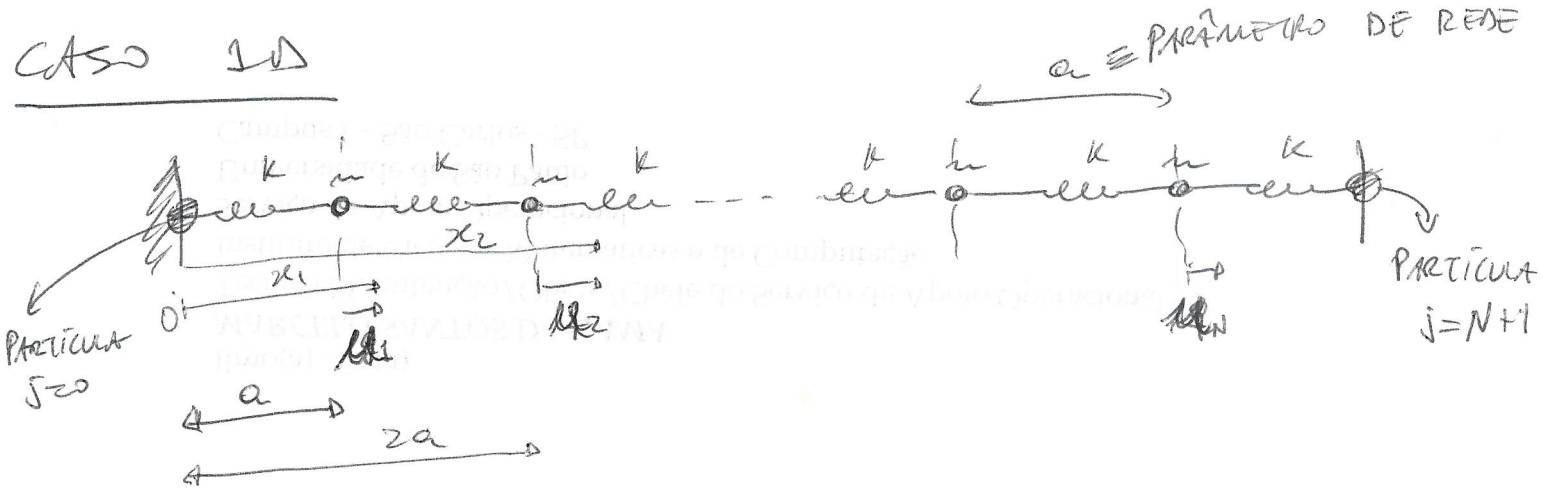
MODELO DE SISTEMAS CONTÍNUOS (SÓLIDOS CRISTALINOS)

- PARTÍCULAS LIGADAS POR MOLAS

P/ PEQUENAS OSCILAÇÕES → MOV. HARMÔNICO

CASO 1D



COMPRIMENTO TOTAL DO SISTEMA : $L = (N+1)a$

$\eta_j = x_j - ja$ ≡ DESLOCAMENTO DA j-ÉSIMA PARTÍCULA EM RELAÇÃO À SUA POSIÇÃO DE EQUILÍBRIO (DEFORMAÇÕES LOCAIS)

LAGRANGIANA

$$\begin{cases} L = \sum_{j=1}^{N} \frac{m}{2}\dot{\eta}_j^2 - \sum_{j=0}^{N} \frac{k}{2}(\eta_{j+1} - \eta_j)^2 \\ H = \sum_{j=1}^{N} \frac{p_j^2}{2m} + \sum_{j=0}^{N} \frac{k}{2}(\eta_{j+1} - \eta_j)^2 \end{cases}$$

#OBS: TOTAL N+1 MOLAS

CONDIÇÕES DE CONTORNO

$\eta_0 = \eta_{N+1} = 0$

EQS. DE EULER-LAGRANGE : $\boxed{m\ddot{\eta}_j = k(\eta_{j-1} - 2\eta_j + \eta_{j+1})}$

$$\Rightarrow \vec{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_N \end{pmatrix} \Rightarrow \ddot{\vec{\eta}} = -\omega^2 M\vec{\eta} \quad ; \quad M = \begin{pmatrix} +2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ \vdots & & & & 2 & -1 \\ 0 & 0 & \cdots & & -1 & 2 \end{pmatrix}$$

$\omega^2 = \frac{k}{m}$

MATRIZ TRIDIAGONAL

COMO RESOLVER O PROBLEMA?

SOLUÇÕES DO TIPO SENOS E COSSENOS

$\Rightarrow$ TENTAR $u_j(t) = \underbrace{e^{ikj}}_{\text{AMPLITUDE COMO FUNÇÃO DA POSIÇÃO}} \; e^{i\omega_k t}$ $\rightarrow$ OSCILAÇÃO HARMÔNICA NO TEMPO

$$\ddot{u}_j = -\omega_k^2 u_j = \Omega^2 u_j \left(e^{-ik} - 2 + e^{ik} \right)$$

$$= \Omega^2 u_j \, 2 \left(\cos(k) - 1 \right) = -4\Omega^2 u_j \sin^2\left(\frac{k}{2}\right)$$

$$\Rightarrow \boxed{\omega_k = \pm\sqrt{4\Omega^2 \sin^2\left(\frac{kj}{2}\right)} = \pm 2\Omega \left| \sin\frac{k}{2} \right|}$$

NOTE QUE $\vec{u}(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_N(t) \end{pmatrix} = \begin{pmatrix} e^{ik} \\ e^{2ik} \\ \vdots \\ e^{Nik} \end{pmatrix} e^{i\omega_k t}$

