Damped physical pendulum

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Solution: *a. The mass of the cylinder is,*

$$M = \int_V \rho_0 dm = \rho_0 \int_0^L \int_0^{2\pi} \int_0^R r dr d\phi dz = \rho_0 \pi R^2 L \ .$$

The moment of inertia for a rotation around the center-of-mass is,

$$I_{cm} = \int_V \rho_0 r^2 dm = \rho_0 \int_0^L \int_0^{2\pi} \int_0^R r^2 r dr d\phi dz = \rho_0 \pi L \frac{R^4}{2} = \frac{M}{2} R^2 \ .$$

The moment of inertia for a rotation about the distant axis is,

$$I = I_{cm} + MR^2 = \frac{3M}{2} R^2 \ .$$

b. The equation of motion is,

$$I\alpha = |\vec{R} \times \vec{\tau}| = -MgR \sin\theta - b\dot{\theta} \ ,$$

or, considering small amplitude oscillations,

$$\ddot{\theta} + \frac{2b}{3MR^2}\dot{\theta} + \frac{2g}{3R}\theta = 0 \ .$$

c. The frequency without friction is,

$$\omega_0 = \sqrt{\frac{2g}{3R}} \ ,$$

the damping coefficient is,

$$\gamma = \frac{b}{3MR^2} \ ,$$

and the frequency with friction is,

$$\omega = \sqrt{\omega_0^2 - \gamma^2} = \sqrt{\omega_0^2 - \frac{b^2}{9M^2R^4}} \ .$$

d. Making the ansatz $\phi(t) = e^{-\gamma t}(Ae^{i\omega t} + Be^{-i\omega t})$, we obtain,

$$\phi(0) = 0 = A + B$$
$$\dot{\phi}(0) = \dot{\phi}_0 = (i\omega - \gamma)A + (-i\omega - \gamma)B \ .$$

Hence,

$$A = -B = \frac{\dot{\phi}_0}{2i\omega} \ ,$$

and finally,

$$\phi(t) = e^{-\gamma t}\left(\frac{\dot{\phi}_0}{2i\omega}e^{i\omega t} - \frac{\dot{\phi}_0}{2i\omega}e^{-i\omega t}\right) = \frac{\dot{\phi}_0}{\omega}e^{-\gamma t}\sin\omega t \ .$$