Accelerated pendulum

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Solution: *a. Effective acceleration is,*

$$a_{ef} = \sqrt{g^2 + a^2} \ .$$

Hence the frequency for small amplitudes is,

$$\omega_0 = \sqrt{\frac{a_{ef}}{L}} = \sqrt{\frac{\sqrt{g^2+a^2}}{L}} \ .$$

We derive the equation of motion in the sketched Cartesian system, where $x = L\sin\alpha$ *and* $y = -L\cos\alpha$*. Here, Newton's equation* $m\ddot{\vec{x}} = m\vec{g} + m\vec{a}$ *becomes,*

$$mL\begin{pmatrix}\ddot{\alpha}\cos\alpha - \dot{\alpha}^2\sin\alpha \\ \ddot{\alpha}\sin\alpha + \dot{\alpha}^2\cos\alpha\end{pmatrix} = m\begin{pmatrix}\ddot{x}\\ \ddot{y}\end{pmatrix} = mg\sin\alpha\begin{pmatrix}-\cos\alpha \\ -\sin\alpha\end{pmatrix} + \begin{pmatrix}ma\\ 0\end{pmatrix} \ .$$

Calculating $m\ddot{x}\cos\alpha + m\ddot{y}\sin\alpha$ *we find,*

$$\ddot{\alpha} = -\frac{g}{L}\sin\alpha + \frac{a}{L}\cos\alpha \ .$$

To find the oscillation frequency, we need to put this equation into a form $\ddot{\alpha} = -A\sin(\alpha-\phi) = A\cos\phi\sin\alpha - A\sin\phi\cos\alpha$*. By comparison,*

$$A\cos\phi = -\frac{g}{L} \qquad , \qquad A\sin\phi = -\frac{a}{L} \ ,$$

or

$$A = \sqrt{\frac{g^2+a^2}{L^2}} \qquad , \qquad \tan\phi = \frac{a}{g} \ .$$

Hence,

$$\ddot{\alpha} = -\frac{\sqrt{g^2+a^2}}{L}\sin(\alpha - \arctan\tfrac{a}{g}) \simeq -\frac{\sqrt{g^2+a^2}}{L}(\alpha - \arctan\tfrac{a}{g}) \ ,$$

confirming the effective acceleration. Still, we transform to the variable $\tilde{\alpha} \equiv \alpha - \arctan\frac{a}{g}$*,*

$$\ddot{\tilde{\alpha}} = -\frac{\sqrt{g^2+a^2}}{L}\sin(\tilde{\alpha}) \ .$$

b. The equation $m\vec{x} = m\vec{g} + m\vec{a}$ *becomes,*

$$m\ddot{x} = -kx + ma \ .$$

Substituting $\tilde{x} = x - \frac{ma}{k}$*,*

$$m\ddot{x} = -k\left(\tilde{x} + \frac{ma}{k}\right) + ma = -k\tilde{x} = m\ddot{\tilde{x}} \ .$$