Buoy in the sea

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Solution: *The upward buoyant force of the water is,*

$$F_a = \sigma A g z \ ,$$

with the water density $\sigma = 1$ *kg/l and the immersion depth of the buoy* z. *Following Archimedes' law the buoy is at balance at the depth* z_0, *when* $F_a - (M+m)g = 0$, *that is, when,*

$$-\sigma A z_0 g = (M+m)g \ .$$

From this we derive an immersion depth with albatross of $z_0 = \frac{M+m}{\sigma A}$. *When the albatross takes off, the weight decreases and we get an acceleration,* $M\ddot{z} = -\sigma A z g - Mg$ *com* $z(0) = z_0$. *That is, the equation of motion is,*

$$\ddot{z} + \frac{\sigma A g}{M} z = -g \ .$$

Hence, the vibration frequency is $\omega = \sqrt{\frac{\sigma A g}{M}}$. *To make this inhomogeneous differential equation a homogeneous one, we make, as in the case of the spring-mass in the gravitational field, the substitution,* $\tilde{z} = z + \frac{M}{\sigma A}$. *With that we get,*

$$\ddot{\tilde{z}} + \frac{\sigma A g}{M} \tilde{z} = 0 \ .$$

The general solution of this equation is $\tilde{z} = C \sin \omega t + D \cos \omega t$, *such that,*

$$z = C \sin \omega t + D \cos \omega t - \frac{M}{\sigma A} \ .$$

With the known initial conditions we have,

$$z(0) = z_0 = C \sin \omega t + D \cos \omega t - \frac{M}{\sigma A} = D - \frac{M}{\sigma A}$$
$$\dot{z}(0) = 0 = \omega C \cos \omega t - \omega D \sin \omega t = \omega C \ ,$$

and we can calculate the vibration amplitude, $D = z_0 + \frac{M}{\sigma A} = -\frac{m}{\sigma A}$. *Hence, the complete solution is,*

$$z = -\frac{m}{\sigma A} \cos \omega t - \frac{M}{\sigma A} \ .$$