Resolution of the forced oscillator equation

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Solution: *Inserting the ansatz $x(t) = A\cos(\omega t - \delta)$ into the differential equation we get,*

$$-\omega^2 mA\cos(\omega t-\delta) - \omega bA\sin(\omega t-\delta) + m\omega_0^2 A\cos(\omega t-\delta) = F_0\cos\omega t\ .$$

Using the trigonometric rules,

$$\begin{aligned}\cos(\omega t-\delta) &= \cos\omega t\cos\delta + \sin\omega t\sin\delta\\ \sin(\omega t-\delta) &= \sin\omega t\cos\delta - \cos\omega t\sin\delta\end{aligned}$$

we obtain,

$$\begin{aligned}&-\omega^2 mA\left[\cos\omega t\cos\delta + \sin\omega t\sin\delta\right] - \omega bA\left[\sin\omega t\cos\delta - \cos\omega t\sin\delta\right]\\ &+m\omega_0^2 A\left[\cos\omega t\cos\delta + \sin\omega t\sin\delta\right] = F_0\cos\omega t\ .\end{aligned}$$

This equation must be valid for $t = \omega\pi/2$ and for $t = 0$. That is, we get two equations,

$$\begin{aligned}-\omega bA\cos\delta + m\left(\omega_0^2-\omega^2\right)A\sin\delta &= 0\\ \omega bA\sin\delta + m\left(\omega_0^2-\omega^2\right)A\cos\delta &= F_0\end{aligned}$$

or

$$\begin{aligned}\tan\delta &= \frac{b\omega}{m(\omega_0^2-\omega^2)}\\ A &= \frac{F_0}{\omega b\sin\delta + m(\omega_0^2-\omega^2)\cos\delta}\ .\end{aligned}$$

To solve the second equation we derive the following trigonometric rules

$$\begin{aligned}\tan\delta &= \frac{\sin\delta}{\cos\delta} = \frac{\sin\delta}{\sqrt{1-\sin^2\delta}} = \frac{1}{\sqrt{\frac{1}{\sin^2\delta}-1}} \qquad\Longrightarrow\qquad \sin\delta = \sqrt{\frac{1}{1+\frac{1}{\tan^2\delta}}}\\ \tan\delta &= \frac{\sin\delta}{\cos\delta} = \frac{\sqrt{1-\cos^2\delta}}{\cos\delta} = \sqrt{\frac{1}{\cos^2\delta}-1} \qquad\Longrightarrow\qquad \cos\delta = \frac{1}{\sqrt{1+\tan^2\delta}}\ .\end{aligned}$$

With this we can calculate:

$$\begin{aligned}A &= \frac{F_0}{\omega b\sqrt{\frac{1}{1+\frac{1}{\tan^2 x}}} + m(\omega_0^2-\omega^2)\frac{1}{\sqrt{1+\tan^2\delta}}}\\ &= \frac{F_0}{\omega b\sqrt{\frac{1}{1+\frac{1}{\left(\frac{b\omega}{m(\omega_0^2-\omega^2)}\right)^2}}} + m(\omega_0^2-\omega^2)\frac{1}{\sqrt{1+\left(\frac{b\omega}{m(\omega_0^2-\omega^2)}\right)^2}}} = \frac{F_0}{\sqrt{m^2(\omega_0^2-\omega^2)^2+b^2\omega^2}}\ .\end{aligned}$$