$\Rightarrow$ AUTO-VETORES DE M

OS AUTO-VALORES CORRESPONDENTES SÃO $\left(2\left|\sin\frac{k}{2}\right|\right)^2$

$$M \begin{pmatrix} e^{ik} \\ \vdots \\ e^{Nik} \end{pmatrix} \Rightarrow \lambda_k \begin{pmatrix} e^{ik} \\ \vdots \\ e^{Nik} \end{pmatrix} \Rightarrow \begin{cases} 2e^{ik} - e^{2ik} = \lambda_k e^{ik} & \rightarrow \text{LINHA } 1 \\ -e^{ik(j-1)} + 2e^{ikj} - e^{ik(j+1)} = \lambda_k e^{ikj} & \rightarrow \text{LINHA } 1<j<N \\ -e^{ik(N-1)} + 2e^{ikN} = \lambda_k e^{ikN} & \rightarrow \text{LINHA } N \end{cases}$$

LINHA $1<j<N$

$\Rightarrow 2 - 2\cos k = \lambda_k \Rightarrow \boxed{\lambda_k = 2(1-\cos k) = 4\sin^2\left(\frac{k}{2}\right)}$

1ª LINHA $\rightarrow 2 - e^{ik} = \lambda_k$

Nª LINHA $\rightarrow 2 - e^{-ik} = \lambda_k$ $\Rightarrow e^{ik} = e^{-ik} \Rightarrow \sin k = 0 \rightarrow$ NÃO FAZ SENTIDO.

CLARO, u_j É REAL $\longrightarrow$ COMO $\lambda_k = \lambda_{-k}$

$\Rightarrow$ PODEMOS TENTAR $A_+ e^{ikj} + A_- e^{-ikj} = A\sin(kj + \delta_k)$

$\Rightarrow$ TENTAR $u_j(t) = \sin(kj + \delta_j)\, e^{i\omega_k t}$

$$\vec{u}(t) = \begin{pmatrix} \sin(k+\delta_k) \\ \sin(2k+\delta_k) \\ \vdots \\ \sin(Nk+\delta_k) \end{pmatrix} e^{i\omega_k t}$$

EQ AUTO-VALORES: $2\sin(k+\delta_k) - \sin(2k+\delta_k) = \delta_k \sin(k+\delta_k)$

$$-\sin((k(j-1)+\delta_k) + 2\sin(kj+\delta_k) - \sin(k(j+1)+\delta_k) = \delta_k \sin(kj+\delta_k)$$

$$-\sin(k(N-1)+\delta_k) + 2\sin(kN+\delta_k) = \delta_k \sin(kN+\delta_k)$$

QUEREMOS QUE AS COND. DE CONTORNO SEJAM SATISFEITAS GENERICAMENTE

$\Rightarrow u_0 = u_{N+1} = 0$

$u_0 = \sin(0+\delta_k) e^{i\omega t} = 0 \Rightarrow \delta_k = m\pi,\ m \in \mathbb{Z}$

NOTE QUE $\sin(kj+\delta_k) = (-1)^m \sin(kj)$ (m: SINAL GLOBAL)

$\Rightarrow$ ESCOLHER $m = 0$

$\Rightarrow \boxed{\delta_k = 0}$

$u_{N+1} = \sin(k(N+1)) e^{i\omega t} = 0$

$\Rightarrow k = \frac{\pi}{N+1} n,\ n \in \mathbb{Z}$

ENTRETANTO, M SÓ POSSUI N AUTO-VALORES E AUTO-VETORES. NOTE QUE $n = 0$ IMPLICA EM $k = 0$, $\vec{u} = 0$

NÃO INTERESSA

NOTE AINDA QUE $n = N+1+m$ É EQUIVALENTE À m

$k_m = \pi + \frac{\pi}{N+1} m = \pi + k_m$

$\Rightarrow \sin(k_m) = \sin(\pi + k_m) = -\sin(k_m)$

MESMO AUTO-VETOR MAS COM SINAL TROCADO

$\Rightarrow$ SÓ ATÉ N VALORES DE n QUE DÃO RESULTADOS DISTINTOS

OU SEJA $\boxed{k = \frac{\pi}{N+1} n,\ \text{com } n = 1, 2, \dots, N}$

DISCRETIZAÇÃO DOS VALORES DE k $\rightarrow$ PELA COND. CONTORNO

$\Rightarrow$ NÃO É QUALQUER COISA QUE "CABE" NA CORDA

FINALMENTE, VOLTANDO À EQ. AUTO-VALORES,

$$\begin{cases} 2\,\mathrm{sen}\,k - \mathrm{sen}(2k) = \lambda_k \,\mathrm{sen}\,k \\ -\mathrm{sen}(k(j-1)) + 2\,\mathrm{sen}(kj) - \mathrm{sen}(k(j+1)) = \lambda_k \,\mathrm{sen}(kj) \\ -\mathrm{sen}(k(N-1)) + 2\,\mathrm{sen}(kN) = \lambda_k \,\mathrm{sen}(kN) \end{cases}$$

1ª $\rightarrow$ $2\,\mathrm{sen}\,k - 2\,\mathrm{sen}\,k\cos k = \lambda_k \,\mathrm{sen}\,k \rightarrow$ $\boxed{\lambda_k = 2 - 2\cos k}$

