# Mecânica Quântica II - FFI 0122

## Lista 3

## Data de Entrega: 26 de Outubro de 2015, segunda-feira (na monitoria)

1. Todos os exercícios do capítulo VIII (Complemento $C_{VIII}$) do livro do Cohen-Tannoudji. Ver cópia anexa.

## 3. Exercises

### a. SCATTERING OF THE $p$ WAVE BY A HARD SPHERE

We wish to study the phase shift $\delta_1(k)$ produced by a hard sphere on the $p$ wave ($l = 1$). In particular, we want to verify that it becomes negligible compared to $\delta_0(k)$ at low energy.

$\alpha$. Write the radial equation for the function $u_{k,1}(r)$ for $r > r_0$. Show that its general solution is of the form:

$$u_{k,1}(r) = C\left[\frac{\sin kr}{kr} - \cos kr + a\left(\frac{\cos kr}{kr} + \sin kr\right)\right]$$

where $C$ and $a$ are constants.

$\beta$. Show that the definition of $\delta_1(k)$ implies that:

$$a = \tan \delta_1(k)$$

$\gamma$. Determine the constant $a$ from the condition imposed on $u_{k,1}(r)$ at $r = r_0$.

$\delta$. Show that, as $k$ approaches zero, $\delta_1(k)$ behaves like* $(kr_0)^3$, which makes it negligible compared to $\delta_0(k)$.

### b. "SQUARE SPHERICAL WELL": BOUND STATES AND SCATTERING RESONANCES

Consider a central potential $V(r)$ such that:

$$V(r) = -V_0 \quad \text{for} \quad r < r_0$$
$$= 0 \quad \text{for} \quad r > r_0$$

where $V_0$ is a positive constant. Set:

$$k_0 = \sqrt{\frac{2\mu V_0}{\hbar^2}}$$

We shall confine ourselves to the study of the $s$ wave ($l = 0$).

$\alpha$. *Bound states* ($E < 0$)

(*i*) Write the radial equation in the two regions $r > r_0$ and $r < r_0$, as well as the condition at the origin. Show that, if one sets:

$$\rho = \sqrt{\frac{-2\mu E}{\hbar^2}}$$

$$K = \sqrt{k_0^2 - \rho^2}$$

the function $u_0(r)$ is necessarily of the form:

$$u_0(r) = A\, e^{-\rho r} \quad \text{for} \quad r > r_0$$
$$= B \sin Kr \quad \text{for} \quad r < r_0$$

(*ii*) Write the matching conditions at $r = r_0$. Deduce from them that the only possible values for $\rho$ are those which satisfy the equation:

$$\tan Kr_0 = -\frac{K}{\rho}$$

(*iii*) Discuss this equation: indicate the number of $s$ bound states as a function of the depth of the well (for fixed $r_0$) and show, in particular, that there are no bound states if this depth is too small.

$\beta$. *Scattering resonances* ($E > 0$)

(*i*) Again write the radial equation, this time setting:

$$k = \sqrt{\frac{2\mu E}{\hbar^2}}$$

$$K' = \sqrt{k_0^2 + k^2}$$

Show that $u_{k,0}(r)$ is of the form:

$$u_{k,0}(r) = A \sin(kr + \delta_0) \quad \text{for} \quad r > r_0$$
$$= B \sin K'r \quad \text{for} \quad r < r_0$$

(*ii*) Choose $A = 1$. Show, using the continuity conditions at $r = r_0$, that the constant $B$ and the phase shift $\delta_0$ are given by:

$$B^2 = \frac{k^2}{k^2 + k_0^2 \cos^2 K'r_0}$$

$$\delta_0 = -kr_0 + \alpha(k)$$

with:

$$\tan \alpha(k) = \frac{k}{K'} \tan K'r_0$$

(*iii*) Trace the curve representing $B^2$ as a function of $k$. This curve clearly shows resonances, for which $B^2$ is maximum. What are the values of $k$ associated with these resonances? What is then the value of $\alpha(k)$? Show that, if there exists such a resonance for a small energy ($kr_0 \ll 1$), the corresponding contribution of the $s$ wave to the total cross section is practically maximal.

* This result is true in general: for any potential of finite range $r_0$, the phase shift $\delta_l(k)$ behaves like $(kr_0)^{2l+1}$ at low energies.

