO DE FASES = PURA DISTRIBUIÇÃO

NR DE PTS (MACROESTADOS) EM TORNO DESTE PT

$$dN = \rho\, dV = \rho\, dq_1 \cdots dq_N\, dp_1 \cdots dp_N$$

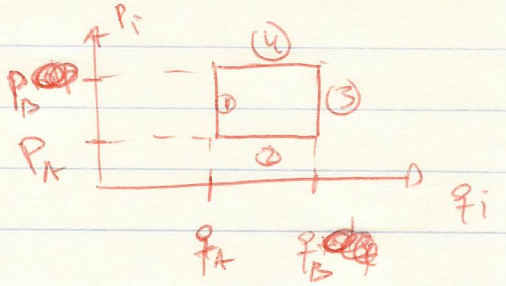
$$\Rightarrow \frac{dN}{dt} = \left(\frac{\partial \rho}{\partial t}\right) dV \qquad \text{p/ } dV \text{ FIXO}$$

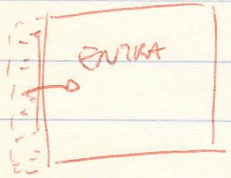
QUEREMOS MOSTRAR QUE $\frac{d\rho}{dt} = 0$

MAS
$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \sum_i \frac{\partial \rho}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial \rho}{\partial p_i}\frac{dp_i}{dt}$$

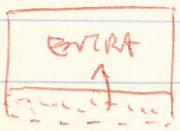
$$= \frac{\partial \rho}{\partial t} + \sum_i \frac{\partial \rho}{\partial q_i}\dot{q}_i + \frac{\partial \rho}{\partial p_i}\dot{p}_i$$

CALCULAR $\frac{\partial \rho}{\partial t}$:

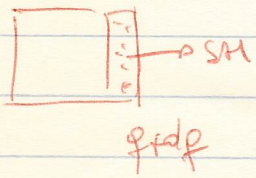




$$dN_1 = \rho_A \left(\dot{q}_A dt\right) dp_i \underbrace{\prod_{j \neq i} dq_j\, dp_j}_{dA_i}$$



$$dN_2 = \rho_A \left(\dot{p}_A dt\right) dq_i\, dA_i$$



$$dN_3 = -\rho_B \left(\dot{q}_B\right) dt\, dp_i\, dA_i$$

$$dN_4 = -\rho_B \dot{p}_B\, dt\, dq_i\, dA_i$$

$$dN_1 + dN_3 = \underbrace{\left(\rho_A \dot{q}_A - \rho_B \dot{q}_B\right) dt\, dp_i\, dA_i}_{\text{fluxo}} - \frac{\partial(\rho \dot{q}_i)}{\partial q_i} dq_i \quad \rightarrow \text{APENAS } q_i \text{ MUDOU}$$

$$\Rightarrow dN = -\left(\frac{\partial(\rho \dot{q}_i)}{\partial q_i} + \frac{\partial(\rho \dot{p}_i)}{\partial p_i} \right) dt\, dV$$

SOMANDO EM TODOS OS q_i's E p_i's

$$\Rightarrow \frac{dN}{dt} = -\sum_{i=1}^{N} \left\{ \frac{\partial(\rho \dot{q}_i)}{\partial q_i} + \frac{\partial(\rho \dot{p}_i)}{\partial p_i} \right\} dV$$

$$\frac{\partial \rho}{\partial t}$$

$$\frac{\partial(\rho \dot{q}_i)}{\partial q_i} + \frac{\partial(\rho \dot{p}_i)}{\partial p_i} = \frac{\partial \rho}{\partial q_i} \dot{q}_i + \frac{\partial \rho}{\partial p_i} \dot{p}_i + \rho \left(\frac{\partial}{\partial q_i} \dot{q}_i + \frac{\partial}{\partial p_i} \dot{p}_i \right)$$

$$\Rightarrow \frac{d\rho}{dt} = -\sum_i \left(\frac{\partial \rho}{\partial q_i} \dot{q}_i + \frac{\partial \rho}{\partial p_i} \dot{p}_i \right) - \rho \sum_i \left(\frac{\partial}{\partial q_i} \dot{q}_i + \frac{\partial}{\partial p_i} \dot{p}_i \right)$$

$$+ \sum_i \left(\frac{\partial \rho}{\partial q_i} \dot{q}_i + \frac{\partial \rho}{\partial p_i} \dot{p}_i \right)$$

$$= -\rho \sum_i \left(\frac{\partial}{\partial q_i} \dot{q}_i + \frac{\partial}{\partial p_i} \dot{p}_i \right)$$

P/ UM SISTEMA DINÂMICO HAMILTONIANO

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$

$$\Rightarrow \boxed{\frac{d\rho}{dt} = 0}$$