2ª $\rightarrow$ $-\mathrm{sen}(kj)\cos k + \cancel{\cos(kj)\,\mathrm{sen}\,k} + 2\,\mathrm{sen}(kj) - \mathrm{sen}(kj)\cos k - \cancel{\cos(kj)\,\mathrm{sen}\,k} = \lambda_k \,\mathrm{sen}(kj)$

$\Rightarrow$ $\boxed{\lambda_k = 2 - 2\cos k}$

3ª $\rightarrow$ ~~$-\mathrm{sen}(kN)\cos k + \cos(kN)\,\mathrm{sen}\,k + 2\,\mathrm{sen}(kN) = \lambda_k \,\mathrm{sen}(kN)$~~

~~$\lambda_k = 2 - \cos k + \dfrac{\cos(kN)}{\mathrm{sen}(kN)}\,\mathrm{sen}\,k = 2 - \cos k + \cos\left(\dfrac{\pi N}{NH}m\right)\dfrac{\mathrm{sen}\,\frac{\pi m}{NH}}{\mathrm{sen}\left(\frac{\pi N}{NH}m\right)}$~~

$\circ\ \mathrm{sen}(k(N-1)) = \mathrm{sen}(\overbrace{k(NH)}^{\pi m} - 2k) = -\cos\pi m\,\mathrm{sen}\,2k$

$\circ\ \mathrm{sen}(kN) = \mathrm{sen}(\overbrace{k(NH)}^{\pi m} - k) = -\cos\pi m\,\mathrm{sen}\,k$

$\Rightarrow$ $\lambda_k = 2 - \left(\dfrac{-\cos\pi m\,\mathrm{sen}\,2k}{-\cos\pi m\,\mathrm{sen}\,k}\right) = \boxed{2 - 2\cos k = \lambda_k}$

SOLUÇÃO GERAL

$$u_j(t) = \sum_{m=1}^{N} \check{A}_{km}\,\mathrm{sen}(k_m j)\,e^{i\omega_{km}t}$$

$$= \sum_{m=1}^{N} A_m \,\mathrm{sen}\left(\frac{\pi}{NH}mj\right)\cos(\omega_m t + \phi_m)$$

ONDE $\omega_m = 2\Omega\left|\mathrm{sen}\,\frac{\pi m}{NH}\right|$

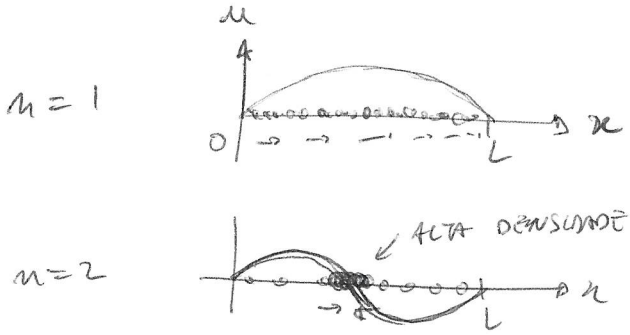
$\boxed{A_m \text{ E } \phi_m \rightarrow \text{ COND. INICIAIS}}$

MODOS NORMAIS

$$K_n = \frac{\pi}{NH}\, n \quad , \quad n = 1, 2, \dots, N$$

↓

DISCRETIZAÇÃO DOS MODOS NORMAIS (E SUAS FREQUÊNCIAS) PELAS COND. CONTORNO



LIMITE CONTÍNUO

NR. PARTÍCULAS $N \to \infty$

DISTÂNCIA INTERATÔMICA $a \to 0$

DENSIDADE FINITA: $\rho = \frac{m}{a} = \frac{M}{Na} = \frac{M}{L}$

OU SEJA $Na \to$ CONSTANTE $= L$

EQ. E-L: $\frac{d^2 u_j}{dt^2} = \frac{k}{m}\left(u_{j+1} - 2u_j + u_{j-1}\right)$; $u_0 = u_{N+1} = 0$

CAMPO DE DEFORMAÇÕES: $u_j(t) \to u(ja, t) \equiv u(x, t)$

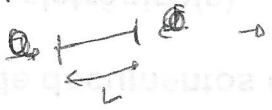
$$\Rightarrow \frac{d^2 u_j}{dt^2} = \frac{\partial^2}{\partial t^2} u(x,t)$$

$$\Rightarrow \frac{k}{m}\left(u_{j+1} - 2u_j + u_{j-1}\right) = \left[\frac{(u_{j+1} - u_j)}{a} - \frac{(u_j - u_{j-1})}{a}\right] \frac{ka}{m}$$

$$= \left(\frac{u(x+a,t) - u(x,t)}{a} - \frac{u(x,t) - u(x-a,t)}{a}\right) \frac{k}{\rho} \to \left(\frac{\partial u(x+a,t)}{\partial x} - \frac{\partial u(x,t)}{\partial x}\right) \frac{k}{\rho}$$

$$= \frac{\partial^2 u(x,t)}{\partial x^2} * \frac{ka}{\rho}$$

MAS MÓDULO DE ELASTICIDADE



$$B = \frac{L\,F}{\Delta L} = \frac{Na \times K\Delta u}{N\Delta u} = ka$$

$$= \frac{B}{\rho} \frac{\partial^2}{\partial x^2} u(x,t)$$

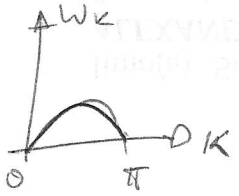
FINALMENTE, $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{B}{\rho}\dfrac{\partial^2 u}{\partial x^2}$