γ. *Relation between bound states and scattering resonances*

Assume that $k_0 r_0$ is very close to $(2n + 1)\frac{\pi}{2}$, where $n$ is an integer, and set:

$$k_0 r_0 = (2n + 1)\frac{\pi}{2} + \varepsilon \qquad \text{with} \qquad |\varepsilon| \ll 1$$

(*i*) Show that, if $\varepsilon$ is positive, there exists a bound state whose binding energy $E = -\hbar^2\rho^2/2\mu$ is given by:

$$\rho \simeq \varepsilon k_0$$

(*ii*) Show that if, on the other hand, $\varepsilon$ is negative, there exists a scattering resonance at energy $E = \hbar^2 k^2/2\mu$ such that:

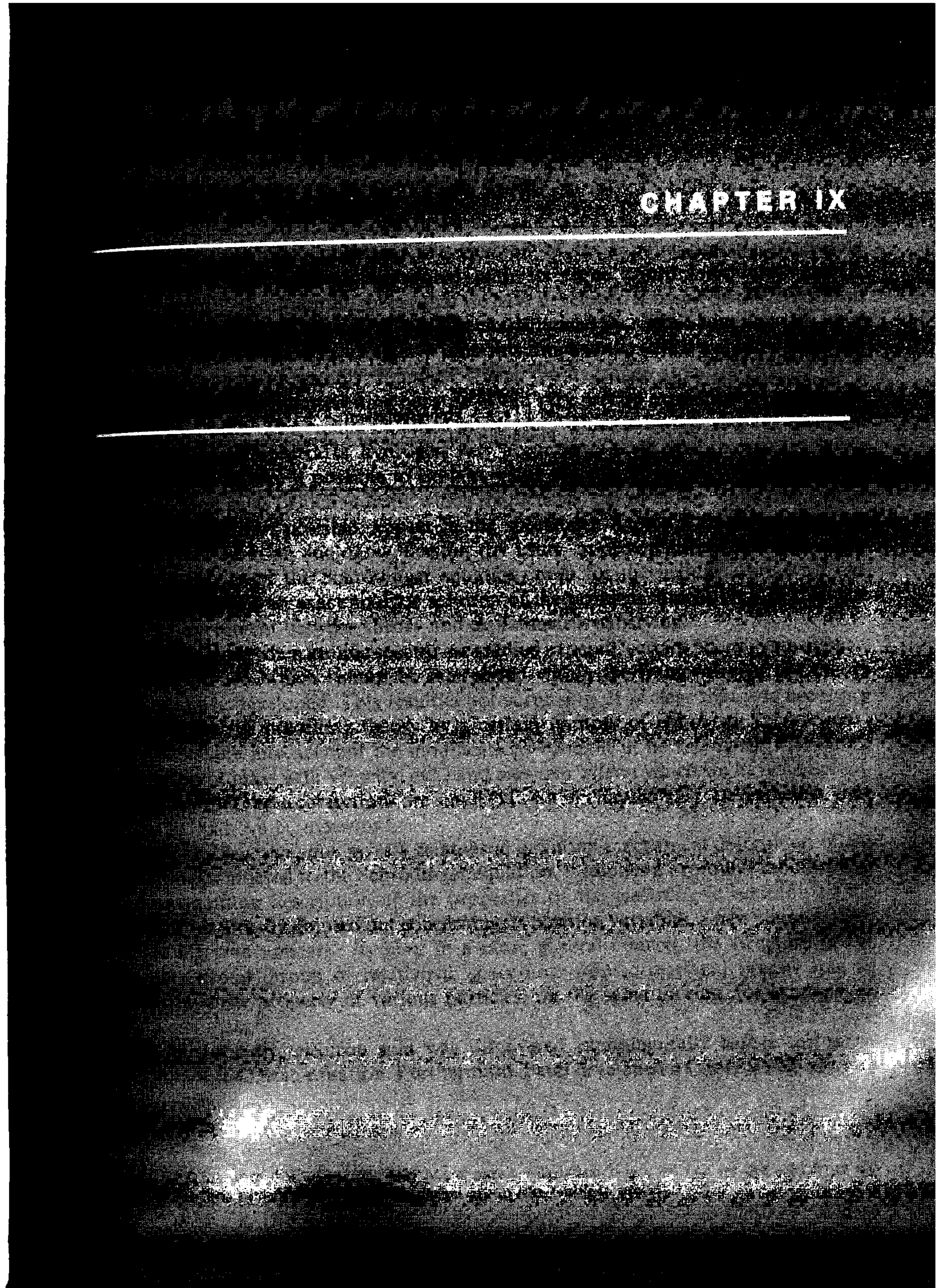
$$k^2 \simeq -\frac{2k_0\varepsilon}{r_0}$$

(*iii*) Deduce from this that if the depth of the well is gradually decreased (for fixed $r_0$), the bound state which disappears when $k_0 r_0$ passes through an odd multiple of $\pi/2$ gives rise to a low energy scattering resonance.

**References and suggestions for further reading**:

Messiah (1.17), chap. IX, § 10 and chap. X, §§ III and IV; Valentin (16.1), Annexe II.





# CHAPTER IX