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0}$$

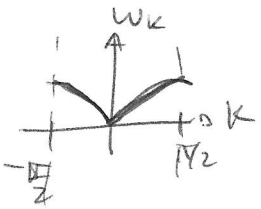
EQ. DE ONDA

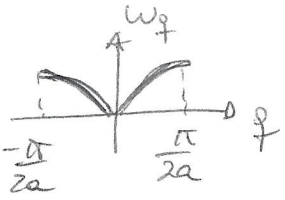
$$\boxed{c^2 = \frac{B}{\rho}}$$

RELAÇÃO DE DISPERSÃO $\omega_k = 2\Omega\left|\sin\frac{k}{2}\right|$ ONDE $k \in]0, \pi[$



COMUMENTE, DEFINI-SE $-\frac{\pi}{2} < k < \pi/2$

$\Rightarrow$ 

$\longrightarrow$ 

PARA $k \ll 1 \Rightarrow \omega_k \approx 2\Omega\left|\frac{k}{2}\right| = \Omega k = \sqrt{\frac{k}{m}} \times \frac{\pi}{(N+1)a} n$

$$\Rightarrow \omega_q = \sqrt{\frac{ka^2}{m}} \times \frac{\pi}{(N+1)a} n = \sqrt{\frac{B}{\rho}}\left(\frac{\pi n}{L}\right)_q = \sqrt{\frac{B}{\rho}}\, q = cq$$

$$\Rightarrow \boxed{\omega = cq}$$ RELAÇÃO DE DISPERSÃO RELATIVÍSTICA

ENERGIA:

$$U = \frac{1}{2}k\sum_j (u_{j+1} - u_j)^2 = \frac{1}{2}\underbrace{ka}_{B}\underbrace{\sum_j a}_{\int dx}\left(\underbrace{\frac{u_{j+1}-u_j}{a}}_{\frac{\partial u}{\partial x}}\right)^2$$

$$\Rightarrow \boxed{U = \frac{1}{2}B\int_0^L dx\left(\frac{\partial u}{\partial x}\right)^2}$$

$$T = \frac{1}{2} m \sum_j \left(\frac{du_j}{dt}\right)^2 = \frac{1}{2} \frac{m}{a} \sum_j a \left(\frac{\partial u}{\partial t}\right)^2$$

$$\Rightarrow \boxed{T = \frac{1}{2} \rho \int_0^L dx \left(\frac{\partial u}{\partial t}\right)^2}$$

$$E = T + U = \frac{1}{2} \left[\int_0^L dx \, \rho \left(\frac{\partial u}{\partial t}\right)^2 + B \left(\frac{\partial u}{\partial x}\right)^2 \right] = \text{CONSTANTE}$$

LEMBRE-SE QUE $u_j = \sum_{n=1}^{N} A_n \sin(k_n j) \cos(\omega_n t + \phi_n)$

$$\rightarrow u(x,t) = \sum_{n=1}^{N} A_n \sin(q_n x) \cos(\omega_n t + \phi_n)$$

P/ A LAGRANGIANA

$$L = \int_0^L dx \, \frac{\rho}{2} \left(\frac{\partial u}{\partial t}\right)^2 + \frac{B}{2} \left(\frac{\partial u}{\partial x}\right)^2$$

$$\Rightarrow \begin{cases} \frac{\partial u}{\partial t} = \sum_n (-\omega_n) A_n \sin(q_n x) \sin(\omega_n t + \phi_n) \\ \frac{\partial u}{\partial x} = \sum_n (q_n) A_n \cos(q_n x) \cos(\omega_n t + \phi_n) \end{cases}$$

$$\Rightarrow E = \frac{1}{2} \sum_{nm} \rho A_n A_m \sin(\omega_n t + \phi_n) \sin(\omega_m t + \phi_m) \left(\int_0^L dx \, \sin(q_n x) \sin(q_m x) \right) \omega_n \omega_m$$

$$+ B A_n A_m \cos(\omega_n t + \phi_n) \cos(\omega_m t + \phi_m) \left(\int_0^L dx \, \cos(q_n x) \cos(q_m x) \right) q_n q_m$$

$\sin\left(\frac{\pi}{L} n x\right) \sin\left(\frac{\pi}{L} m x\right) \rightarrow \frac{L}{2} \delta_{n,m}$

$\frac{2}{L} \delta_{n,m}$

$$\Rightarrow E = \frac{1}{2} \sum_n A_n^2 \left(\rho \omega_n^2 \sin^2(\omega_n t + \phi_n) + B q_n^2 \cos^2(\omega_n t + \phi_n) \right) \frac{L}{2}$$

$$= \frac{L}{4} B \sum_n A_n^2 \left(\frac{\omega_n^2}{c^2} \sin^2(\omega_n t + \phi_n) + q_n^2 \cos^2(\omega_n t + \phi_n) \right)$$

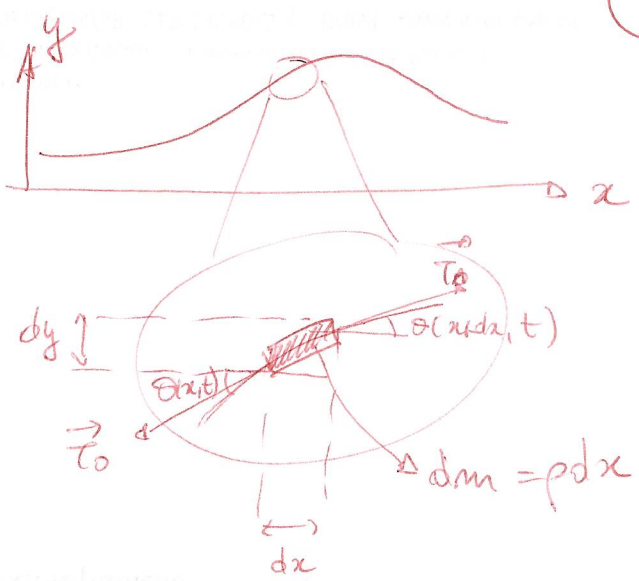
$$= \frac{L}{4} B \sum_n A_n^2 q_n^2 = \text{cte no tempo}$$

$$= \frac{L}{4} B \sum_n A_n^2 \left(\frac{\pi n}{L}\right)^2 = \frac{L}{4} \rho \sum_n A_n^2 \omega_n^2$$

$$\Rightarrow \frac{L}{4} B \sum_n \frac{\Delta q}{\Delta q} A_q q^2 \rightarrow \frac{L^2}{4\pi} B \int dq \, A^2(q) \, q^2$$

CORDA ESTICADA

$y = y(x,t)$



PEQUENAS PERTURBAÇÕES:

$$\frac{\partial y}{\partial x} \equiv \tan\theta \ll 1$$

$$\vec{T} = (T_0, T_{y(x)})$$

↳ ok → APENAS MOV. VERTICAIS

EQ. DO MOVIMENTO: $dm\,\ddot{y} = \rho\,dx\,\ddot{y} = T_0\left[\sin\theta(x+dx,t) - \sin\theta(x,t)\right]$

$$= T_0\,dx\,\frac{\partial}{\partial x}\sin\theta(x,t)$$

$$\approx T_0\,dx\,\frac{\partial}{\partial x}\underbrace{\tan\theta(x,t)}_{\frac{\partial y}{\partial x}}$$

$$= T_0\,dx\,\frac{\partial^2 y}{\partial x^2}$$

$$\Rightarrow \rho\frac{\partial^2 y}{\partial t^2} = T_0\frac{\partial^2 y}{\partial x^2}$$

$$\Rightarrow \boxed{\frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = 0}$$ EQ DE ONDA

$$c = \sqrt{T_0/\rho}$$

SOLUÇÕES ≡ ONDAS PROPAGANTES: $y(x,t) = y(x \pm ct)$

$$= f(x-ct) + g(x+ct)$$

CONDIÇÕES INICIAIS

$$f(x) + g(x) = y(x,0) = y_0(x)$$

$$v(x,t) = \frac{\partial y}{\partial t} = c(g' - f')$$

$$\Rightarrow g(x) - f(x) = c^{-1}\int_{-\infty}^{x} v_0(z)\,dz - \underbrace{(g(-\infty) - f(-\infty))}_{\text{ok} \Rightarrow \text{DESPREZAR}}$$

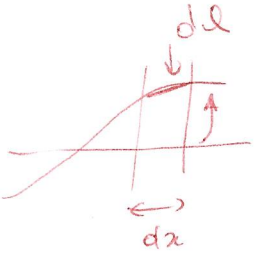
$$\Rightarrow \begin{cases} f(x) = \frac{y_0(x)}{2} - \frac{1}{2c}\int_{-\infty}^{x} v_0(z)\,dz \\ g(x) = \frac{y_0(x)}{2} + \frac{1}{2c}\int_{-\infty}^{x} v_0(z)\,dz \end{cases}$$

ENERGIA $\quad dT = \frac{1}{2}\, dm\, v^2 = \frac{1}{2}\rho \left(\frac{\partial y}{\partial t}\right)^2 dx$

$dU = T_0 (ds - dx)$ ENERGIA P/ ESTICAR

$= T_0 \left(\sqrt{dy^2 + dx^2} - dx\right)$

$\approx \frac{T_0\, dx}{2}\left(\frac{\partial y}{\partial x}\right)^2$



LAGRANGIANA → $L = \int dx \left(\frac{1}{2}\rho \left(\frac{\partial y}{\partial t}\right)^2 - \frac{1}{2} T_0 \left(\frac{\partial y}{\partial x}\right)^2 \right)$

DENSIDADE DE LAGRANGIANA

$$\mathcal{L} \equiv \mathcal{L}\left(\frac{\partial y}{\partial x}, \frac{\partial y}{\partial t}\right)$$

GENERICAMENTE, PARA UM CAMPO $\phi(x,t)$,

ESPERAMOS QUE $\mathcal{L} \equiv \mathcal{L}\left(\phi, \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial t}, x, t\right)$

$\Rightarrow \boxed{S = \int dt\, L = \int dt \int dx\, \mathcal{L}(\phi, \phi', \dot{\phi}, x, t)}$ ← AÇÃO

↓ AUTO-INTERAÇÃO (LOCAL) DO CAMPO

PRINCÍPIO DE HAMILTON (MÍNIMA AÇÃO)

$$\delta S = \delta\left(\int_{t_1}^{t_2} dt \int_{x_1}^{x_2} dx\, \mathcal{L}\right) = 0$$

$$\Rightarrow \delta S = \int dt\, dx \left\{ \frac{\partial \mathcal{L}}{\partial \phi}(\delta\phi) + \frac{\partial \mathcal{L}}{\partial \phi'}(\delta\phi') + \frac{\partial \mathcal{L}}{\partial \dot{\phi}}\delta\dot{\phi} \right\}$$

ONDE $\begin{cases} \delta\phi' = \delta\left(\frac{\partial \phi}{\partial x}\right) = \frac{\partial}{\partial x}(\delta\phi) \\ \delta\dot{\phi} = \delta\left(\frac{\partial \phi}{\partial t}\right) = \frac{\partial}{\partial t}(\delta\phi) \end{cases}$

CONDIÇÕES P/ EXTREMIZAÇÃO

- $\delta\phi(x, t_1) = \delta\phi(x, t_2) = 0 \quad \forall x$

(CONDIÇÕES INICIAL E FINAL IGUAIS P/ TODAS AS TRAJETÓRIAS)

- $\delta\phi(x_1, t) = \delta\phi(x_2, t) = 0 \quad \forall t$

(CONDIÇÕES DE CONTORNO, DEPENDENDO DO PROBLEMA ESSAS CONDIÇÕES PODEM SER OUTRAS. VAMOS USAR AS DA "CORDA [illegible] FIXA" NOS EXTREMOS)

DEVEMOS TRABALHAR OS TERMOS

$$\star \int dt\,dx\, \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \frac{\partial}{\partial t}(\delta\phi) = \int dx \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \delta\phi \Big|_{t_1}^{t_2 \to \infty} - \int dt\,dx \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) \delta\phi$$

$$\star \int dt\,dx \frac{\partial \mathcal{L}}{\partial \phi'} \frac{\partial}{\partial x}(\delta\phi) = \int dt \frac{\partial \mathcal{L}}{\partial \phi'} \delta\phi \Big|_{x_1}^{x_2 \to \infty} - \int dt\,dx \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \phi'}\right) \delta\phi$$

LOGO, $$\delta S = \int dt\,dx \left\{ \frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial \phi'}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) \right\} \delta\phi = 0$$

OU SEJA, A EQ. DE EULER-LAGRANGE FICA

$$\boxed{\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \phi}{\partial t}\right)}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \phi}{\partial x}\right)}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0}$$

EXEMPLO: COMO FICA A EQ. E-L P/ CORDA VIBRANTE?

$$\mathcal{L} = \frac{1}{2}\rho\left(\frac{\partial y}{\partial t}\right)^2 - \frac{1}{2}T_0\left(\frac{\partial y}{\partial x}\right)^2$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\rho \frac{\partial y}{\partial t}\right) + \frac{\partial}{\partial x}\left(T_0 \frac{\partial y}{\partial x}\right) - 0 = 0$$

$$\Rightarrow \boxed{\rho \frac{\partial^2 y}{\partial t^2} - T_0 \frac{\partial^2 y}{\partial x^2} = 0} \rightarrow \text{EQ DE ONDAS}$$

$$\text{com } c = \sqrt{\frac{T_0}{\rho}}$$

*OBS: $\phi = \phi(x,t)$, x E t SÃO VARIÁVEIS INDEPENDENTES $\Rightarrow$ NÃO EXISTE $\frac{\partial x}{\partial t}$

ALÉM DISSO, $\frac{d\phi}{dt} = \frac{\partial \phi}{\partial t}$, $\frac{d\phi}{dx} = \frac{\partial \phi}{\partial x}$ MAS $\frac{d}{dt}\mathcal{L} \neq \frac{\partial \mathcal{L}}{\partial t}$

$\frac{d}{dx}\mathcal{L} \neq \frac{\partial \mathcal{L}}{\partial x}$

DE FATO $\frac{d}{dt}\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi}\dot{\phi} + \frac{\partial \mathcal{L}}{\partial \dot{\phi}}\ddot{\phi} + \frac{\partial \mathcal{L}}{\partial \phi'}\dot{\phi}' + \frac{\partial \mathcal{L}}{\partial t}$

VÁRIOS CAMPOS EM VÁRIAS DIMENSÕES

$$\vec{\phi} = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_N \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}$$

$$\Rightarrow \mathcal{L} \equiv \mathcal{L}\left(\vec{\phi}, \frac{\partial \vec{\phi}}{\partial t}, \nabla \vec{\phi}, \vec{x}, t\right)$$

EQS. EULER-LAGRANGE:

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \phi_\alpha}{\partial t}\right)}\right) + \nabla \cdot \left(\frac{\partial \mathcal{L}}{\partial(\nabla \phi_\alpha)}\right) - \frac{\partial \mathcal{L}}{\partial \phi_\alpha} = 0, \quad \alpha = 1, \ldots, N$$

NOTAÇÃO: $\cdot\ x_0 = ct$

$\cdot\ \partial_\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}\right) = \left(\frac{\partial}{\partial x_0}, \nabla\right)$

$\Rightarrow$ EQ. E-L:

$$\boxed{\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_\alpha)}\right) - \frac{\partial \mathcal{L}}{\partial \phi_\alpha} = 0}, \quad \alpha = 1, \ldots N$$

MAIS NOTAÇÃO $\frac{\partial}{\partial x_j}\phi_\alpha = \phi_{\alpha,j}$

$\Rightarrow$ EQ. E-L:

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial \phi_{\alpha,\mu}}\right) - \frac{\partial \mathcal{L}}{\partial \phi_\alpha} = 0$$

FORMULAÇÃO HAMILTONIANA OU HAMILTONIANA ASSOCIADA

MOMENTO CONJUGADO $\pi_\alpha(\vec{x},t) = \dfrac{\partial \mathcal{L}}{\partial \dot{\phi}_\alpha}$

DENSIDADE DE HAMILTONIANA (ou DE ENERGIA)

$$\mathcal{H} = \sum_\alpha \pi_\alpha \dot{\phi}_\alpha - \mathcal{L} = \mathcal{H}(\vec{\phi}, \nabla\vec{\phi}, \vec{\pi}, \nabla\vec{\pi}, \vec{x}, t)$$

E A HAMILTONIANA

$$H = \int d\vec{x}\, \mathcal{H}(\vec{\phi}, \nabla\vec{\phi}, \vec{\pi}, \nabla\vec{\pi}, \vec{x}, t)$$

caso em que $\mathcal{H} \equiv \mathcal{H}(\vec{\phi}, \nabla\vec{\phi}, \vec{\pi})$

EQS. DE HAMILTON: $S = \int dt\, d\vec{x} \left(\sum_\alpha \pi_\alpha \dot{\phi}_\alpha - \mathcal{H} \right)$

$$\delta S = 0 = \int dt\, d\vec{x} \sum_\alpha \left\{ \left[-\dot{\pi}_\alpha - \frac{\partial \mathcal{H}}{\partial \phi_\alpha} + \nabla \cdot \left(\frac{\partial \mathcal{H}}{\partial \nabla \phi_\alpha} \right) \right] \delta\phi_\alpha + \left[\dot{\phi}_\alpha - \frac{\partial \mathcal{H}}{\partial \pi_\alpha} + \nabla \cdot \left(\frac{\partial \mathcal{H}}{\partial (\nabla \pi_\alpha)} \right) \right] \delta\pi_\alpha \right\}$$

$$\Rightarrow \begin{cases} \dot{\phi}_\alpha = \dfrac{\partial \mathcal{H}}{\partial \pi^\alpha} - \nabla \cdot \dfrac{\partial \mathcal{H}}{\partial (\nabla \pi_\alpha)} \equiv \dfrac{\delta \mathcal{H}}{\delta \pi_\alpha} \\ \dot{\pi}_\alpha = -\dfrac{\partial \mathcal{H}}{\partial \phi_\alpha} + \nabla \cdot \dfrac{\partial \mathcal{H}}{\partial (\nabla \phi_\alpha)} \equiv -\dfrac{\delta \mathcal{H}}{\delta \phi_\alpha} \end{cases}$$

CORRENTES CONSERVADAS

ANTES, NO SIST. DE PARTÍCULAS, QDO $\frac{\partial L}{\partial t} = 0 \Rightarrow$ ENERGIA CONSERVADA

(EXCETO FORÇA LORENTZ)

AGORA, PARA CAMPOS, O MESMO ACONTECE. ENTRETANTO, COMO ESTAMOS TRABALHANDO COM A DENSIDADE DE LAGRANGIANA, A ENERGIA PODE NÃO SER CONSERVADA LOCALMENTE, MAS GLOBALMENTE $\Rightarrow$ HÁ CORRENTES DE CONSERVAÇÃO QUE TRANSFORMAM QUANTIDADES GLOBALMENTE CONSERVADAS DE UM PONTO A OUTRO

FAÇAMOS A DERIVADA

$$\frac{d\mathcal{L}}{\partial x_j} = \sum_\alpha \frac{\partial \mathcal{L}}{\partial \phi_\alpha}\frac{\partial \phi_\alpha}{\partial x_j} + \sum_k \frac{\partial \mathcal{L}}{\partial(\partial_k \phi_\alpha)}\frac{\partial(\partial_k \phi_\alpha)}{\partial x_j} + \frac{\partial \mathcal{L}}{\partial x_j}\,, \quad j = 0, 1, \ldots, 4$$

- NOTAÇÃO $\rightarrow$ ÍNDICES REPETIDOS SIGNIFICAM SOMA (NOTAÇÃO DE EINSTEIN)

$$\sum_\alpha \frac{\partial \mathcal{L}}{\partial \phi_\alpha}\frac{\partial \phi_\alpha}{\partial x_j} \equiv \frac{\partial \mathcal{L}}{\partial \phi_\alpha}\frac{\partial \phi_\alpha}{\partial x_j}$$

- NOTAÇÃO: $\frac{d\phi_\alpha}{\partial x_j} = \frac{\partial \phi_\alpha}{\partial x_j} \equiv \phi_{\alpha,j}$

$$\Rightarrow \frac{d\mathcal{L}}{d x_j} = \frac{\partial \mathcal{L}}{\partial \phi_\alpha}\phi_{\alpha,j} + \frac{\partial \mathcal{L}}{\partial \phi_{\alpha,k}}\phi_{\alpha,kj} + \frac{\partial \mathcal{L}}{\partial x_j}$$

AGORA USAMOS A EQ. E-L:

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_\alpha}\right) + \nabla\cdot\left(\frac{\partial \mathcal{L}}{\partial \nabla\phi_\alpha}\right) - \frac{\partial \mathcal{L}}{\partial \phi_\alpha} = 0$$

NA NOVA NOTAÇÃO

$$\sum_{j=0}^{d} \frac{\partial}{\partial x_j}\left(\frac{\partial \mathcal{L}}{\partial \phi_{\alpha,j}}\right) - \frac{\partial \mathcal{L}}{\partial \phi_\alpha} = 0$$

$$\Rightarrow \quad \frac{\partial}{\partial x_j}\left(\frac{\partial \mathcal{L}}{\partial \phi_{\alpha,j}}\right) - \frac{\partial \mathcal{L}}{\partial \phi_\alpha} = 0$$

VOLTANDO P/ $\dfrac{d\mathcal{L}}{dx_j}$, $\dfrac{\partial \mathcal{L}}{\partial \phi_{\alpha,\ell}}\,\phi_{\alpha,\ell j}$

$$\frac{d\mathcal{L}}{dx_j} = \frac{\partial}{\partial x_\ell}\left(\frac{\partial \mathcal{L}}{\partial \phi_{\alpha,\ell}}\right)\phi_{\alpha,j} + \frac{\partial \mathcal{L}}{\partial \phi_{\alpha,k}}\phi_{\alpha,kj} + \frac{\partial \mathcal{L}}{\partial x_j}$$

$$\frac{d}{dx_\ell}\left(\frac{\partial \mathcal{L}}{\partial \phi_{\alpha,\ell}}\phi_{\alpha,j}\right)$$

$$\Rightarrow \quad \frac{d}{dx_\ell}\left[\frac{\partial \mathcal{L}}{\partial \phi_{\alpha,\ell}}\phi_{\alpha,j} - \mathcal{L}\,\delta_{\ell,j}\right] = -\frac{\partial \mathcal{L}}{\partial x_j}$$

RECORDANDO QUE j É FIXO E $\ell \in \alpha$ SÃO SOMADOS

DEFINIÇÃO: TENSOR ENERGIA-TENSÃO (STRESS-ENERGY TENSOR)

$$T_{j\ell} \equiv \frac{\partial \mathcal{L}}{\partial \phi_{\alpha,\ell}}\phi_{\alpha,j} - \mathcal{L}\,\delta_{j,\ell}$$

SE $\dfrac{\partial \mathcal{L}}{\partial x_j} = 0 \quad \Rightarrow \quad \boxed{\dfrac{d}{dx_\ell} T_{j\ell} = 0}$

↓

LAGRANGIANA NÃO DEPENDER EXPLICITAMENTE DE x_j

INTERPRETAÇÃO:

$$j=0 \rightarrow \boxed{\frac{1}{c}\frac{d}{dt}T_{00} + \nabla\cdot\vec{T}_0 = 0}$$

EQ. CONTINUIDADE

ONDE $\vec{T}_0 = (T_{01}, T_{02}, T_{03} - T_{0d})$

$$T_{00} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_\alpha}\dot{\phi}_\alpha - \mathcal{L}$$

$$= \pi_\alpha \dot{\phi}_\alpha - \mathcal{L} = \mathcal{H}$$

⇒ DENSIDADE DE ENERGIA

$$T_{01} = \frac{\partial \mathcal{L}}{\partial(\partial_x \phi_\alpha)}\dot{\phi}_\alpha$$ → 1ª COMPONENTE DA CORRENTE DE ENERGIA

P/ AS OUTRAS ~~EQS~~ , $j = 1, 2, 3$

$$\frac{1}{c}\frac{\partial T_{j0}}{\partial t} + \nabla\cdot\vec{T}_j = 0 \quad , \quad \vec{T}_j = (T_{j1}, T_{j2}, T_{jd})$$

↓

CONSERVAÇÃO DO MOMENTO EM CADA DIREÇÃO

$$T_{j0} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_\alpha}\partial_j \phi_\alpha$$

= j-ÉSIMA COMPONENTE DA DENSIDADE DE MOMENTO

EX: CORDA VIBRANTE:

$$\mathcal{L} = \frac{1}{2}\rho\left(\frac{\partial y}{\partial t}\right)^2 - \frac{1}{2}T_0\left(\frac{\partial y}{\partial x}\right)^2$$

$$\Rightarrow T_{00} = \rho\left(\frac{\partial y}{\partial t}\right)^2 - \mathcal{L} = \frac{\rho}{2}\left(\frac{\partial y}{\partial t}\right)^2 + \frac{T_0}{2}\left(\frac{\partial y}{\partial x}\right)^2 = \mathcal{H}$$

$$T_{01} = -T_0\left(\frac{\partial y}{\partial x}\right)\left(\frac{\partial y}{\partial t}\right)$$

DE MANEIRA GERAL $\frac{1}{c}\frac{d}{dt}T_{j0} + \nabla\cdot\vec{T}_j = 0$

$$\Rightarrow \frac{1}{c}\frac{d}{dt}\underbrace{\int_V dV\, T_{j0}}_{P_j} + \underbrace{\int_V dV\, \nabla\cdot\vec{T}_j}_{\oint \vec{T}_j\cdot d\vec{A}} = 0$$

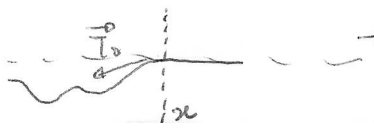
$j=0$

$$\Rightarrow \frac{1}{c}\frac{d}{dt}\underbrace{\int dV\, \mathcal{H}}_{H} + \oint \vec{T}_0\cdot d\vec{A} = 0$$

→ ENERGIA É CONSERVADA SE O SISTEMA NÃO IRRADIA NAS BORDAS

$T_{01} = -T_0\left(\frac{\partial y}{\partial x}\right)\left(\frac{\partial y}{\partial t}\right)$ = POTÊNCIA COM QUE O LADO ESQUERDO DA CORDA TRANSFERE ENERGIA P/ O LADO DIREITO

$\frac{\partial y}{\partial x}$ ↑



→ POTÊNCIA $= \vec{F}\cdot\vec{v} = F_y v_y = -T_0 \sin\theta \frac{\partial y}{\partial t}$

$$= -T_0\left(\frac{\partial y}{\partial x}\right)\left(\frac{\partial y}{\partial t}\right)$$